Classification of bivariate configurations with simple Lagrange interpolation formulae *

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Abstract. In 1977 Chung and Yao introduced a geometric characterization in multivariate interpolation in order to identify distributions of points such that the Lagrange functions are products of real polynomials of first degree. We discuss and describe completely all these configurations up to degree 4 in the bivariate case. The number of lines containing more nodes than the degree is used for classifying these configurations.

Keywords: bivariate interpolation, Lagrange formula, geometric characterization

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1. Introduction

Multivariate interpolation is a more difficult subject than univariate interpolation. One of the difficulties is that the solvability of the interpolation problem depends on the geometric properties of the set of interpolation points (nodes) [6]. For solvable problems, it is interesting to express the solution by simple interpolation formulae. In 1977 Chung and Yao introduced a geometric characterization (GC) for a set of nodes whose associated Lagrange polynomials are products of polynomials of first degree. If a set of nodes satisfies the GC condition, the Lagrange interpolation formula can be explicitly given by a simple expression in terms of the nodes, the data values and the lines used for checking the GC condition.

Many configurations of points satisfying the GC condition in the plane have been described [4, 2, 7]. However we are still far from knowing all possible configurations. A relevant conjecture was stated by Gasca and Maeztu in 1982: *a set of nodes satisfying the GC condition for interpolation with polynomials of degree n must contain $n+1$ collinear points.*

Busch [1] verified the conjecture for degrees less than or equal to 4. Trying to shed more light on the conjecture, we have recently obtained another proof [3] with different arguments. However it seems difficult to extend those arguments to $n > 4$.

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The information derived from the verification of the conjecture is crucial for analyzing the possible GC configurations of points because it introduces some simplifications in the analysis of the geometric characterization.

The purpose of this paper is to describe all GC configurations of points in the plane for degree ≤ 4. We think that knowing all possible GC configurations of low degree can be helpful to solve more cases of the conjecture and this in turn can be used for classifying partially or completely the GC configurations for higher degree.

In [2], the authors stated a conjecture stronger than the above mentioned one: *in a GC configuration of nodes of degree n, the number of lines containing $n+1$ nodes must be at least 3*.

In section 2, we state both conjectures and review some auxiliary results for arbitrary degree. In section 3, we describe the GC configurations up to degree 3. The cases of degree $n = 1, 2$ are very simple and so, the section is especially devoted to the case $n = 3$. We show that the number of lines containing 4 nodes is at least 3 and this information is used for describing all GC configurations for degree 3. In section 4, we show that the number of lines containing 5 nodes in GC configurations of degree 4 is at least 3 and we describe all these configurations.

2. Some conjectures on the geometric characterization

In the bivariate case, the GC condition can be stated as follows.

Definition 1. Let $X \subseteq \mathbb{R}^2$, $|X| = (n+2)(n+1)/2$. The set X satisfies the geometric characterization GC_n if for each $x \in X$, there exist lines $L_1^x, \dots, L_n^x$ such that

$$x \notin L_1^x \cup \cdots \cup L_n^x, \quad X \setminus \{x\} \subseteq L_1^x \cup \cdots \cup L_n^x.$$

We say that the lines $L_1^x, \dots, L_n^x$ are *used* by the node $x \in X$. The set of lines used by $x \in X$ is denoted by Γ_x and $\Gamma_X := \bigcup_{x \in X} \Gamma_x$ is the set of lines used by some node.

Observe that if X satisfies GC_n, then the Lagrange interpolation problem on the set X is unisolvent in $\Pi_n(\mathbb{R}^2)$. This implies that the set of lines Γ_x is uniquely defined by the node x.

Let us precisely state the conjecture on the GC condition stated by Gasca and Maeztu [5].

CONJECTURE 1. *Let $X \subseteq \mathbb{R}^2$ satisfy the GC_n condition. Then there exists a line in the plane containing $n+1$ points of X.*

The lines containing $n+1$ points seem to be relevant for describing the GC configurations. The following Proposition summarizes some properties of these lines stated in Proposition 2.1 of [2] and Proposition 2.2 of [3] that will be useful in the rest of the paper.

PROPOSITION 1. *Let X be a set of nodes satisfying the* GC_n *condition. Then*

(a) No line contains more than $n+1$ points of X.

(b) If $n \geq 1$ and a line L contains $n+1$ points of X, then $L \in \Gamma_X$ and it is used by each point of X not lying on it.

(c) Two lines, each containing $n+1$ points of X, cannot be parallel. Their intersection belongs to X.

(d) Three lines, each containing $n+1$ points of X, cannot be concurrent.

(e) For any line L containing $n+1$ points of X, the set $X \setminus L$ satisfies the GC_{n-1} *condition.*

In order to describe the possible configurations we shall use the following notation

$$X = \Big\{ x_{ij} \in I\!\!R^2 \mid i < j \in \{1, \dots, n+2\} \Big\}, \tag{2.1}$$

$$\mathcal{K}_X := \{K \in \Gamma_X \mid |K \cap X| = n+1\}, \tag{2.2}$$

$$m = |\mathcal{K}_X|, \quad \mathcal{K}_X := \{K_1, \dots, K_m\}. \tag{2.3}$$

Taking into account Proposition 1, we can assume that the indices have been arranged in such a way that that

$$x_{ij} \in K_l \Longleftrightarrow l = i \text{ or } l = j, \tag{2.4}$$

which means that $x_{ij} = K_i \cap K_j$ for all $i < j \leq m$, $x_{ij} \in K_i \setminus \bigcup_{r \neq i} K_r$, for all $i \leq m < j$ and $x_{ij} \notin \bigcup_{r=1}^{m} K_r$ for $m < i < j$.

The number $m = |\mathcal{K}_X|$ of lines containing $n+1$ points can be used for classifying all GC configurations of nodes. In [2], the concept of *default* was introduced for this purpose. It seems that the term *defect* is more appropriate. We thank the referee for warning us at this point.

Definition 2. Let $X \subseteq I\!\!R^2$ be a set satisfying the GC_n condition. We say that X has *defect* d if the number of lines in $\mathcal{K}_X$ is $n+2-d$.

If X is a GC_n set of defect 0, then X is a *natural lattice*, that is, the set of all pairwise intersections of $n+2$ lines in general position.

In [2] all possible configurations of defect 0, 1, 2 were described for arbitrary degree.

In all known GC configurations, we always find 3 lines with $n+1$ points, that is the defect is always less than or equal to $n-1$. The following conjecture was launched in [2].

CONJECTURE 2. *Let $X \subseteq \mathbb{R}^2$ satisfy the* GC_n *condition. Then there exist at least three lines in the plane containing $n+1$ points of X, that is, $|\mathcal{K}_X| \geq 3$.*

This conjecture is equivalent to saying that each node uses a line with $n+1$ nodes.

PROPOSITION 2. *Let $X \subseteq \mathbb{R}^2$ satisfy the* GC_n *condition. Then the following statemens are equivalent*

(a) There exist at least three lines in Γ_X containing $n+1$ points of X.

(b) For each $x \in X$, there exist a line in Γ_x containing $n+1$ points of X.

Proof. If (a) holds, then by Proposition 1 (c), (d), the 3 lines are in general position. So, for each $x \in X$, one of the 3 lines, say K, satisfies $x \notin K$. By Proposition 1 (b), x uses K, that is, $K \in \Gamma_x$.

Conversely if (b) holds, then we take $x_1 \in X$ and K_1 a line with $n+1$ nodes used by x_1. Now take $x_2 \in K_1 \cap X$ and K_2 a line with $n+1$ nodes used by x_2. Since $x_2 \notin K_2$, $K_2 \neq K_1$. By Proposition 1 (c), $K_1 \cap K_2$ is a node. Take $x_3 = K_1 \cap K_2$ and K_3 a line with $n+1$ nodes used by x_3. Since $x_3 \notin K_3$, the line K_3 must be different from K_1, K_2 and then (a) holds. ∎

The following auxiliary result is well-known (see for instance Lemma 1 of [1]). We provide the proof for the sake of completeness.

LEMMA 1. *Let L_1, L_2, L_3, M_1, M_2, M_3 be lines such that*

$$|(L_1 \cup L_2 \cup L_3) \cap (M_1 \cup M_2 \cup M_3)| = 9.$$

If a cubic polynomial vanishes at 8 of the points in $(L_1 \cup L_2 \cup L_3) \cap (M_1 \cup M_2 \cup M_3)$, then it also vanishes at the remaining point.

Proof. Let $x_{ij} = L_i \cap M_j$, $i, j = 1, 2, 3$ and let p be a cubic polynomial vanishing at 8 of these points. Without loss of generality we may assume that p vanishes at all points of $(L_1 \cup L_2 \cup L_3) \cap (M_1 \cup M_2 \cup M_3) \setminus \{x_{33}\}$. Let x_{14} be any point lying on $M_1 \setminus (L_1 \cup L_2 \cup L_3)$ and let x_{41} be any point in $L_1 \setminus (M_1 \cup M_2 \cup M_3)$. Then the set $X = \{x_{ij} \mid i + j \leq 5\}$ is a $\Pi_3(\mathbb{R}^2)$-unisolvent set (because it is a system of order 3 as defined in [5]) and using the Lagrange formula we obtain

$$p = \frac{p(x_{14})}{L_1(x_{14})L_2(x_{14})L_3(x_{14})} L_1 L_2 L_3 + \frac{p(x_{41})}{M_1(x_{41})M_2(x_{41})M_3(x_{41})} M_1 M_2 M_3,$$

which shows that p also vanishes at x_{33}. ∎

3. GC configurations of degree less than 4

In this section we verify Conjecture 2 and describe all GC_n sets of nodes for $n = 1, 2, 3$.

PROPOSITION 3. *Let $X \subseteq \mathbb{R}^2$ be a GC_n set.*

(a) If $n = 1$, then X is formed by 3 nodes not in a line, $|\mathcal{K}_X| = 3$ and the defect is 0.

(b) If $n = 2$, then $3 \leq |\mathcal{K}_X| \leq 4$ and the defect is 0 or 1.

(c) If $n = 3$, then $3 \leq |\mathcal{K}_X| \leq 5$ and the defect is either 0, 1 or 2.

Proof. (a) It is straightforward. A GC_1 set X is formed by 3 points not in a line and then the 3 lines joining each pair of them are the lines with 2 nodes. Therefore, X is a natural lattice.

(b) Let X be a GC_2 set and let $x \in X$. The GC_2 condition implies that the 5 points of $X \setminus \{x\}$ must be distributed along 2 lines and one of them will have 3 points and no more by Proposition 1. So each point of X uses a line in $\mathcal{K}_X$. By Proposition 2, there exist at least 3 lines containing 3 nodes, that is the defect is 0 or 1.

(c) From [1, 3] we know that there exists at least a line K_1 containing 4 nodes.

Each point $x \in X \cap K_1$ uses 3 lines, each of them containing one point of K_1. The three lines of Γ_x must also contain the 6 points in $X \setminus K_1$. By Proposition 1 (e) $X \setminus K_1$ is a GC_2 set. By Proposition 3 (b), there are 3 lines M_1, M_2, M_3, each of them containing 3 points of $X \setminus K_1$.

Let us assume that K_1 is the only line containing 4 nodes. Then each line in Γ_x contains exactly 2 nodes in $X \setminus K_1$. Each of the 3 lines in Γ_x must contain at most (and at least) one vertex of the triangle with sides M_1, M_2, M_3. Let N_{ij} be the line joining the vertex $M_i \cap M_j$ with the only point in $X \setminus (K_1 \cup M_i \cup M_j)$, $i \neq j \in \{1, 2, 3\}$ (see Figure 1). The lines N_{12}, N_{13}, N_{23} are the only lines containing exactly two points of $X \setminus K_1$ (one vertex of the triangle with sides M_1, M_2, M_3 and a second node). Therefore $\Gamma_x = \{N_{12}, N_{13}, N_{23}\}$, for each $x \in K_1 \cap X$. This is a contradiction because different points cannot use the same 3 lines.

So, we deduce that there exist at least two lines K_1, K_2 containing 4 nodes. By Proposition 1 (c), K_1 and K_2 intersect at a node. By Proposition 1 (e), the set $X \setminus (K_1 \cup K_2)$ is GC_1, that is, it consists of the 3 vertices of a triangle whose sides are denoted by L_1, L_2, L_3. Take $i \in \{1, 2\}$ and let $j \neq i$, $j \in \{1, 2\}$. Since, by Proposition 1 (e), $X \setminus K_j$ is GC_2, there exist three lines containing 3 nodes in $X \setminus K_j$. One of them is K_i; the other 2 lines must be in $\{L_1, L_2, L_3\}$ and intersect $K_i \setminus K_j$

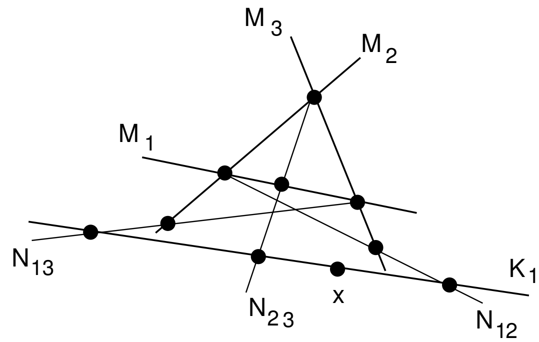

Figure 1. Only the line K_1 contains 4 nodes.

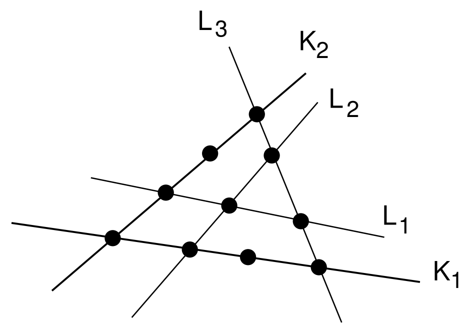

Figure 2. Two lines K_1 and K_2 contain 4 nodes.

at a node. So we have

$$|(L_1 \cup L_2 \cup L_3) \cap (K_i \setminus K_j) \cap X| \geq 2, \quad j \in \{1,2\} \setminus \{i\}, \quad i \in \{1,2\}.$$

Then one of the lines in $\{L_1, L_2, L_3\}$ intersects both K_1 at a node and K_2 at another node (see Figure 2) and that line contains 4 nodes.

So, we have deduced that there exist at least three lines containing 4 nodes. The defect can be 0, 1, or 2. ∎

Now, let us use Proposition 3 in order to describe all sets of nodes satisfying GC_n, $n = 1, 2, 3$. All possible configurations with defect 0, 1, 2 and arbitrary degree have already been described in [2]. We use this information for describing completely all GC configurations up to degree 3.

Proposition 3 (a) means that a GC_1 set of nodes in the plane is just a set of 3 noncollinear points. Let us describe now all possible GC_2 configurations. If the defect is 0, then X is a natural lattice and the 6 points are the vertices of a complete quadrilateral. If the defect is 1, we use the notations (2.1-4) for the set of nodes and lines. The 3 vertices

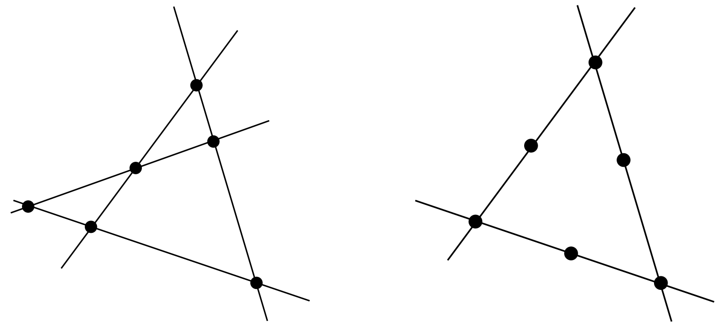

Figure 3. GC_2 configurations of defect 0 and 1.

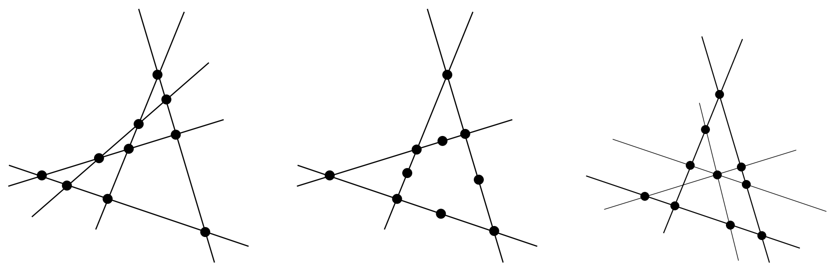

Figure 4. GC_3 configurations of defect 0, 1 and 2.

x_{12}, x_{13}, x_{23} of the triangle whose sides are the 3 lines K_1, K_2, K_3 containing 3 nodes are in X and the remaining nodes x_{14}, x_{24}, x_{34} are not collinear and each one lies on a different side of the triangle, that is, $x_{i4} \in K_i$. Figure 3 shows all possible GC_2 configurations of defect 0,1.

Finally, let us describe all possible GC_3 configurations. If the defect is 0, then we have a natural lattice and the vertices are the 10 intersections of 5 lines in general position. If the defect is 1, then we have 4 lines intersecting in 6 nodes. Each of the 4 lines contains an additional node and these 4 nodes are not collinear. If the defect is 2, we use the notations (2.1-4) in order to simplify the description. Then we have 3 lines K_1, K_2, K_3 with 4 nodes and 3 concurrent lines L_1, L_2, L_3, each of them containing three nodes, $L_1 \cap X = \{x_{24}, x_{35}, x_{45}\}$, $L_2 \cap X = \{x_{15}, x_{34}, x_{45}\}$, $L_3 \cap X = \{x_{14}, x_{25}, x_{45}\}$.

Figure 4 shows all possible GC_3 configurations of defect 0, 1 and 2.

4. GC configurations of degree 4

In order to verify Conjecture 2 for $n = 4$, we proceed in two steps. First we show that there exist at least two lines with 5 nodes.

LEMMA 2. *Let $X \subseteq \mathbb{R}^2$ be a* GC_4 *set. Then* $|\mathcal{K}_X| \geq 2$.

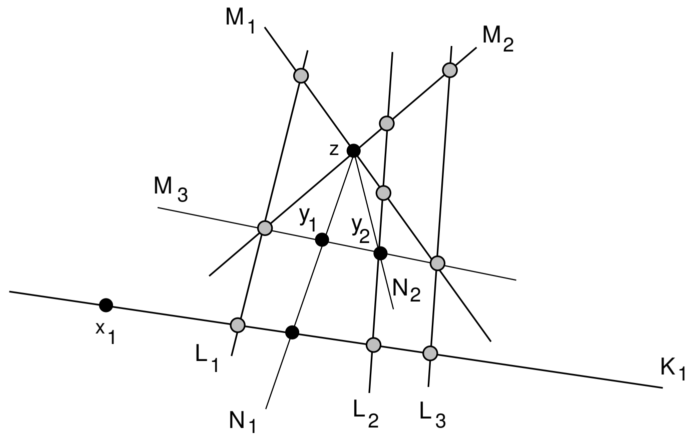

Figure 5. Only K_1 contains 5 nodes.

Proof. From [1, 3], we know that $\mathcal{K}_X$ contains at least one line K_1. By Proposition 1 (e), $X \setminus K_1$ is a GC_3 set. By Proposition 3 (c) there are 3 lines M_1, M_2, M_3, each of them containing 4 points of $X \setminus K_1$. By Proposition 1 (c), $M_i \cap M_j \in X$, $i \neq j$ and therefore the set $M_3 \cap X \setminus (M_1 \cup M_2)$ consists of two points y_1, y_2. Let N_1 (respectively N_2) be the line passing through $z = M_1 \cap M_2$ and y_1 (respectively y_2).

Each point $x \in X \cap K_1$ uses 4 lines, each of them containing one point of $K_1 \setminus \{x\}$. If $M_i \in \Gamma_x$, then $M_i \cap K_1 \cap X \neq \emptyset$ and $|M_i \cap X| \geq 5$. So, in this case, a second line contains 5 nodes and the statement of the Lemma holds.

Let us assume that K_1 is the only line containing 5 nodes. Then no line in Γ_x, $x \in K_1 \cap X$, is in the set $\{M_1, M_2, M_3\}$. Since any line M_i contains exactly 4 nodes and Γ_x consists of 4 lines none of them being M_i, we have

$$|L \cap M_i \cap X| = 1, \quad \forall L \in \Gamma_x, \quad i = 1, 2, 3.$$

On the other hand, one of the lines in Γ_x must contain $z = M_1 \cap M_2$. It follows that either N_1 or N_2 is in Γ_x.

Taking into account that $K_1 \cap X$ contains 5 points and each one uses N_1 or N_2, there must exist two nodes $x_i \in K_1 \cap X$, $i = 1, 2$, using the same line, say N_1. Each of the three lines in $\Gamma_{x_i} \setminus \{N_1\}$ intersects each of the three lines K_1, M_1, M_2 at points of X. The three nodes placed in each M_i are $M_i \cap X \setminus (M_1 \cap M_2)$, $i = 1, 2$ and the three points in K_1 are $K_1 \setminus (N_1 \cup \{x_i\})$ (See Figure 5).

So, the three lines $L_1, L_2, L_3 \in \Gamma_{x_1} \setminus \{N_1\}$ intersect the lines K_1, M_1, M_2 at 9 points. The Lagrange function associated to x_2 is of the form $N_1 p$, where p is a cubic polynomial vanishing on the three lines of $\Gamma_{x_2} \setminus \{N_1\}$. So, it vanishes at eight of the points of $(K_1 \cup M_1 \cup M_2) \cap (L_1 \cup L_2 \cup L_3)$

but not at the other one, x_2. This contradicts Lemma 1, showing that the assumption that K_1 is the only line containing 5 nodes leads to a contradiction. So, $|\mathcal{K}_X| \geq 2$. ∎

PROPOSITION 4. *Let $X \subseteq \mathbb{R}^2$ be a* GC_4 *set. Then* $3 \leq |\mathcal{K}_X| \leq 6$ *and the defect is either 0, 1, 2 or 3.*

Proof. By Lemma 2, there exist two lines K_1, K_2 containing at least 5 nodes. By Proposition 1 (c), $|K_1 \cap K_2 \cap X| = 1$. By Proposition 1 (e), $X \setminus (K_1 \cup K_2)$ is a GC_2 set and, by Proposition 3 (b), the set $\mathcal{K}_{X\setminus(K_1\cup K_2)}$ is formed by 3 or 4 lines. On the other hand, $X \setminus K_i$ is a GC_3 set, and, by Proposition 3 (c), there exist 3 lines each of them containing 4 points of $X \setminus K_i$. One of these lines must be K_j, $j \in \{1, 2\} \setminus \{i\}$ and the other two L_{i1}, L_{i2} must be in $\mathcal{K}_{X\setminus(K_1\cup K_2)}$ and have a node in $K_j \setminus K_i$.

If any of the lines L_{i1}, L_{i2}, $i = 1, 2$ intersects K_i at a node then this line would have 5 nodes and the Proposition follows.

Otherwise, the lines L_{i1}, L_{i2}, $i = 1, 2$ have only 4 nodes and $L_{i1} \cap K_i \cap X = L_{i2} \cap K_i \cap X = \emptyset$, $i = 1, 2$. This means that $|\mathcal{K}_{X\setminus(K_1\cup K_2)}| = 4$ and the set $X \setminus (K_1 \cup K_2)$ is a natural lattice. Let us take $x := L_{21} \cap K_1 \in X$ and identify the lines in Γ_x (see Figure 6).

By Proposition 1 (b), the line K_2 (containing 5 nodes) belongs to Γ_x. By Proposition 1 (a), $X \setminus K_2$ is GC_3 and, using Proposition 1 (b), it is easy to see that the line L_{22} (containing 4 nodes) belongs to Γ_x. The 5 remaining nodes form the set $X \setminus (K_2 \cup L_{22} \cup \{x\})$. Two of them are in $K_1 \setminus (L_{21} \cup L_{22})$, other two in $L_{21} \cap (L_{11} \cup L_{12})$ and the last one is $L_{11} \cap L_{22}$. These 5 nodes must be placed in the 2 lines of $\Gamma_x \setminus \{K_2, L_{22}\}$, which cannot be either K_1 or L_{21}. So there exists a line in $\Gamma_x \setminus \{K_2, L_{22}\}$ containing 3 nodes: one in $K_1 \setminus (L_{21} \cup L_{22})$, another one in $L_{21} \cap (L_{11} \cup L_{12})$ and the third one is $L_{11} \cap L_{22}$. Then the line must be L_{11} or L_{12} and it intersects K_1 at a node, contradicting our assumption. Therefore $|\mathcal{K}_X| \geq 3$. ∎

Now we are ready to describe all GC_4 sets. All these sets have 15 nodes. The cases of defect 0,1,2 have been described in [2]. If the defect is 0, then we have a natural lattice, that is the 15 pairwise intersections of 6 lines in general position. The case of defect 1 can also be easily described: 5 lines K_i, $i = 1, 2, 3, 4, 5$, in general position intersecting at 10 points and the remaining 5 points are not collinear each of them placed on a different line K_i, $i = 1, 2, 3, 4, 5$.

In order to simplify the description of the GC_4 sets with defect 2, let us use the notations (2.1–4). We have 4 lines K_i containing 5 nodes, $i = 1, 2, 3, 4$. Three of the 5 nodes in $X \cap K_i$ are the intersection with the remaining lines K_j, $x_{ij} = K_i \cap K_j$, $j \in \{1, 2, 3, 4\} \setminus \{i\}$, $i \in$

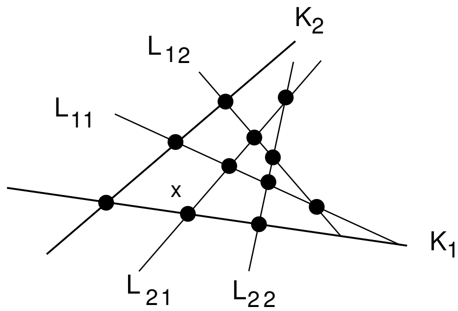

Figure 6. Proof of the existence of a third line containing 5 nodes.

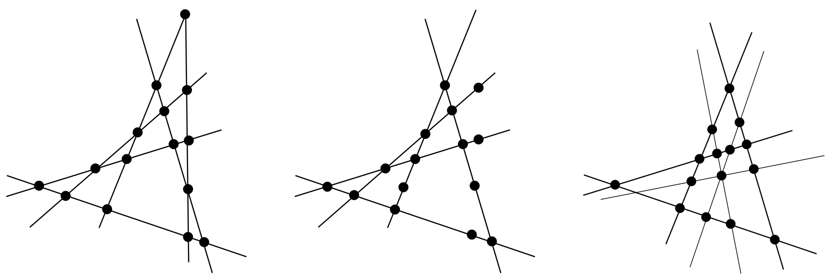

Figure 7. GC_4 configurations of defect 0, 1 and 2.

$\{1,2,3,4\}$. Only one node x_{56} lies outside the lines containing 5 nodes. The additional nodes x_{i5}, x_{i6} on the lines K_i, $i = 1,2,3,4$, are contained in 3 lines L_1, L_2, L_3 which are concurrent at the node $x_{56} \in L_1 \cap L_2 \cap L_3$. Each line L_i contains 3 or 4 nodes. More precisely, two of them contain 4 nodes and the remaining one 3 nodes.

Figure 7 shows GC_4 configurations of defect 0, 1 and 2.

Let us now describe the GC_4 sets with defect 3.

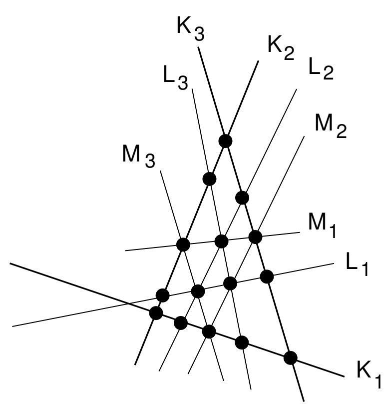

Figure 8. A GC_4 configuration of defect 3.

PROPOSITION 5. *A set X is* GC_4 *with defect 3 if and only if the nodes can be labelled in the form (2.1) and the following properties simultaneously hold:*

(i) There exist lines K_1, K_2, K_3 in general position, containing 5 nodes, such that (2.4) holds.

(ii) There exist lines L_1, L_2, L_3 in general position, containing 4 nodes, such that

$$x_{ij} \in L_k \Longleftrightarrow (j = k+3,\ 1 \leq i \leq 3,\ i \neq k) \text{ or } (4 \leq i < j \leq 6,\ k+3 \in \{i, j\}) \tag{4.1}$$

for each $k \in \{1, 2, 3\}$.

(iii) There exist lines M_1, M_2, M_3 in general position, containing 3 nodes, such that

$$x_{ij} \in M_k \Longleftrightarrow (j = i+3,\ 1 \leq i \leq 3,\ i \neq k) \text{ or } (4 \leq i < j \leq 6,\ k+3 \notin \{i, j\}) \tag{4.2}$$

for each $k \in \{1, 2, 3\}$.

Proof. Let X be a GC_4 with defect 3. Then the nodes can be labelled such that (2.1–4) hold for $m = 3$. By Proposition 1 (e), the set $X \setminus (K_1 \cup K_2 \cup K_3)$ is GC_1 and it consists of 3 points x_{45}, x_{46}, x_{56}, forming a triangle. The 3 sides of the triangle are the lines in $\mathcal{K}_{X \setminus (K_1 \cup K_2 \cup K_3)}$ each of them containing 2 points of $X \setminus (K_1 \cup K_2 \cup K_3)$. For $k \in \{1, 2, 3\}$, $X \setminus K_k$ is a GC_3 set and by Proposition 3 (c), there exist 3 lines containing 4 points in $X \setminus K_k$. Two of them are K_i, $i \neq k$ and we denote the remaining one by L_k. By Proposition1 (c) applied to the GC_3 set $X \setminus K_k$, each line L_k intersects the lines K_i, $i \neq k$, at a node in $X \setminus K_k$. Let us observe that L_1, L_2, L_3 are the three lines in $\mathcal{K}_{X \setminus (K_1 \cup K_2 \cup K_3)}$. Again by Proposition 1 (c) applied to the GC_1 set $X \setminus (K_1 \cup K_2 \cup K_3)$, each line L_k intersects L_i, $i \neq k$, at a node in $X \setminus (K_1 \cup K_2 \cup K_3)$. Therefore each line L_k contains at least 4 nodes: two of them are intersections with the lines L_i, $i \neq k$, and the other two are the intersections with the lines K_i, $i \neq k$. By assumption, only K_1, K_2 and K_3 contain 5 nodes and then $|L_k \cap X| = 4$. So we can label the points in X in such a way that (4.1) holds.

The 5 points in the line K_i are the intersections $K_i \cap K_j$ with the other lines K_j, $j \neq i$, the intersections $K_i \cap L_j$ with the lines L_j, $j \neq i$, and one point $x_{i,i+3}$ not placed on other lines L_1, L_2 or L_3, or K_j, $j \neq i$.

Let M_k be the line passing through $x_{i,i+3}$, $x_{j,j+3}$, where $i, j, k \in \{1, 2, 3\}$, $i, j \neq k$, $i < j$. Using Proposition 1 (e) and taking into account that K_k and L_k do not intersect at a node, we see that set $X \setminus (K_k \cup L_k)$ is GC_2. By Proposition 3 (b), there exist 3 lines, each of them containing 3 points of the set $X \setminus (K_k \cup L_k)$. Two of them are K_i, K_j, $i, j \neq k$,

$i < j$. The third line must contain the points $x_{i,i+3}$, $x_{j,j+3}$, $x_{i+3,j+3}$, $i, j, k \in \{1, 2, 3\}$, $i, j \neq k$, $i < j$ and must therefore be M_k. So, each line M_k contains a third node $x_{i+3,j+3} = L_i \cap L_j$, $i, j, k \in \{1, 2, 3\}$, $i, j \neq k$, $i < j$. This shows that (4.2) holds.

Conversely, if X is a set (2.1) such that (2.4), (4.1) and (4.2) holds, then GC_4 holds. A node $x_{i+3,j+3}$, $i, j \in \{1, 2, 3\}$, $i < j$, uses the lines K_1, K_2, K_3 and the line L_k, $k \in \{1, 2, 3\} \setminus \{i, j\}$, passing through $\{x_{45}, x_{46}, x_{56}\} \setminus \{x_{i+3,j+3}\}$. A node $x_{i,i+3}$, $i \in \{1, 2, 3\}$ uses the lines K_j, L_j, $j \in \{1, 2, 3\} \setminus \{i\}$. A node $x_{i,j+3}$, $i, j \in \{1, 2, 3\}$, $i \neq j$, uses the lines K_l, $l \neq \{1, 2, 3\} \setminus \{i\}$ and L_k, M_k for $k \in \{1, 2, 3\} \setminus \{i, j\}$. Finally, the nodes x_{ij}, $i, j \in \{1, 2, 3\}$, $i < j$, use the lines K_k, L_k, M_k, $k \in \{1, 2, 3\} \setminus \{i, j\}$ and the line passing through $x_{i,j+3}$ and $x_{j,i+3}$. ∎

The previous proposition shows an easy way of constructing a GC_4 set with defect 3 (see Figure 8). Take any 3 straight lines K_1, K_2, K_3 in general position in the plane and let $x_{ij} = K_i \cap K_j$, $1 \leq i < j \leq 3$. Take one point on each K_i different from the intersections with the other two lines and denote it by $x_{i,i+3}, 1 \leq i \leq 3$. Let M_1 be the straight line joining x_{25}, x_{36}, M_2 the line joining x_{14}, x_{36} and M_3 the line joining x_{14}, x_{25}. Take now any three points $x_{56} \in M_1$, $x_{46} \in M_2$ and $x_{45} \in M_3$, none of them on the lines $K_j, 1 \leq j \leq 3$. Finally let L_1 be the line joining x_{45}, x_{46}, L_2 the line joining x_{45}, x_{56}, and L_3 the line joining x_{46}, x_{56}. If no line L_i, $1 \leq i \leq 3$, is parallel to any K_j, $j \neq i$, and $K_j \cap L_i \notin \{x_{12}, x_{13}, x_{23}\}$, $j \neq i$, then the intersections $x_{j,i+3} = K_j \cap L_i$, $j \neq i$, are the 6 remaining nodes of the GC_4 set with defect 3.

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