Generation of lattices of points for bivariate interpolation *

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Abstract. Principal lattices in the plane are distributions of points particularly simple to use Lagrange, Newton or Aitken-Neville interpolation formulas. Principal lattices were generalized by Lee and Phillips, introducing three-pencil lattices, that is, points which are the intersection of three lines, each one belonging to a different pencil. In this contribution, a semicubical parabola is used to construct lattices of points with similar properties. For the construction of new lattices we use cubic pencils of lines and an addition of lines on them.

Keywords: bivariate interpolation, principal lattices, Lagrange formula, geometric characterization

AMS Subject Classification: 41A05, 65D05, 41A63

1. Introduction

Principal lattices in the plane are the most well-known distributions of points which give rise to unisolvent interpolation problems in $\Pi_n(\mathbb{R}^2)$, the space of bivariate polynomials of total degree not greater than n. They are the intersection points of three pencils of equidistant parallel lines. By a suitable change of coordinates the three pencils of lines are given by the equations

$$x = i/n, \quad y = j/n, \quad 1 - x - y = k/n, \qquad 0 \leq i, j, k \leq n. \qquad (1.1)$$

Three lines each belonging to one pencil are concurrent if and only if $i + j + k = n$. Principal lattices form a simple structure of nodes useful for posing interpolation problems in the plane because they lead to solvable interpolation problems with simple interpolation formulae.

Lee and Phillips [6] described principal lattices as particular cases of three-pencil lattices. The incidence properties (collinearity of points and concurrence of lines) are invariant under projective transformations of the plane. Pencils of concurrent lines are all lines passing through a given vertex. From a projective point of view, pencils of parallel lines are pencils of concurrent lines whose vertex lies on the ideal line. So principal lattices are lattices generated by three pencils of

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lines whose vertices lie on the ideal line. Projective transformations of principal lattices provide lattices generated by three pencils with arbitrary collinear vertices. In [6] the general case of three pencils with noncollinear vertices was also analyzed.

The aim of this paper is to show some distributions of points in the plane with similar properties to those of three pencil lattices. In Section 2 we show that the set of tangents to a semicubical parabola can be used for a construction of that type, which is called a generalized principal lattice.

An operation called addition of points can be defined on the set of nonsingular points of an algebraic irreducible cubic curve, giving rise to an abelian group. Inspired by this idea and using duality, we introduced in [3] an addition on some sets of lines as a way of constructing generalized principal lattices. In the same paper we studied the general properties of this composition law in the case of cubic pencils of lines. We also showed an example of construction of generalized principal lattices from the (irreducible) cubic pencil of tangents to a deltoid.

In Section 3 of the present paper we show generalized principal lattices generated by reducible cubic pencils of lines, complementing in this way the theory provided in [3].

2. Lattices generated by semicubical parabolas

First we have a look to three pencil lattices in a form convenient to our purposes. Let $\Lambda_0, \Lambda_1, \Lambda_2$ be three pencils of lines concurrent at the noncollinear vertices v_0, v_1, v_2, respectively. Let $p_0(x,y) = 0$, $p_1(x,y) = 0$, $p_2(x,y) = 0$ be the equations of the sides of the triangle formed by the vertices. Then the lines of each of the pencils $p_1(x,y) - \mu_0 p_0(x,y) = 0$, $p_2(x,y) - \mu_1 p_1(x,y) = 0$, $p_0(x,y) - \mu_2 p_2(x,y) = 0$, are concurrent if and only if $\mu_0\mu_1\mu_2 = 1$. This condition allows us to select $3n+3$ lines

$$L_i^r \in \Lambda_r, \quad i = 0, \ldots, n, \quad r = 0, 1, 2,$$

such that L_i^0, L_j^1, L_k^2 are concurrent if and only if $i + j + k = n$. This can be done, for instance, in the following form

$$\begin{aligned}
L_i^0 &\equiv p_1(x,y) - \exp(t_0 + ih) p_0(x,y) = 0, \quad i = 0, \ldots, n, \\
L_j^1 &\equiv p_2(x,y) - \exp(t_1 + jh) p_1(x,y) = 0, \quad j = 0, \ldots, n, \\
L_k^2 &\equiv p_0(x,y) - \exp(t_2 + kh) p_2(x,y) = 0, \quad k = 0, \ldots, n,
\end{aligned}$$

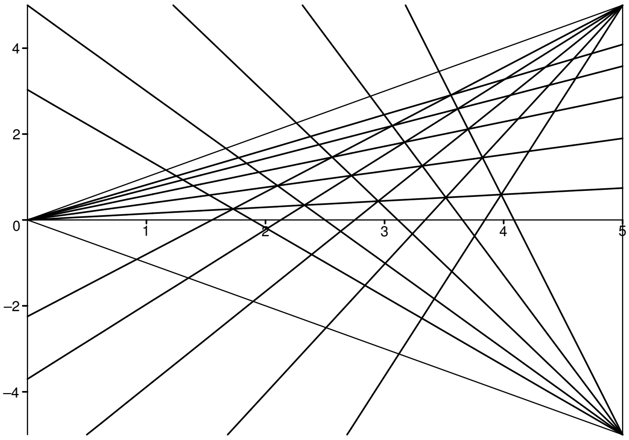

Figure 1. A three-pencil lattice

where $t_0, t_1, t_2 \in I\!R$, $h \in I\!R \setminus \{0\}$. The condition of concurrence for three lines each belonging to one pencil is

$$\exp(t_0 + ih) \exp(t_1 + jh) \exp(t_2 + kh) = 1,$$

that is, $t_0 + t_1 + t_2 + (i + j + k)h = 0$. If the following condition holds

$$t_0 + t_1 + t_2 + nh = 0 \qquad (2.1)$$

then the lines are concurrent if and only if $i+j+k = n$. Figure 1 shows an example with $p_0(x, y) = x - y$, $p_1(x, y) = x + y$, $p_2(x, y) = 5 - x$, $n = 4$, $h = 0.5$, $t_0 = 0.3$, $t_1 = -1.5$, $t_2 = -0.8$.

A similar situation can be described for the set of tangents to a semicubical parabola. Let us consider the semicubical parabola $(y/2)^2 = (x/3)^3$, which can be parameterized in the form

$$x(t) = 3t^2, \quad y(t) = 2t^3.$$

The parameter t represents the slope at $(x(t), y(t))$, $t \in I\!R$. Let $L(t)$ be the tangent line at $(x(t), y(t))$ whose equation is $tx - y - t^3 = 0$. Then three lines $L(t_0), L(t_1), L(t_2)$ of the family are concurrent if and only if

$$\begin{vmatrix} t_0 & -1 & t_0^3 \\ t_1 & -1 & t_1^3 \\ t_2 & -1 & t_2^3 \end{vmatrix} = 0,$$

that is, $(t_1 - t_0)(t_2 - t_0)(t_2 - t_1)(t_1 + t_2 + t_3) = 0$. If the lines are distinct, we have the condition

$$t_0 + t_1 + t_2 = 0.$$

So we have seen that if $t_0 + t_1 + t_2 = 0$, the three lines $t_i x - y - t_i^3 = 0$, $i = 0, 1, 2$, meet at the point $(-(t_0 t_1 + t_1 t_2 + t_2 t_0), -t_0 t_1 t_2)$, interior to the

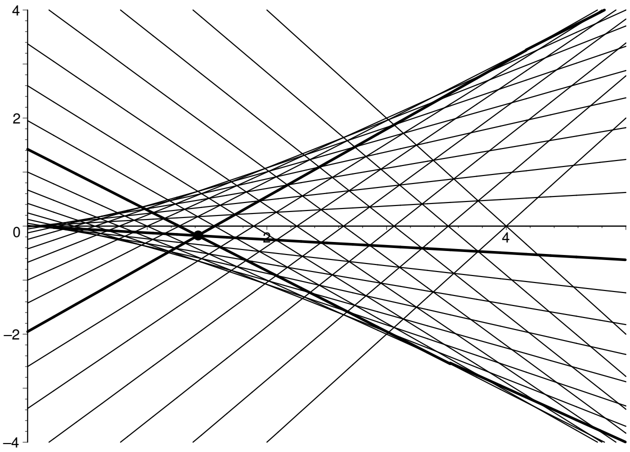

Figure 2. Tangents to a semicubical parabola

semicubical parabola, that is, lying in $(y/2)^2 \leq (x/3)^3$. Figure 2 shows the tangents to a semicubical parabola. The slopes of three tangents passing through the same point add up to 0.

Due to the analytic condition of concurrence, we can construct configurations of points which resemble principal lattices. In this case, there are no different pencils $\Lambda_0, \Lambda_1, \Lambda_2$. Any two lines intersect and no more than three lines can intersect at the same point. All the lines belong to the same one parameter family

$$L(t) \equiv tx - y - t^3 = 0, \quad t \in I\!R. \tag{2.2}$$

However we can again take $3n+3$ distinct lines of the family, which we denote by L_i^r, $i = 0, \dots, n$, $r = 0, 1, 2$, such that L_i^0, L_j^1, L_k^2 are concurrent if and only if $i + j + k = n$. This can be done with the choice

$$\begin{aligned} L_i^0 &\equiv L(t_0 + ih) = 0, \quad i = 0, \dots, n, \\ L_j^1 &\equiv L(t_1 + jh) = 0, \quad j = 0, \dots, n, \\ L_k^2 &\equiv L(t_2 + kh) = 0, \quad k = 0, \dots, n, \end{aligned}$$

where n and t_0, t_1, t_2, h satisfy (2.1). Figure 3 shows an example with $n = 4$, $h = 0.1$, $t_0 = -1.9$, $t_1 = 0.24$, $t_2 = 1.26$.

The common properties of the three-pencil lattices and the lattices generated by the tangents to a semicubical parabola suggest the following definition.

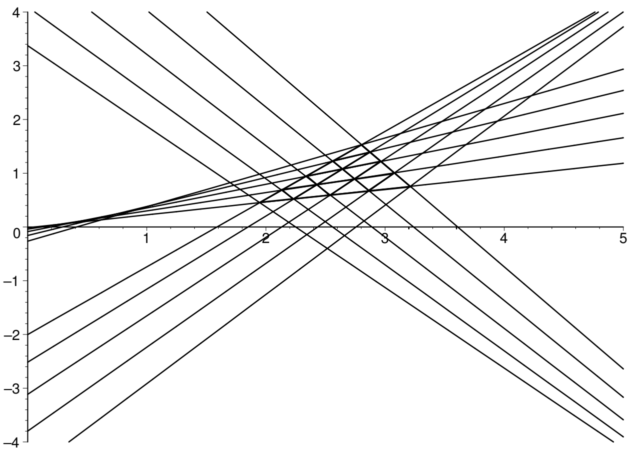

Figure 3. A lattice generated by a semicubical parabola

Definition 1. Let L_i^r, $i = 0, \ldots, n$, $r = 0, 1, 2$, be $3n+3$ distinct lines such that

$$L_i^0, L_j^1, L_k^2 \text{ are concurrent } \iff i+j+k=n. \tag{2.3}$$

A *generalized principal lattice* is the set of points

$$X = \{x_{ijk} \mid x_{ijk} := L_i^0 \cap L_j^1 \cap L_k^2,\ i,j,k \in \{0,1,\ldots,n\},\ i+j+k=n\}, \tag{2.4}$$

lying on exactly one line of each family.

Chung and Yao [4] introduced the *geometric characterization* to identify sets of nodes such that the Lagrange interpolation problem has a unique solution in $\Pi_n(\mathbb{R}^2)$ and the corresponding Lagrange polynomials are products of linear factors.

Definition 2. A set of $\binom{n+2}{2}$ nodes $X \subseteq \mathbb{R}^2$ satisfies the geometric characterization GC_n if for each node $x \in X$, there exist n lines containing all nodes in $X \setminus \{x\}$ but not x.

Generalized principal lattices satisfy the GC_n condition because for any point x_{ijk}, we have that

$$L_{i'}^0,\ i'<i, \quad L_{j'}^1,\ j'<j, \quad L_{k'}^2,\ k'<k,$$

is a set of n lines containing all nodes except x_{ijk}.

The corresponding Lagrange interpolation formula follows immediately.

3. Addition on cubic pencils and lattices generated by reducible pencils

The set of lines $ax + by + c = 0$ satisfying $F(a, b, c) = 0$, where F is a homogeneous cubic polynomial in 3 variables, is called a *cubic pencil* of lines.

An example of a cubic pencil is the set of three pencils of lines with vertices (x_0, y_0), (x_1, y_1) and (x_2, y_2), corresponding to the equation

$$(ax_0 + by_0 + c)(ax_1 + by_1 + c)(ax_2 + by_2 + c) = 0.$$

Analogously, it is easy to see that the set of tangents to the semicubic parabola $(y/2)^2 = (x/3)^3$ corresponds to the equation

$$a^3 + b^2c = 0.$$

In both cases we have constructed *generalized principal lattices* associated to those pencils. Let us indicate how to construct them from any cubic pencil.

It is well-known that the set of nonsingular points of an irreducible algebraic cubic curve [1], [5], can be equipped with a composition law called addition of points such that three points are collinear if and only if they add up to zero. By duality, this means that in the set of nonsingular lines of an irreducible cubic pencil we can define a composition law called *addition of lines* such that three lines are concurrent if and only if they add up to zero. This idea can be extended to reducible cubic curves (see [2]) and then to reducible cubic pencils (see [3]).

In the general case, a set V of vertices is defined such that three lines not meeting at a vertex add up to 0 if and only if they are concurrent at a point of $\mathbb{R}^2 \setminus V$. In the irreducible case, $V = \emptyset$. In the reducible case, V is the set of vertices of the linear pencils contained in the cubic pencil.

In each case, we can establish a parameterization $L : G \to \Lambda$ which is a group isomorphism between an abelian group G and the set of nonsingular lines of the pencil. The fact that the parameterization L is an isomorphism means that for $\gamma_1, \gamma_2, \gamma_3 \in G$,

$$\begin{aligned}&\gamma_1 + \gamma_2 + \gamma_3 = 0 \text{ and } L(\gamma_1) \cap L(\gamma_2) \cap L(\gamma_3) \cap V = \emptyset \iff \\ &L(\gamma_1), L(\gamma_2), L(\gamma_3) \text{ are concurrent at a point in } \mathbb{R}^2 \setminus V.\end{aligned}$$

Since L is an isomorphism, the addition of lines satisfies

$$L(\gamma_1) + L(\gamma_2) = L(\gamma_1 + \gamma_2), \quad \gamma_1, \gamma_2 \in G,$$

and then it can be seen as the binary operation on Λ induced by the group G through the bijection L.

In the case of all tangents to the semicubical parabola, $G = I\!R$, the group isomorphism is given by (2.2) and the addition of lines is $L(t_1) + L(t_2) = L(t_1 + t_2)$.

The construction of generalized principal lattices given in [3] can be formulated using the addition of lines. Consider the lines

$$\begin{aligned}
&L^0_i = L(g_0 + ih), \quad i = 0, \dots, n,\\
&L^1_j = L(g_1 + jh), \quad j = 0, \dots, n,\\
&L^2_k = L(g_2 + kh) = 0, \quad k = 0, \dots, n,\\
&g_0, g_1, g_2, h \in G, \quad g_0 + g_1 + g_2 + nh = 0,
\end{aligned} \tag{3.1}$$

where $ih \neq 0$, $0 < i \leq 2n$, and all elements

$$g_0+ih,\ i = 0, \dots, n,\ g_1+jh,\ j = 0, \dots, n,\ g_2+kh,\ k = 0, \dots, n, \tag{3.2}$$

are distinct. In the reducible case, there exist linear factors corresponding to pencils with given vertices and the set of intersection points (nodes) should not contain any vertex.

Remark 1. A projective mapping of the plane transforms a cubic pencil into another cubic pencil. The set of nonsingular lines of one pencil Λ will be transformed into the set of nonsingular lines of the other one $\tilde{\Lambda}$. The projective mapping acts as a group isomorphism between Λ and $\tilde{\Lambda}$ and so the associated abelian group is essentially the same. Each set of lines of Λ (3.1) giving rise to a generalized principal lattice will be transformed into a set of lines of $\tilde{\Lambda}$ with the same incidence properties. So, $\tilde{\Lambda}$ also generates a generalized principal lattice. In this sense, we consider both sets of lines Λ and $\tilde{\Lambda}$ equivalent.

The lattice generated by a completely reducible pencil (product of three linear pencils) is the three-pencil lattice studied in [6]. According to Remark 1, we have two essentially different cases. The first case corresponds to three noncollinear vertices and an example is discussed in the previous section. The second case corresponds to three collinear vertices and gives rise to all projective transformations of principal lattices (1.1). Both cases are considered in more detail in [3].

The remaining reducible cases are lattices generated by two pencils: a linear pencil and an irreducible quadratic pencil (that is, the set of tangents to an irreducible conic). According to Remark 1, we must distinguish three essentially different cases. The vertex in the interior of the conic, the vertex lying on the conic and the vertex in the exterior of the conic. The set of nonsingular lines are all lines of the pencil excluding the tangents to the conic from the vertex.

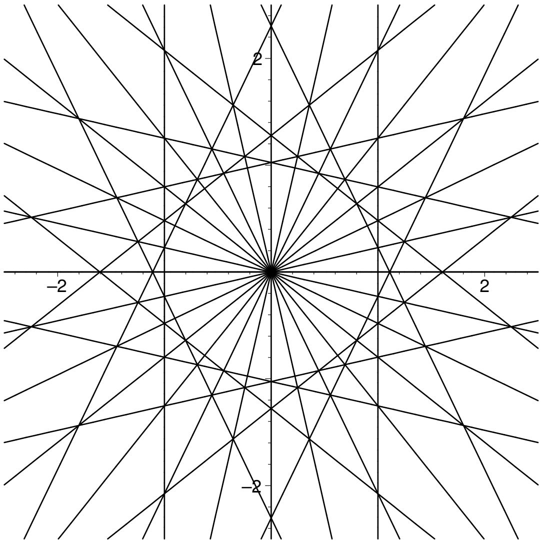

Figure 4. Tangents to a circle and lines through its center

We are going to analyze the three cases and then we shall sketch the construction of generalized principal lattices from them, taking into account the different group structures. Let us analyze first the reducible cubic pencil formed by the tangents to a conic and a linear pencil with a vertex interior to the conic. Without loss of generality, we may take as the conic the circle

$$x^2 + y^2 - 1 = 0$$

and the vertex at the origin $(0, 0)$. The equation of the pencil is given by

$$(a^2 + b^2 - c^2)c = 0.$$

We can parameterize the tangents to a circle and the lines through the origin by

$$\begin{aligned} &L(\alpha, 1) \equiv \cos\alpha x + \sin\alpha y - 1 = 0, \\ &L(\alpha, 0) \equiv \sin(\alpha/2)x + \cos(\alpha/2)y = 0, \quad \alpha \in I\!R. \end{aligned} \tag{3.3}$$

The line $L(\alpha, 1)$ is the tangent to the circle at $(\cos\alpha, \sin\alpha)$ and the line $L(\alpha, 0)$ is the line through the origin with slope $-\tan(\alpha/2)$. Let us remark that $L(\alpha + 2\pi, r) = L(\alpha, r)$, for all $\alpha \in I\!R$, $r = 0, 1$. Figure 4 shows a set of tangents to the circle and lines through the origin.

Two tangents to the conic $L(\alpha_1, 1), L(\alpha_2, 1)$ and one line through the origin $L(\alpha_0, 0)$ are concurrent if and only if

$$\begin{vmatrix} 0 & \sin(\alpha_0/2) & \cos(\alpha_0/2) \\ -1 & \cos(\alpha_1) & \sin(\alpha_1) \\ -1 & \cos(\alpha_2) & \sin(\alpha_2) \end{vmatrix} = 0. \tag{3.4}$$

The left hand side of (3.4) is

$$2 \sin\left((\alpha_0 + \alpha_1 + \alpha_2)/2\right) \sin\left((\alpha_2 - \alpha_1)/2\right)$$

and, assuming that the tangents to the conic are distinct, $\alpha_1 \neq \alpha_2$, (3.4) reduces to $\alpha_0 + \alpha_1 + \alpha_2 \in 2\pi\mathbb{Z}$.

Let us denote by $\mathbb{Z}_2$ the additive group of integers modulo 2. Then L can be regarded as a bijection between the group $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{Z}_2$ and the cubic pencil

$$(\alpha + 2\pi\mathbb{Z}, r) \in \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{Z}_2 \mapsto L(\alpha, r),$$

such that if the corresponding group elements add up to zero then the three lines are concurrent. Conversely, if three lines are concurrent at a point which is not the origin, then the corresponding group elements add up to zero.

Now, let us analyze the reducible cubic pencil formed by the tangents to a conic and a linear pencil with a vertex on the conic. Without loss of generality, we may take as the conic the parabola

$$y = x^2$$

and as the vertex of the linear pencil the ideal point corresponding to the direction of the vector $(0, 1)$. Then the equation of the pencil is given by

$$(a^2 - 4bc)b = 0.$$

We can parameterize the tangents to the parabola and the lines of the linear pencil by

$$L(t, 1) \equiv y - 2tx + t^2 = 0, \quad L(t, 0) \equiv x + t/2 = 0, \quad t \in \mathbb{R}. \tag{3.5}$$

The line $L(t, 1)$ is the tangent to the parabola at (t, t^2) and the line $L(t, 0)$ is a line parallel to the Y axis. Figure 5 shows a set of tangents to the parabola and lines parallel to the Y axis.

Two tangents to the conic $L(t_1, 1), L(t_2, 1)$ and one line through the origin $L(t_0, 0)$ are concurrent if and only if

$$\begin{vmatrix} t_0/2 & 1 & 0 \\ t_1^2 & -2t_1 & 1 \\ t_2^2 & -2t_2 & 1 \end{vmatrix} = 0. \tag{3.6}$$

The left hand side of (3.6) is $(t_2 - t_1)(t_0 + t_1 + t_2)$ and, assuming that the tangents to the parabola are distinct, $t_1 \neq t_2$, (3.6) reduces to $t_0 + t_1 + t_2 = 0$. Then L can be regarded as a bijection between the group $\mathbb{R} \times \mathbb{Z}_2$ and the cubic pencil

$$(t, r) \in \mathbb{R} \times \mathbb{Z}_2 \mapsto L(t, r),$$

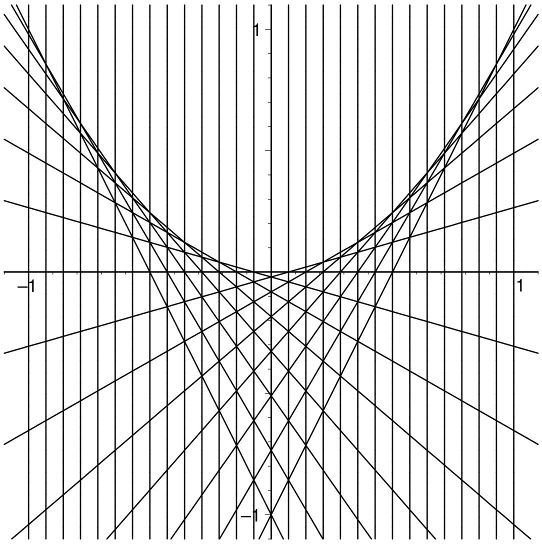

Figure 5. Tangents to a parabola and lines parallel to the Y axis

such that if the corresponding group elements add up to zero then the three lines are concurrent. Conversely, if three lines are concurrent at a point which is not the origin, then the corresponding group elements add up to zero.

Finally, let us consider the reducible cubic pencil formed by the tangents to a conic and a linear pencil with a vertex exterior to the conic. Without loss of generality, we may take as the conic the hyperbola

$$xy = 1$$

and as the vertex the intersection points of the asymptotes, that is the origin. Then the equation of the pencil is given by

$$(c^2 - 4ab)c = 0.$$

The singular lines of the pencil are the asymptotes. We can parameterize the tangents to the hyperbola and the lines of the linear pencil by

$$\begin{aligned} &L(t,s,1) \equiv (-1)^s \exp(t)y + (-1)^s \exp(-t)x - 2 = 0, \\ &L(t,s,0) \equiv y - (-1)^s \exp(t)x = 0, \quad t \in I\!R, s \in Z\!\!Z_2. \end{aligned} \tag{3.7}$$

The line $L(t,s,1)$ is the tangent to the hyperbola at the point

$$((-1)^s \exp(t), (-1)^s \exp(-t))$$

and the line $L(t,s,0)$ is a line through the origin, different from the asymptotes. Figure 6 shows a set of tangents to the hyperbola and lines through the origin.

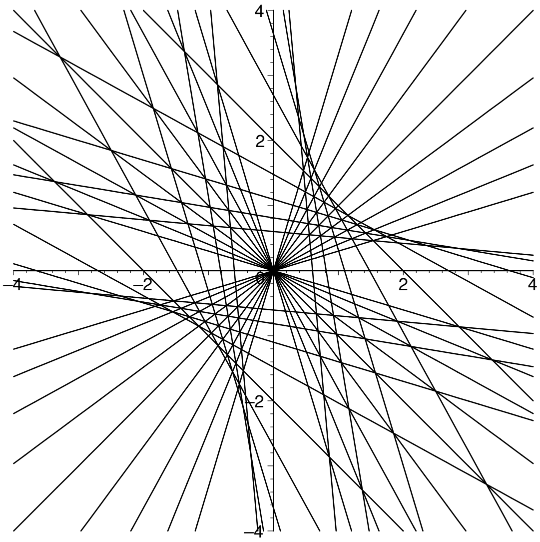

Figure 6. Tangents to a hyperbola and lines through the origin

Two tangents to the conic $L(t_1, s_1, 1), L(t_2, s_2, 1)$ and one line through the origin $L(t_0, s_0, 0)$ are concurrent if and only if

$$\begin{vmatrix} 0 & -\mu_0 & 1 \\ -2 & \mu_1^{-1} & \mu_1 \\ -2 & \mu_2^{-1} & \mu_2 \end{vmatrix} = 0, \tag{3.8}$$

where $\mu_i = (-1)^{s_i} \exp(t_i)$, $i = 0, 1, 2$. The left hand side of (3.8) is $2\mu_1^{-1}\mu_2^{-1}(\mu_2 - \mu_1)(1 - \mu_0\mu_1\mu_2)$ and, assuming that the tangents to the hyperbola are distinct, $\mu_1 \neq \mu_2$, (3.8) reduces to $\mu_0\mu_1\mu_2 = 1$, that is $t_0 + t_1 + t_2 = 0$ and $s_0 + s_1 + s_2 = 0$. Then L can be regarded as a bijection between the group $\mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and the cubic pencil

$$(t, s, r) \in \mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \mapsto L(t, s, r),$$

such that if the corresponding group elements add up to zero then the three lines are concurrent. Conversely, if three lines are concurrent at a point which is not the origin, then the corresponding group elements add up to zero.

Let us now sketch the construction of generalized principal lattices in the three cases. Formula (3.1) combined with (3.3) (resp., (3.5) and (3.7)) allows us to generate them. We choose n, g_0, g_1, g_2, h, provided that $ih \neq 0$, $0 < i \leq 2n$, all elements (3.2) are distinct and the set of nodes does not include the vertex. This last condition is satisfied if the following condition holds: h and exactly one element among g_0, g_1, g_2 belong to the set $\mathbb{R}/2\pi\mathbb{Z} \times \{0\}$, (resp., $\mathbb{R} \times \{0\}$, $\mathbb{R} \times \mathbb{Z}_2 \times \{0\}$).

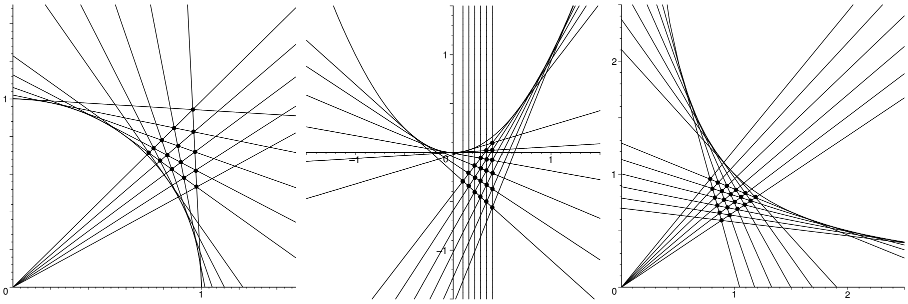

Figure 7. Generalized principal lattices generated by reducible cubic pencils

Figure 7 left shows a lattice generated by (3.1) using the lines (3.3) and taking $n = 4$, $h = (0.14, 0)$, $g_0 = (-1.56, 0)$, $g_1 = (0.05, 1)$, $g_2 = (0.95, 1)$. Figure 7 center shows a lattice generated by (3.1) using the lines (3.5) and taking $n = 5$, $h = (0.12, 0)$, $g_0 = (-0.80, 0)$, $g_1 = (-0.45, 1)$, $g_2 = (0.65, 1)$. Figure 7 right shows a lattice generated by (3.1) using the lines (3.7) and taking $n = 5$, $h = (0.12, 0, 0)$, $g_0 = (-0.40, 0, 0)$, $g_1 = (-0.65, 0, 1)$, $g_2 = (0.45, 0, 1)$.

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