Cubic pencils of lines and bivariate interpolation

J. M. Carnicer* and M. Gasca*
University of Zaragoza, Spain

Dedicated to Claude Brezinski in occasion of his 65th birthday with friendship

Abstract: Cubic pencils of lines are classified up to projectivities. Explicit formulae for the addition of lines on the set of nonsingular lines of the pencils are given. These formulae can be used for constructing planar generalized principal lattices, which are sets of points giving rise to simple Lagrange formulae in bivariate interpolation. Special attention is paid to the irreducible nonsingular case, where elliptic functions are used in order to express the addition in a natural form.
Mathematics Subject Classification: 41A05, 65D05, 41A63

1. Introduction

Principal lattices of order n are classical distributions of points in multivariate interpolation, structured uniformly according to the geometry of a simplex. They give rise to Lagrange interpolation problems unisolvent in the space of polynomials of degree not greater than n. These lattices have often been used for the construction of finite elements [8]. Lee and Phillips [9] extended principal lattices to a more general setting as points of intersection of linear pencils of hyperplanes. Although the points need not to be regularly distributed, the simplicial structure of the nodes is preserved. In the bivariate case, the lattices introduced by Lee and Phillips are called three pencil lattices.

Given an irreducible cubic curve, a binary operation, addition of points, can be defined in the set of nonsingular points of the cubic, with the property that the addition of three points of the cubic is zero if they are collinear (cf. [2]). In [4], a definition of addition of points was provided for the reducible cases.

In [3, 4], it was shown that cubic pencils of lines in the plane can be used to generate sets of nodes with a triangular structuration and leading to simple Lagrange interpolation formulae. A main tool was the addition of lines, which is a dualization of the addition of points in a cubic curve. In those papers, it was shown that the principal lattice and three-pencil lattice constructions can be regarded as particular cases of reducible cubic pencils. The remaining cases of reducible cubic pencils were also studied. Generalized principal lattices in the plane were introduced in [3, 4] to describe all sets of nodes with a triangular structure including all lattices obtained from cubic pencils. They have been extended to several variables in [5].

In [7], it was shown that planar generalized principal lattices can be obtained from the addition of lines in a cubic pencil. In other words, planar generalized principal lattices are just lattices coming from cubic pencils. Therefore the task of describing and classifying all generalized principal lattices in the plane can be reduced to describing the addition of lines in each cubic pencil.

In [4], a classification of cubic curves was recalled. By duality, a classification of cubic pencils could be deduced. The aim of the present paper is to provide a classification of

* Partially supported by the Spanish Research Grant MTM2006-03388, by Gobierno de Aragón and Fondo Social Europeo.

cubic pencils and explicit formulae for the addition of lines, which may be convenient for practical uses. Moreover, in this task we have realized that the description of the addition of lines in nonsingular irreducible cubic pencils of lines requires elliptic functions. We use the Weierstrass $\mathcal{P}$ function to obtain a natural parameterization of irreducible nonsingular cubic pencils. We have compiled some complementary information on the Weierstrass $\mathcal{P}$ function which is scattered in the literature and can be useful for the reader (for related additional information see [1, 10]).

Section 2 is devoted to relevant definitions and to recall some related results. Section 3 offers a classification of all cubic pencils, analyzing a representative pencil of each equivalence class. An addition of lines is defined in the set of nonsingular lines of each pencil in order to describe all generalized principal lattices.

2. Definitions and related results

In [3,4] a definition of generalized principal lattices was given. In [6], a slightly more general definition was introduced. Let us remind this definition and some notations.

Definition 1. Let

$$L_i^r, \quad i = 0, \ldots, n, \quad r = 0, 1, 2,$$

be $3n+3$ distinct lines of the plane such that

$$L_i^0 \cap L_j^1 \cap L_k^2 \neq \emptyset, \quad \forall i, j, k \in \{0, \ldots, n\}, i + j + k = n.$$

The set of points

$$X = \{(x_{ijk}, y_{ijk}) \mid (x_{ijk}, y_{ijk}) := L_i^0 \cap L_j^1 \cap L_k^2, \ i, j, k \in \{0, \ldots, n\}, i + j + k = n\},$$

is a *generalized principal lattice of order* n if

$$i + j + k \neq n \implies L_i^0 \cap L_j^1 \cap L_k^2 \cap X = \emptyset.$$

Proposition 2.2 of [6] shows that if a point in X lies on a line L_i^r, then it cannot lie on any other line L_j^r, with $j \neq i$. This implies that different indices (i, j, k), $i + j + k = n$, correspond to different points of X. We also observe that, as shown in Proposition 2.5 (c) of [6], the lines L_i^r, $i \in \{0, \ldots, n-1\}$, $r = 0, 1, 2$, are uniquely determined by the set X. Conversely, the reduced set of $3n$ lines L_i^r, $i \in \{0, \ldots, n-1\}$, $r = 0, 1, 2$, is sufficient to determine all points of the set X. However, the lines L_n^r, $r = 0, 1, 2$, are not completely determined by the set X. For instance, the line L_n^0 can be replaced by any other line containing no point of X but (x_{n00}, y_{n00}).

Definition 2. A *polynomial pencil of degree* n in the plane is the set of all lines

$$\{(x, y) \in \mathbf{R}^2 \mid ax + by + c = 0\}, \quad (a, b, c) \neq (0, 0, 0),$$

such that

$$F(a, b, c) = 0,$$

where F is a homogeneous polynomial of degree n in 3 variables. A polynomial pencil of degree 1, (resp., 2, 3) is called a *linear* (resp., *quadratic*, *cubic*) pencil. A *vertex* V of a polynomial pencil is a point such that the linear pencil of all lines passing through V is contained in the polynomial pencil. If a polynomial pencil has a vertex, then a homogeneous linear polynomial is a factor of $F(a,b,c)$ and F is a reducible polynomial. A *nonsingular line* of a pencil is any line $ax+by+c=0$ such that

$$F(a,b,c)=0, \quad \Big(\frac{\partial F}{\partial a}(a,b,c), \frac{\partial F}{\partial b}(a,b,c), \frac{\partial F}{\partial c}(a,b,c)\Big) \neq (0,0,0).$$

According to sections 3 and 5 of [4], we can state the following proposition.

Proposition 3. *Let Λ be the set of nonsingular lines and V the set of vertices of a cubic pencil of lines. Then a binary operation $\oplus$ called an addition of lines can be defined on Λ such that the following properties hold*

(a) $(\Lambda, \oplus)$ is an abelian group,

(b) for any distinct lines $L_1, L_2, L_3 \in \Lambda$ such that $L_1 \cap L_2 \cap L_3 \cap V = \emptyset$, then

$$L_1 \oplus L_2 \oplus L_3 = 0 \Leftrightarrow L_1 \cap L_2 \cap L_3 \neq \emptyset.$$

In Theorem 3.1 of [4], the construction of generalized principal lattices was analyzed assuming that an addition of lines has been defined. We recall that Definition 1 is more general than the one given in [4]. For this reason, in Theorem 2.4 of [7], a new version of that result was given that we restate below. We need the following notation: Λ denotes any set of lines equipped with a binary operation $\oplus$ such that (a) and (b) of Proposition 3 hold, iH denotes the sum of i equal terms $H \oplus \cdots \oplus H$, $\ominus H$ denotes the opposite of H in the abelian group and 0 denotes the neutral element of the group.

Theorem 4. *Let $\oplus$ be an addition of lines defined on Λ with a set of vertices V.*

(i) Let H, K_1, K_2 be three lines of Λ. Then the $3n+3$ lines

$$\begin{aligned} L_i^0 &= K_1 \oplus iH, \quad i \in \{0,\dots,n\}, \\ L_j^1 &= K_2 \oplus jH, \quad j \in \{0,\dots,n\}, \\ L_k^2 &= \ominus K_1 \ominus K_2 \oplus (k-n)H, \quad k \in \{0,\dots,n\}, \end{aligned} \tag{1}$$

are distinct if and only if

$$mH \neq 0, \quad 0 < m \leq n, \tag{2}$$

and

$$K_1 \ominus K_2 \oplus nH,\ \ominus 2K_1 \ominus K_2,\ \ominus K_1 \ominus 2K_2 \notin \{mH \mid m \in \{0,\dots,2n\}\}. \tag{3}$$

Moreover, if $i,j,k \in \{0,\dots,n\}$, $i+j+k=n$, then we have that $L_i^0 \oplus L_j^1 \oplus L_k^2 = 0$.

(ii) Let H, K_1, K_2 be three lines of Λ satisfying (2) and (3) and define L_i^0, L_j^1 and L_k^2 by (1). If $i,j,k \in \{0,\dots,n\}$, $i+j+k=n$, then L_i^0, L_j^1 and L_k^2 are concurrent at a point $(x_{ijk}, y_{ijk}) := L_i^0 \cap L_j^1 \cap L_k^2$. Let $X := \{(x_{ijk}, y_{ijk}) \mid i,j,k \in \{0,\dots,n\}, i+j+k=n\}$. If $X \cap V = \emptyset$, then X is a generalized principal lattice.

(iii) Let X be a generalized principal lattice of order n contained in $\mathbf{R}^2 \setminus V$, defined by lines L_i^r, $i \in \{0, \ldots, n\}$, $r = 0, 1, 2$, belonging to the set Λ. Then there exist $H, K_1, K_2 \in \Lambda$ satisfying (2) and (3) such that (1) holds.

In Theorem 3.5 of [7], it was shown that, if X is a generalized principal lattice, then there exists a cubic pencil containing all the lines in the reduced determining set of $3n$ lines $\{L_i^r \mid i = 0, \ldots, n-1, r = 0, 1, 2\}$ mentioned after Definition 1. We restate it here.

Theorem 5. *Let X be a generalized principal lattice of order $n \geq 4$. Then there exists a unique cubic pencil containing all the lines L_i^r, $i \in \{0, \ldots, n-1\}$, $r = 0, 1, 2$.*

Remark 6. If X is a generalized principal lattice of order $n < 4$, then the results in [7] show the existence of a cubic pencil containing all the lines L_i^r, $i \in \{0, \ldots, n-1\}$, $r = 0, 1, 2$, but it is not unique.

Using the above results, we can obtain a general formula for the lines L_i^r defining a generalized principal lattice. Take any generalized principal lattice X, defined by the lines L_i^r, $i \in \{0, \ldots, n\}$, $r = 0, 1, 2$. By Theorem 5 and Remark 6, there exists a cubic pencil containing all the lines L_i^r, $i \in \{0, \ldots, n-1\}$, $r = 0, 1, 2$. Let Λ be the set of nonsingular lines of the cubic pencil. If the lines L_i^r, $i \in \{0, \ldots, n-1\}$, $r = 0, 1, 2$, belong to Λ (i.e., they are nonsingular), then, by Proposition 3, we can define an addition of lines on Λ. Moreover, an abelian group G can be associated to Λ and we can parameterize Λ by a bijection $L : G \to \Lambda$ such that $L(g_1 + g_2) = L(g_1) \oplus L(g_2)$, $g_1, g_2 \in G$.

If, in addition, the set X does not contain any vertex, we deduce from Theorem 4 (iii), that there exist suitable group elements g_0, g_1, g_2 ($g_0 + g_1 + g_2 = 0$) and h such that the lines

$$L(g_r + ih), \quad i \in \{0, \ldots, n\}, \quad r = 0, 1, 2, \tag{4}$$

generate the generalized principal lattice X. Since, by Proposition 2.5 (c) of [6], the lines L_i^r, $i \in \{0, \ldots, n-1\}$, $r = 0, 1, 2$ are uniquely determined by the set X, we have that

$$L_i^r = L(g_r + ih), \quad i \in \{0, \ldots, n-1\} \quad r = 0, 1, 2.$$

Let us remark that a projective transformation of the plane maps a cubic pencil into another cubic pencil, so that, the set of nonsingular lines Λ of one of them corresponds to the set of nonsingular lines $\tilde{\Lambda}$ of the other one. The projective transformation gives rise to a group isomorphism between the abelian groups Λ and $\tilde{\Lambda}$. Each set of $3n + 3$ lines of Λ defining a generalized principal lattice, corresponds to a set of $3n + 3$ lines of $\tilde{\Lambda}$ with the same incidence properties. Therefore, we can consider both sets of lines Λ and $\tilde{\Lambda}$ equivalent. This induces a partition on the set of all cubic pencils into equivalence classes. Hence, it is sufficient to analyze a representative of each class, as done in the next section.

3. Classification

A projective classification of cubic pencils can be obtained by duality of the projective classification of cubic curves. Since we are interested in classifying generalized principal lattices in the real plane rather than in the complex plane, we shall use the real projective classification.

A first criterion is reducibility. Reducible pencils can be completely reducible with 3 linear factors (3 vertices), or can have 2 irreducible factors, one linear (1 vertex) and one quadratic (the tangents to a conic). Irreducible singular pencils can be classified according to the number of cusps of the cubic or quartic enveloped by the pencil. Another criterion of classification is the number of connected components of the pencil, which coincides with the number of connected components of the group and is used for classifying irreducible nonsingular pencils. So we have the following classification.

- A. Completely reducible pencils
  - A.1. Three collinear vertices (principal lattices)
  - A.2. Three noncollinear vertices (three pencil lattices)
- B. Reducible pencils with one vertex and the set of tangents to a conic
  - B.1. Vertex interior to the conic
  - B.2. Vertex on the conic
  - B.3. Vertex exterior to the conic
- C. Irreducible singular pencils
  - C.1. Tangents to a cubic with 1 real cusp (semicubical parabola)
  - C.2. Tangents to a tricuspidal quartic with 3 real cusps (deltoid)
  - C.3. Tangents to a tricuspidal quartic with 1 real cusp
- D. Irreducible nonsingular pencils
  - D.1. One connected component
  - D.2. Two connected components

In each of the cases, we shall describe the group, a canonical cubic pencil parameterized by the group and an addition of lines. Moreover we shall depict a representative set of lines of the pencil in order to visualize the vertices and/or envolvent of the pencil and an example of construction which has been computed using formula (4).

A. COMPLETELY REDUCIBLE PENCILS

They consist of three linear pencils with vertices V_1, V_2, V_3. Two different cases arise. If the three vertices are collinear then we can take them without loss of generality in the ideal line and recover principal lattices, an example providing interpolation sets classical in bivariate interpolation. The case of noncollinear vertices is essentially the Lee and Phillips construction of three pencil lattices introduced in [9].

The interpolation points are intersections of three lines adding up to zero. In order to apply Theorem 4, we need to choose lines not intersecting at a vertex, that is, not belonging to the same linear pencil. Then we need to take three lines $L(t_i, r_i)$ in A.1 (resp., $L(t_i, s_i, r_i)$ in A.2), $i = 0, 1, 2$, with three different indices $r_i \in \mathbf{Z}_3$ $i = 0, 1, 2$. Without loss of generality, we can take $r_0 = 0$, $r_1 = 1$ and $r_2 = 2$. In A.1, it is sufficient to choose $t_0, t_1, t_2 \in \mathbf{R}$ such that $t_0 + t_1 + t_2 = 0$. In A.2, we also have to choose $s_0, s_1, s_2 \in \mathbf{Z}_2$ such that $s_0 + s_1 + s_2 = 0$ and different cases may arise according to four different choices of the $s_i \in \mathbf{Z}_2$, $i = 0, 1, 2$. Either $s_0 = s_1 = s_2 = 0$ or only one s_i is zero, $i = 0, 1, 2$. The four cases correspond to intersections in each of the four quadrants (connected components) in which the axes divide the plane in Figure A.2 (left).

A.1. Completely reducible cubic pencils. Principal lattices.

We take as vertices the direction of the axes and the direction of the bisector of the 2nd and 4th quadrant.
— Group: $\mathbf{R} \times \mathbf{Z}_3$.
— Pencil: $L(t, 0) \equiv x = t$, $L(t, 1) \equiv y = t$, $L(t, 2) \equiv x + y = -t$, $t \in \mathbf{R}$.
— Addition of lines: $L(t_1, r_1) \oplus L(t_2, r_2) = L(t_1 + t_2, r_1 + r_2)$, $t_1, t_2 \in \mathbf{R}$, $r_1, r_2 \in \mathbf{Z}_3$.

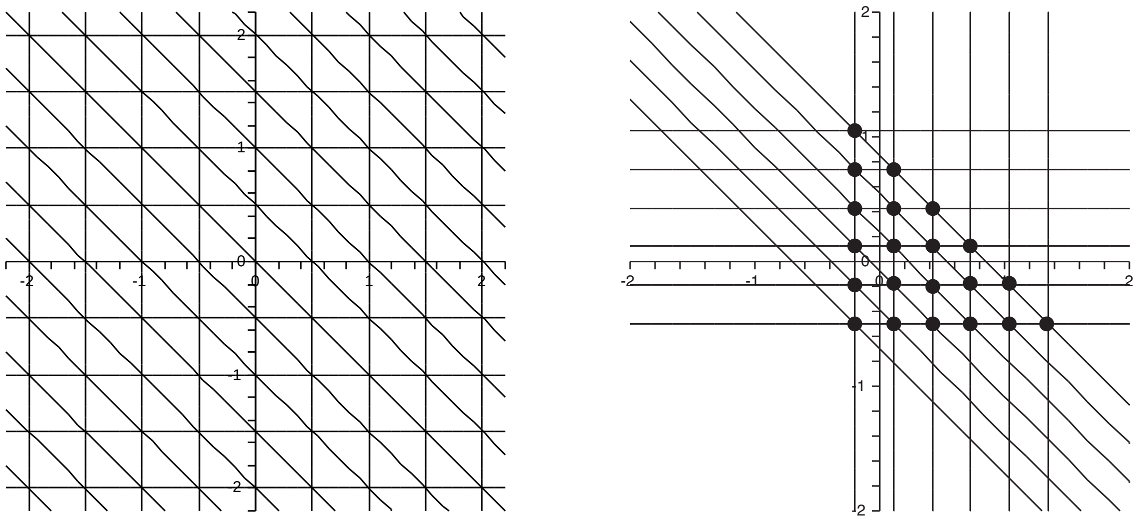

Figure A.1. Three directional mesh (left) and a principal lattice (right)

A.2. Completely reducible cubic pencils. Three pencil lattices.

The vertices are taken now to be the origin and the directions of the axes.
— Group: $\mathbf{R} \times \mathbf{Z}_2 \times \mathbf{Z}_3$.
— Pencil: $L(t, s, 0) \equiv x = (-1)^s \exp(t)$, $L(t, s, 1) \equiv y = (-1)^s \exp(-t)$, $L(t, s, 2) \equiv y = (-1)^s \exp(t)x$, $t \in \mathbf{R}$, $s \in \mathbf{Z}_2$.
— Addition of lines: $L(t_1, s_1, r_1) \oplus L(t_2, s_2, r_2) = L(t_1 + t_2, s_1 + s_2, r_1 + r_2)$, $t_1, t_2 \in \mathbf{R}$, $s_1, s_2 \in \mathbf{Z}_2$, $r_1, r_2 \in \mathbf{Z}_3$.

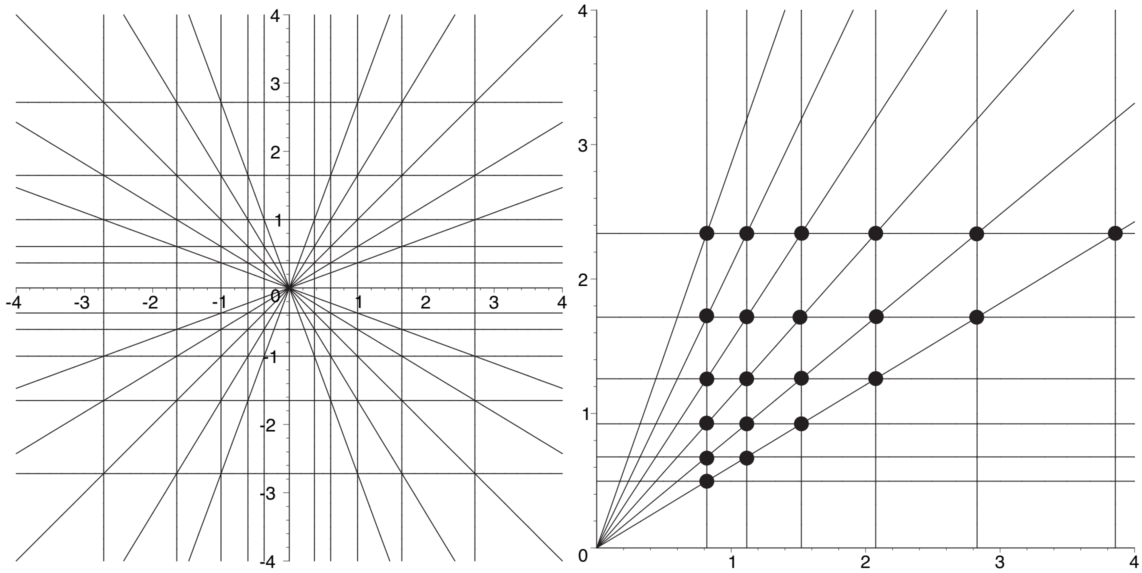

Figure A.2. A mesh generated by three pencils (left) and a three pencil lattice (right)

B. REDUCIBLE PENCILS: ONE QUADRATIC AND ONE LINEAR PENCIL

Now let us discuss the classification of lattices generated by (reducible) cubic pencils consisting of an irreducible quadratic pencil and a linear pencil. Let V be the vertex of the linear pencil and C the envolvent of the quadratic pencil, which is an irreducible conic. There are three different cases according to the relative position of V and C. The first case, V interior to C, will be illustrated with the tangents to a circle and the lines through its center. The second case, where V lies on C, will be illustrated with the tangents to a parabola and its diameters (lines parallel to the axis). The third case, V exterior to C, will be illustrated with the tangents to an equilateral hyperbola and the lines through its center. All these cases have been analyzed in [3].

According to Theorem 4, we need to choose three lines not intersecting at a vertex. In this case, there is only one vertex. This implies that exactly one of the three lines belongs to the linear pencil and the other two belong to the quadratic pencil. Then we need to take three lines $L(t_i, r_i)$ in B.1 or B.2 (resp., $L(t_i, s_i, r_i)$ in B.3), $i = 0, 1, 2$ with one of the r_i's equal to 0 and the other two r_i's equal to 1. Without loss of generality, we can take $r_0 = 0$, $r_1 = r_2 = 1$. In B.1, it is sufficient to choose $t_0, t_1, t_2 \in \mathbf{R}$ such that $t_0 + t_1 + t_2 \in 2\pi\mathbf{Z}$. In B.2 and B.3, we have to choose $t_0, t_1, t_2 \in \mathbf{R}$ such that $t_0 + t_1 + t_2 = 0$. In B.3 we also have to choose $s_0, s_1, s_2 \in \mathbf{Z}_2$ such that $s_0 + s_1 + s_2 = 0$ and we have three possible choices of the $s_i \in \mathbf{Z}_2$, $i = 0, 1, 2$. Either $s_0 = 0$ and $s_1 = s_2 = 0$, or $s_0 = 0$ and $s_1 = s_2 = 1$, or $s_0 = 1$ and $s_1 \neq s_2$.

B.1. Reducible pencil. Vertex interior to the conic.

C is a circle and V its center.

— Group: $(\mathbf{R}/2\pi\mathbf{Z}) \times \mathbf{Z}_2$.
— Curve: $x(t) = \cos t$, $y(t) = \sin t$, $t \in \mathbf{R}$.
— Pencil: $L(t, 0) \equiv \sin(t/2)x + \cos(t/2)y = 0$, $L(t, 1) \equiv \cos t\, x + \sin t\, y = 1$, $t \in \mathbf{R}/2\pi\mathbf{Z}$.
— Addition of lines: $L(t_1, r_1) \oplus L(t_2, r_2) = L(t_1 + t_2, r_1 + r_2)$, $t_1, t_2 \in \mathbf{R}/2\pi\mathbf{Z}$, $r_1, r_2 \in \mathbf{Z}_2$.

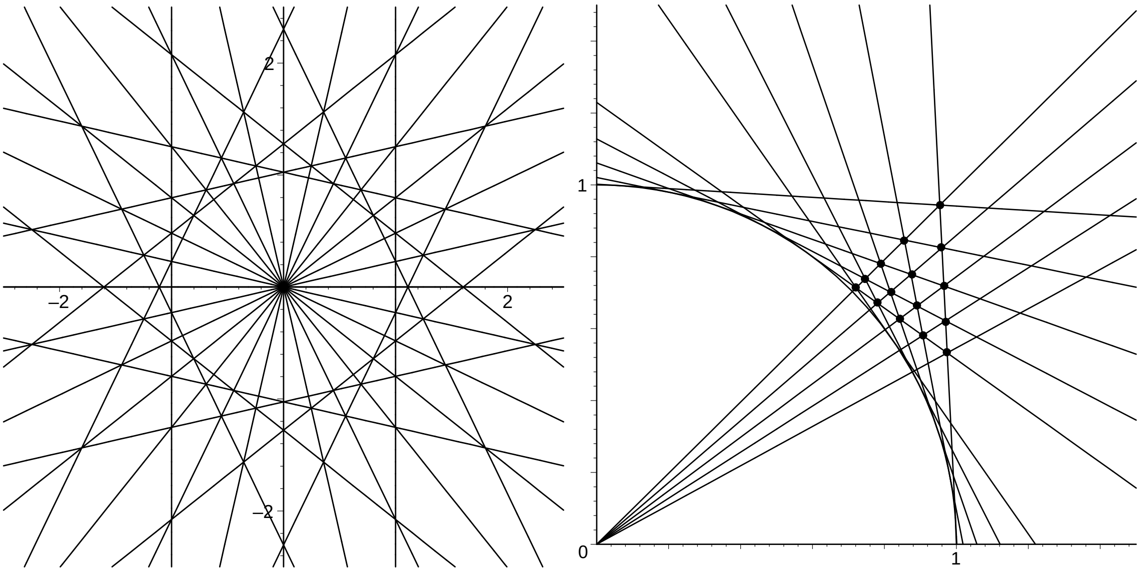

Figure B.1. Tangents to a circle and its diameters (left)
and a generalized principal lattice (right)

B.2. Reducible pencil. Vertex on the conic.

C is a parabola and V the direction of its axis.

— Group: $\mathbf{R} \times \mathbf{Z}_2$.

— Curve: $x(t) = t$, $y(t) = t^2$, $t \in \mathbf{R}$.

— Pencil: $L(t, 0) \equiv x + t/2 = 0$, $L(t, 1) \equiv y - 2tx + t^2 = 0$, $t \in \mathbf{R}$.

— Addition of lines: $L(t_1, r_1) \oplus L(t_2, r_2) = L(t_1 + t_2, r_1 + r_2)$, $t_1, t_2 \in \mathbf{R}$, $r_1, r_2 \in \mathbf{Z}_2$.

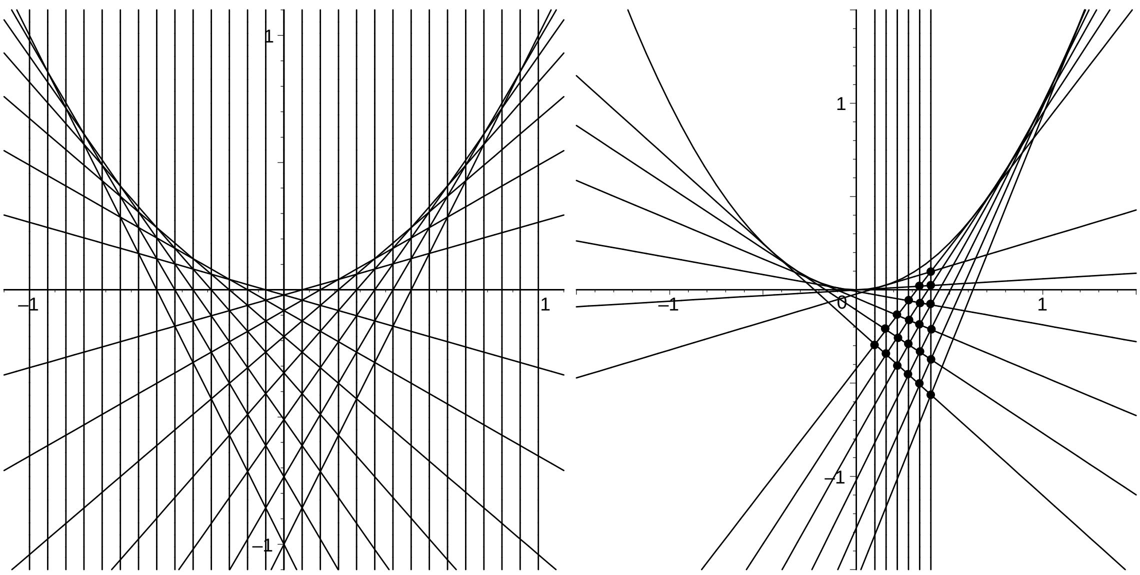

Figure B.2. Tangents to a parabola and its diameters (left)
and a generalized principal lattice (right)

B.3. Reducible pencil. Vertex exterior to the conic.

C is a hyperbola and V the intersection of its asymptotes.

— Group: $\mathbf{R} \times \mathbf{Z}_2 \times \mathbf{Z}_2$.

— Curve: $x(t) = (-1)^s \exp(t)$, $y(t) = (-1)^s \exp(-t)$, $t \in \mathbf{R}$, $s \in \mathbf{Z}_2$.

— Pencil: $L(t, s, 0) \equiv y - (-1)^s \exp(t) x = 0$, $L(t, s, 1) \equiv (-1)^s \exp(t) y + (-1)^s \exp(-t) x - 2 = 0$, $t \in \mathbf{R}$, $s \in \mathbf{Z}_2$.

— Addition of lines: $L(t_1, s_1, r_1) \oplus L(t_2, s_2, r_2) = L(t_1 + t_2, s_1 + s_2, r_1 + r_2)$, $t_1, t_2 \in \mathbf{R}$, $s_1, s_2 \in \mathbf{Z}_2$, $r_1, r_2 \in \mathbf{Z}_2$.

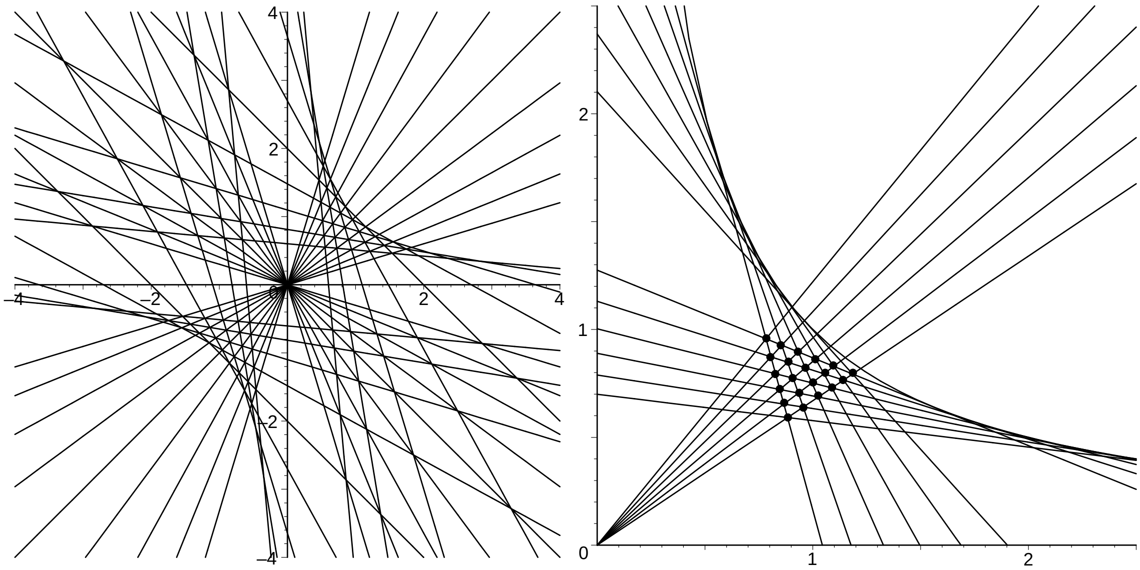

Figure B.3. Tangents to a hyperbola and its diameters (left)
and a generalized principal lattice (right)

C. IRREDUCIBLE SINGULAR CUBIC PENCILS

Irreducible pencils with singular lines admit a rational parameterization, which leads to a simple description of the groups in terms of polynomial, trigonometric and hyperbolic functions, similarly to the reducible cases. There are three distinct cases. The first case corresponds to the tangents to a semicubical parabola and has associated a particularly simple group, the additive group of the real numbers. The envelop of the pencil is a cubic with 1 real cusp, that is, another semicubical parabola. In the second case the lines envelop a tricuspidal quartic with 3 real cusps. Finally, the third case corresponds to the tangents to a tricuspidal quartic with two complex cusps. The first case was analyzed in [3] and the second case in [4].

In order to obtain an interpolation point, we need to choose three lines of the pencil. In C.2, it is sufficient to choose $t_0, t_1, t_2 \in \mathbf{R}$ such that $t_0 + t_1 + t_2 \in 2\pi\mathbf{Z}$. In C.1 and C.3, we have to choose $t_0, t_1, t_2 \in \mathbf{R}$ such that $t_0 + t_1 + t_2 = 0$. In C.3 we also have to choose $s_0, s_1, s_2 \in \mathbf{Z}_2$ such that $s_0 + s_1 + s_2 = 0$ and we have two possible choices of the $s_i \in \mathbf{Z}_2$, $i = 0, 1, 2$. Either $s_0 = s_1 = s_2 = 0$, or exactly one of the s_i's is 0 and the other two are 1, corresponding to each of the connected components in which the interpolation points may appear, as shown in Figure C.3 (left)

C.1. Irreducible singular cubic pencils. Tangents to a semicubical parabola.

— Group: $\mathbf{R}$.
— Curve: $x(t) = 3t^2$, $y(t) = 2t^3$, $t \in \mathbf{R}$.
— Pencil: $L(t) \equiv y = tx - t^3$, $t \in \mathbf{R}$.
— Addition of lines: $L(t_1) \oplus L(t_2) = L(t_1 + t_2)$, $t_1, t_2 \in \mathbf{R}$.

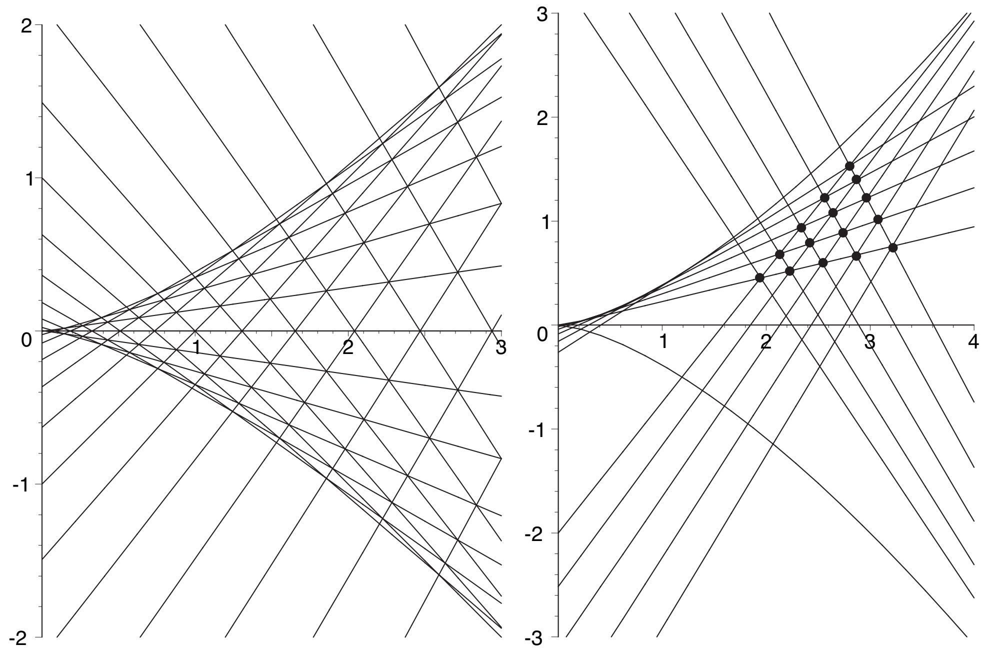

Figure C.1. Tangents to a semicubical parabola (left) and a generalized principal lattice (right)

C.2. Irreducible singular cubic pencils. Tangents to a tricuspidal quartic with three real cusps.

— Group: $\mathbf{R}/2\pi\mathbf{Z}$.
— Curve: $x(t) = \cos t(\cos t + 1)$, $\quad y(t) = \sin t(\cos t - 1)$.
— Pencil: $L(t) \equiv y = \tan(t/2)x - \sin(t)$, $t \in \mathbf{R}/2\pi\mathbf{Z}$.
— Addition of lines: $L(t_1) \oplus L(t_2) = L(t_1 + t_2)$, $t_1, t_2 \in \mathbf{R}/2\pi\mathbf{Z}$.

C.3. Irreducible singular cubic pencils. Tangents to a tricuspidal quartic with only one real cusp.

— Group: $\mathbf{R} \times \mathbf{Z}_2$.
— Curve: $x(t, s) = \cosh t(\cosh t + (-1)^s)$, $y(t, s) = \sinh t((-1)^s \cos t - 1)$, $t \in \mathbf{R}$, $s \in \mathbf{Z}_2$.
— Pencil: $L(t, 0) \equiv y = \tanh(t/2)x - \sinh(t)$, $L(t, 1) \equiv y = -\tanh(t/2)^{-1}x - \sinh(t)$, $t \in \mathbf{R}$.
— Addition of lines: $L(t_1, s_1) \oplus L(t_2, s_2) = L(t_1 + t_2, s_1, s_2)$, $t_1, t_2 \in \mathbf{R}$, $s_1, s_2 \in \mathbf{Z}_2$.

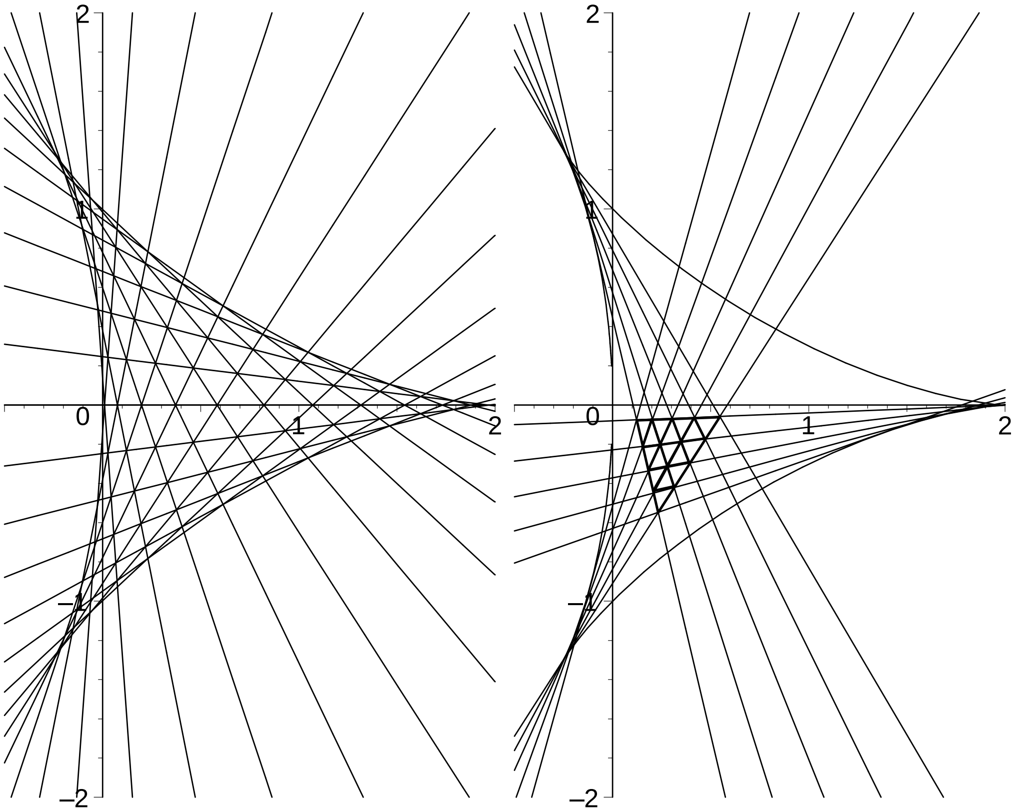

Figure C.2. Tangents to a tricuspidal quartic with three real cusps (left) and a generalized principal lattice (right)

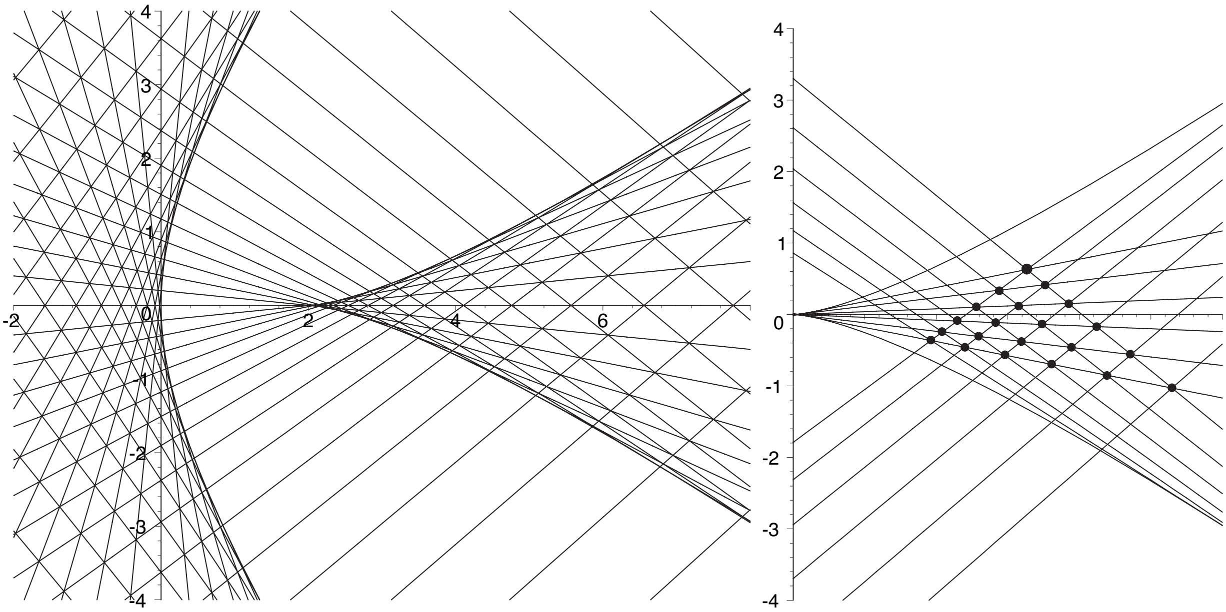

Figure C.3. Tangents to a tricuspidal quartic with one real cusp (left) and a generalized principal lattice (right)

D. IRREDUCIBLE NONSINGULAR CUBIC PENCILS

Irreducible nonsingular cubic pencils cannot be parameterized with rational functions. A natural parameterization can be done in terms of elliptic functions, that is doubly periodic meromorphic functions with half-periods $\omega_1, \omega_3 \in \mathbf{C}$. A third half-period $\omega_2 = -(\omega_1 + \omega_3)$ is used for symmetry reasons. There are different kinds of irreducible nonsingular cubic pencils from the projective point of view, according to the relation between both half-periods of the function. However, the associated complex groups are all isomorphic to the toroidal group $(\mathbf{R}/2\pi\mathbf{Z}) \times (\mathbf{R}/2\pi\mathbf{Z})$.

In the case of real cubic pencils, we can distinguish up to isomorphisms two different groups, the connected group $\mathbf{R}/2\pi\mathbf{Z}$ and the group with two connected components $(\mathbf{R}/2\pi\mathbf{Z}) \times \mathbf{Z}_2$. In each case, one has a one-parameter family of essentially different projective cubic pencils, with additions of lines described by the corresponding group.

Let us recall some facts about the parameterization of real cubic curves using the Weierstrass $\mathcal{P}$ function, which is a solution of the differential equation

$$\mathcal{P}'(t)^2 = 4\mathcal{P}^3(t) - g_2\mathcal{P}(t) - g_3,$$

where $g_2, g_3 \in \mathbf{R}$. The set of all complex points of the algebraic cubic with equation

$$y^2 = 4x^3 - g_2 x - g_3.$$

is given by the complex parameterization

$$\gamma(t) = (x(t), y(t)) = (\mathcal{P}(t), \mathcal{P}'(t)), \quad t \in \mathbf{C}.$$

The above parameterization has the advantage that an addition (with neutral element the ideal point corresponding to the direction of the Y axis) on the set of nonsingular points of the cubic is given by

$$\gamma(t_1) \oplus \gamma(t_2) = \gamma(t_1 + t_2),$$

because of the addition formula for Weierstrass $\mathcal{P}$ functions

$$\begin{vmatrix} 1 & \mathcal{P}(t_0) & \mathcal{P}'(t_0) \\ 1 & \mathcal{P}(t_1) & \mathcal{P}'(t_1) \\ 1 & \mathcal{P}(t_2) & \mathcal{P}'(t_2) \end{vmatrix} = 0, \quad t_0 + t_1 + t_2 = 0.$$

Let us observe that, although it is convenient to work with two parameters $g_2, g_3 \in \mathbf{C}$, the set of curves $y^2 = 4x^3 - g_2 x - g_3$ is essentially uniparametric, because the change of variables $\tilde{x} = \mu^2 x$, $\tilde{y} = \mu^3 y$, transforms the cubic into a projectively equivalent curve

$$\tilde{y}^2 = 4\tilde{x}^3 - \mu^{-4} g_2 \tilde{x} - \mu^{-6} g_3.$$

The number of connected components of the set of real points of the cubic coincides with the number of connected components of the group. If t ranges over $\mathbf{R}$, then

$$\gamma(t), \quad t \in \mathbf{R},$$

describes one connected component of the real cubic. If the curve has two connected components, we also need to consider a parameterization of the second component of the real cubic

$$\gamma(t+\omega_3), \quad t \in \mathbf{R},$$

where ω_3 denotes the pure imaginary semiperiod. By duality we derive the corresponding properties of the cubic pencils which we describe below. We also observe that the envolvent of the cubic pencil is a sextic curve with nine complex cusps (three of them real) with one or two connected components according to the number of components of the group.

In order to obtain an interpolation point, we need to choose three lines of the pencil. In D.1, it is sufficient to choose $t_0, t_1, t_2 \in \mathbf{R}$ such that $t_0 + t_1 + t_2 \in 2\omega_2\mathbf{Z}$. In D.2, we have to choose $t_0, t_1, t_2 \in \mathbf{R}$ such that $t_0 + t_1 + t_2 \in 2\omega_1\mathbf{Z}$ and $s_0, s_1, s_2 \in \mathbf{Z}_2$ such that $s_0 + s_1 + s_2 = 0$. We have two possible choices of the $s_i \in \mathbf{Z}_2$, $i = 0, 1, 2$. Either $s_0 = s_1 = s_2 = 0$ or exactly one of the s_i's is 0, corresponding to each of the connected components in which the interpolation points may appear, as shown in Figure D.2.

D.1. Irreducible nonsingular cubic pencils with one connected component.

In this case, ω_1 and ω_3 are complex conjugates and ω_2 is real and positive.

— Group: $\mathbf{R}/2\omega_2\mathbf{Z}$.
— Pencil: $L(t) \equiv \mathcal{P}'(t)y = \mathcal{P}(t)x + 1$, $t \in \mathbf{R}/2\omega_2\mathbf{Z}$.
— Addition of lines: $L(t_1) \oplus L(t_2) = L(t_1 + t_2)$, $t_1, t_2 \in \mathbf{R}/2\omega_2\mathbf{Z}$.

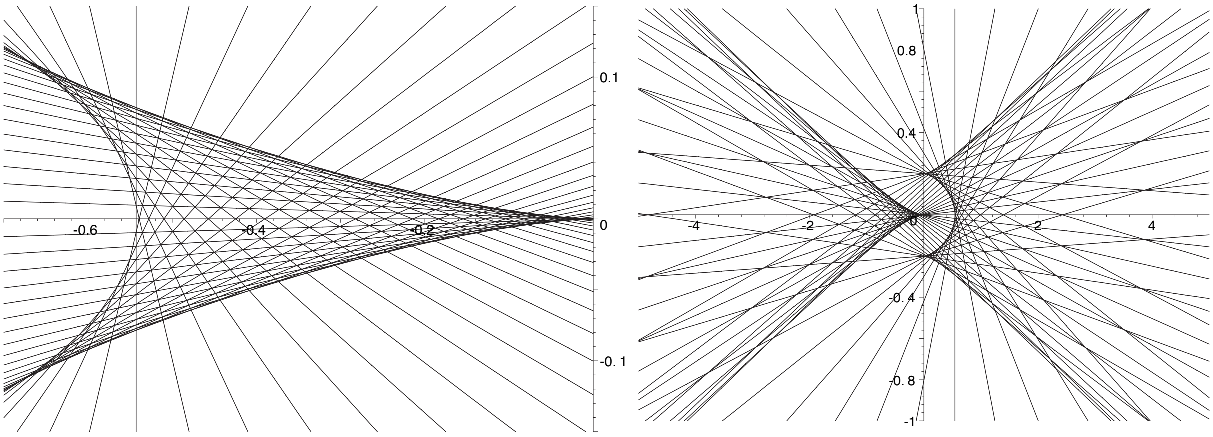

Figure D.1. Irreducible nonsingular cubic pencils with $g_2 = 0$, $g_3 = 25$ (left) and with $g_2 = 0$, $g_3 = -25$ (right)

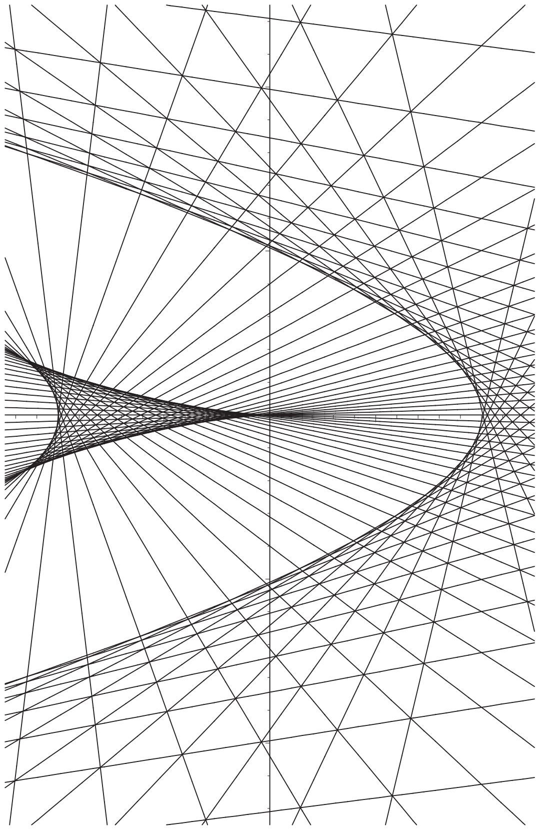

Figure D.2. Irreducible nonsingular cubic pencils with $g_2 = 25$, $g_3 = 0$

D.2. Irreducible nonsingular cubic pencils with two connected components.

In this case ω_1 is real and positive and ω_3 is pure imaginary.

— Group: $(\mathbf{R}/2\omega_1\mathbf{Z}) \times \mathbf{Z_2}$.

— Pencil: $L(t,0) \equiv \mathcal{P}'(t)y = \mathcal{P}(t)x + 1$. $L(t,1) \equiv \mathcal{P}'(t+\omega_3)y = \mathcal{P}(t+\omega_3)x + 1$, $t \in \mathbf{R}/2\omega_3\mathbf{Z}$.

— Addition of lines: $L(t_1,s_1) \oplus L(t_2,s_2) = L(t_1+t_2, s_1+s_2)$, $t_1,t_2 \in \mathbf{R}/2\omega_1\mathbf{Z}$, $s_1,s_2 \in \mathbf{Z_2}$.

Finally, let us remark that evaluation of the Weierstrass $\mathcal{P}$ function is only available in some computer algebra systems and needs longer computation time than other usual operations. However, the addition of lines in a cubic pencil can be expressed as a rational function of the line coordinates (a,b). The condition on three lines

$$b_i y = a_i x + 1, \quad i = 0,1,2,$$

of the pencil

$$b^2 = 4a^3 - g_2 a - g_3,$$

to sum up to zero is

$$\begin{vmatrix} a_0 & b_0 & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0.$$

Then we have

$$b_i = b_1 + m(a_i - a_1), \quad i = 0, 1, 2, \quad \text{where } m = \frac{b_2 - b_1}{a_2 - a_1},$$

which implies that a_i, $i = 0, 1, 2$, are the roots of the cubic equation

$$(b_1 + m(a - a_1))^2 = 4a^3 - g_2 a - g_3.$$

Therefore $a_0 + a_1 + a_2 = m^2/4$. So we obtain a_0, b_0 in terms of a_i, b_i, $i = 1, 2$,

$$a_0 = -(a_1 + a_2) + \frac{1}{4}m^2, \quad b_0 = b_1 + m(a_0 - a_1).$$

Taking into account that the opposite of the line $b_0 y = a_0 x + 1$ is the line $-b_0 y = a_0 x + 1$, we see that the sum of the lines $b_i y = a_i x + 1$, $i = 1, 2$ is the line $by = ax + 1$ with

$$a = -(a_1 + a_2) + \frac{1}{4}m^2, \quad b = -b_1 - m(a - a_1), \quad m = \frac{b_2 - b_1}{a_2 - a_1}.$$

This addition formula allows us to construct easily generalized principal lattices corresponding to cubic pencils of the form $b^2 = 4a^3 - g_2 a - g_3$.

Acknowledgement: The authors wish to thank Prof. Enrique Artal of the University of Zaragoza for his useful remarks concerning Algebraic Geometry and Elliptic Functions.

References

[1] Abramowitz, M. and Stegun, I. A., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 1964.

[2] Bix, R., *Conics and Cubics. A Concrete Introduction to Algebraic Curves*, Springer Verlag, New York, 1998.

[3] Carnicer, J. M. and Gasca, M., Generation of lattices of points for bivariate interpolation, *Numer. Algorithms* **39** (2005), 69–79

[4] Carnicer, J. M. and Gasca, M., Interpolation on lattices generated by cubic pencils, *Adv. Comput. Math.* **24** (2006), 113–130.

[5] Carnicer, J. M., Gasca, M. and Sauer, T., Interpolation lattices in several variables, *Numer. Math.* **102** (2006), 559–581.

[6] Carnicer, J. M. and Godés, M., Geometric characterization and generalized principal lattices, *J. Approx. Theory* **143** (2006), 2–14.

[7] Carnicer, J. M. and Godés, M., Generalized principal lattices and cubic pencils, to appear in *Numer. Algorithms*

[8] Ciarlet, P. G. and Raviart, P. A., General Lagrange and Hermite interpolation in $\mathbf{R}^n$ with applications to finite element methods, *Arch. Rational Mech. Anal.*, **46** (1972) 177–199.

[9] Lee, S. L. and Phillips, G. M., Construction of Lattices for Lagrange Interpolation in Projective Space, *Constr. Approx.* **7** (1991), 283–297.

[10] Walker, P. L., *Elliptic Functions: a constructive approach*, John Wiley and Sons, Chichester, 1996.