Bivariate polynomial interpolation with asymptotic conditions

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Abstract. Bivariate Hermite-Birkhoff interpolation problems with an asymptotic condition on some straight lines are studied, using a Newton approach. The existence and uniqueness of solution in an adequate polynomial space is proved. Special attention is paid to the case of vanishing asymptotic conditions, which allows us to describe by a characteristic property the interpolation space when there is no asymptotic condition.

§1. Introduction

Let f be a function $\mathbb{R} \to \mathbb{R}$ such that there exist a positive integer M and a real number A_M satisfying

$$f(x) = A_M x^M + o(|x|^M), \quad |x| \to \infty. \tag{1}$$

If we want to approximate $f(x)$ by a polynomial, one of the possibilities which can be considered natural is to choose as approximant a polynomial of the form

$$p(x) = A_M x^M + \hat{p}(x) \tag{2}$$

with $\hat{p}$ a polynomial of degree not greater than $M-1$. The polynomial p can be determined, for example, by M interpolation conditions. More precisely, let us consider a Hermite-Birkhoff (or Birkhoff) interpolation problem with M data of the type $f^{(i)}(x_j)$ which is poised in Π_{M-1} (space of polynomials of degree not greater than $M-1$), i.e. there exists a unique solution of the problem in Π_{M-1} for any f. The problem of finding p of the form (2) such that

$$p^{(i)}(x_j) = f^{(i)}(x_j) \qquad \forall i, j, \tag{3}$$

has, obviously, a unique solution (2), with $\hat{p}$ satisfying

$$\hat{p}^{(i)}(x_j) = g^{(i)}(x_j) \qquad \forall i, j,$$

for $g(x) = f(x) - A_M x^M$.

Since (1) and (2) mean that

$$\lim_{x\to\infty} \frac{p(x)}{x^M} = \lim_{x\to\infty} \frac{f(x)}{x^M}, \tag{4}$$

we can call (4) an *asymptotic condition* for p with respect to f. If we denote by f^{*M} the value

$$f^{*M} = \lim_{x\to\infty} \frac{f(x)}{x^M}$$

for a function f when this limit is finite, as for example for $f \in \Pi_M$, then p can be considered as an interpolation polynomial of degree not greater than M which matches the function f at the data $f^{(i)}(x_j)$ and f^{*M}, i.e.

$$\begin{aligned} p^{(i)}(x_j) &= f^{(i)}(x_j), \quad \forall i, j, \\ p^{*M} &= f^{*M}. \end{aligned}$$

We can say that p is the solution of an interpolation problem with an asymptotic condition. As we have explained above, this problem is immediately reducible to an ordinary Hermite-Birkhoff problem with one degree less.

Similarly we may consider bivariate functions satisfying

$$f(x,y) = f^{*M}(x,y) + o\left(|(x,y)|^M\right), \quad |(x,y)| \to \infty, \tag{5}$$

where $|(x,y)|$ denotes the euclidean norm of (x,y) and $f^{*M}(x,y)$ is a homogenous polynomial of degree M. If we know f^{*M}, the problem of approximating f by polynomials

$$p(x,y) = f^{*M}(x,y) + \hat{p}(x,y),$$

with $\hat{p} \in \Pi_{M-1}(x,y)$, can be reduced to an ordinary interpolation problem as in the univariate case.

The problem becomes more interesting for functions such that (5) holds when we do not know explicitly f^{*M}, but the asymptotic values of f along some given directions. In this case the problem is not so easily reducible to a simpler one.

Some years ago, Gasca and Maeztu [4] introduced a Newton approach to deal with bivariate Hermite-Birkhoff interpolation problems, by considering the data distributed along a set of straight lines. The aim of this paper is twofold: on one hand, we want to study bivariate interpolation problems with asymptotic conditions along some straight lines, using the techniques of [4]. On the other hand, we want to show that some interpolation problems considered by Dyn and Ron in [3] and by Bojanov, Hakopian and Sahakian in [1] and [6], can be stated in the present framework in a simpler form. Those authors provided a Lagrange formula and a compact characterization of the interpolation space, while our technique provides a Newton formula and a description of the space as the space generated by the Newton basis.

In the next section we state the problem and prove the existence and uniqueness of the solution, showing that the matrix of the linear system associated to the problem is lower triangular and nonsingular. In Section 3 we take advantage of the triangularity of the matrix to construct the coefficients of the polynomial which solves the problem by a recurrence formula. That is , we find a Newton formula for this problem. In the last section we show the connections of [3] with the results of the precedent sections and we give some examples.

§2. Statement of the problem

Let $r(x,y) = ax+by+c$, $(a,b) \neq (0,0)$ be a bivariate polynomial of exact degree one. Then the set of points $\{(x,y) \in \mathbb{R}^2 \mid ax+by+c=0\}$ is a straight line whose equation is $r=0$. Conversely, given any line of the plane there exists a polynomial of degree one $r(x,y) = ax+by+c$, $(a,b) \neq (0,0)$, such that $r=0$ is the equation of that line. Since the polynomial r is defined only up to a constant factor, we may choose r such that, for example $(a,b) = (\cos\theta, \sin\theta)$, for some $\theta \in (-\pi/2, \pi/2]$ or any other suitable choice. Once the polynomial r has been chosen, it will remain unchanged. In the rest of the paper r will indistinctly denote the polynomial of degree one and also, by abuse of notation, the straight line whose equation is $r=0$.

A general interpolation problem can be understood as the problem of finding a function of a certain linear space satisfying a set of linear conditions. In bivariate problems the set of conditions and the elements of a basis of the space can be indexed taking into account the relative position of the data points. For this reason we shall introduce a set of indices I with a special structure:

$$I = \{(i,j) | i = 0,1,\ldots,m; j = 0,1,\ldots,m(i)\}, \tag{6}$$

where $m(0),\ldots,m(n)$ are nonnegative integers. We shall use the lexicographical order

$$(i,j) < (k,l) \iff i < k \text{ or } (i = k \text{ and } j < l). \tag{7}$$

The interpolation space will be a subspace of dimension $\#I$ of the space of bivariate polynomials. The degree of all polynomials of the space will be less than or equal to

$$M := \max\{i+j \mid (i,j) \in I\}. \tag{8}$$

Usually the set of indices is rearranged in order to have $m(0) \geq \cdots \geq m(n)$ to keep the maximal degree of the interpolating functions as low as possible.

Each element L_{ij}, $(i,j) \in I$ of the data set is supported on a single point u_{ij}, unless an asymptotic condition is introduced. We shall use a set of flags w_{ij} indicating which kind of condition is introduced. If $w_{ij} = 1$ then $L_{ij}f$ is the value of f or some derivative of f at u_{ij}, whereas if $w_{ij} = 0$ we associate to the index (i,j) an asymptotic condition. Generally this association of indices and points in the plane will be helpful, as suggested by [4], only if we are able to arrange the points in some structure. A first requirement is that all points $\{u_{ij} \mid j = 0,\ldots,m(i)\}$ are collinear, belonging to the same line r_i. In order to mark the position of the different points u_{ij}, $j = 0,\ldots,m(i)$, on the line r_i we shall use a set of *transversal* (that is, neither parallel nor coincident) lines r_{ij} such that $r_i \cap r_{ij} = u_{ij}$.

Extending some ideas introduced in [4] [5] among other papers, we define an *interpolation system* as a set of triples

$$S = \{(r_i, r_{ij}, w_{ij}) \mid (i,j) \in I)\}, \tag{9}$$

satisfying the following conditions:

Condition 1. *I is a set of the form (6) and $w_{ij} = 1$ for all $(i, j) \in I$ with $i + j < M$, where M is given by (8).*

Condition 2. *For each $(i, j) \in I$ such that $w_{ij} = 1$,the lines r_i, r_{ij} intersect at exactly one point u_{ij}.*

Condition 3. *If $w_{i,M-i} = 0$ for some i with $(i, M - i) \in I$, then r_i is transversal to r_h for all $h < i$.*

Let us remark that parallel or coincident lines are allowed in S, the only lines which have to be transversal are r_i and r_{ij} for all $(i, j) \in I$ with $w_{ij} = 1$.

If $w_{ij} = 1$ for all (i, j), the interpolation system will give rise to a Hermite–Birkhoff interpolation problem of the type considered in [4]. Else we will have a Hermite–Birkhoff interpolation problem with additional asymptotic conditions along some lines.

As usual, for any vector $v = (a, b)$ we denote by

$$D_v f(x, y) = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}$$

the directional derivative along v. For a given interpolation system S we define the *Newton basis*, $B(S)$, as the set of polynomials

$$\phi_{ij} = \prod_{h=0}^{i-1} r_h \prod_{k=0}^{j-1} r_{ik}, \quad (i, j) \in I, \tag{10}$$

where the empty products (when $i = 0$ or $j = 0$) equal 1 by convention. We shall see later that these polynomials are linearly independent and why we may call this set a Newton basis.

The interpolation space $V(S)$ of our problem is the space spanned by the polynomials ϕ_{ij} of $B(S)$. Consequently, it will be a subspace of Π_M, the space of bivariate polynomials of total degree not greater than M.

We associate to each $(i, j) \in I$ with $w_{ij} = 1$ the numbers s_i and t_{ij}. The number of lines r_h, $h < i$, which are coincident with r_i will be denoted by s_i and the number of lines $r_0, r_1, \ldots, r_{i-1}, r_{i0}, r_{i1}, \ldots, r_{i,j-1}$, which contain u_{ij} but are not coincident with r_i will be denoted by t_{ij}.

Let us now consider the space V_M of bivariate functions f satisfying (5). Obviously, the space V_M contains the space Π_M.

For $f \in V_M$, a given vector v different from zero and a point $u_0 \in \mathbb{R}^2$, $f(u_0 + \tau v)$ is a function of the parameter $\tau \in \mathbb{R}$, which gives the values of f at the points of the line passing by u_0 with the direction of v. Then we have

$$f^{*M}(v) = \lim_{|\tau|\to\infty} \frac{f(u_0 + \tau v)}{\tau^M}. \tag{11}$$

We observe that the value of $f^{*M}(v)$ is independent of the choice of u_0.

Let us define now the set of interpolation data associated to the system S:

$$L_{ij}f := \begin{cases} D^{s_i}_{\rho_{ij}} D^{t_{ij}}_{\rho_i} f(u_{ij}), & \text{if } w_{ij} = 1, \\ f^{*M}(\rho_i), & \text{if } w_{ij} = 0, \end{cases} \qquad \forall (i,j) \in I, \tag{12}$$

where ρ_i, ρ_{ij} is a suitable choice of the directional vectors of the lines r_i, r_{ij}, respectively. We denote by $P(S)$ the interpolation problem defined by the linear forms L_{ij}, $(i,j) \in I$ on the space $V(S)$ generated by the polynomials ϕ_{ij}, $(i,j) \in I$.

Let us see an example including all these notations.

Example 1. Take

$$r_0(x,y) = x, \quad r_1(x,y) = y, \quad r_2(x,y) = y, \quad r_3(x,y) = x - y,$$

$$\begin{array}{ll} r_{00}(x,y) = r_{01}(x,y) = y, & r_{02}(x,y) = r_{03}(x,y) = r_{04}(x,y) = y - 1, \\ r_{10}(x,y) = x, & \\ r_{20}(x,y) = x, & r_{21}(x,y) = x - 1, \\ r_{30}(x,y) = r_{31}(x,y) = x - 1. & \end{array}$$

Then we have

$$\begin{array}{l} u_{00} = u_{01} = (0,0), \quad u_{02} = u_{03} = u_{04} = (0,1), \\ u_{10} = u_{20} = (0,0), \quad u_{21} = (1,0), \quad u_{30} = u_{31} = (1,1), \end{array}$$

and our choice for the directional vectors is

$$\begin{array}{l} \rho_0 = \rho_{10} = \rho_{20} = \rho_{21} = \rho_{30} = \rho_{31} = (0,1), \\ \rho_1 = \rho_2 = \rho_{00} = \rho_{01} = \rho_{02} = \rho_{03} = \rho_{04} = (1,0), \\ \rho_3 = (1,1). \end{array}$$

The Newton basis $B(S)$ is

$$\{1, y, y^2, y^2(y-1), y^2(y-1)^2, x, xy, x^2y, xy^2, xy^2(x-1)\}.$$

Here the maximal degree is $M = 4$, and the corresponding multiplicities of the derivatives are:

$$\begin{array}{l} s_0 = s_1 = 0, \quad s_2 = 1, \quad s_3 = 0, \\ t_{00} = 0, \quad t_{01} = 1, \quad t_{02} = 0, \\ t_{10} = t_{20} = t_{21} = 1, \quad t_{30} = 0, \quad t_{31} = 1. \end{array}$$

According to condition 2, w_{ij} must be 1 except for $(i,j) \in \{(0,4), (3,1)\}$, where we can take the value 1 or 0.

The interpolation space $V(S)$ is spanned by $B(S)$ and consequently is a subspace of Π_4. It can be easily shown that $V(S)$ can also be generated by the monomial basis

$$\{1, x, y, xy, y^2, x^2y, xy^2, y^3, x^2y^2, y^4\},$$

but this is not a Newton basis as $B(S)$ is.

The linear forms L_{ij} are, for $i+j \le 4$,

$$L_{00}f = f(0,0), \quad L_{01}f = \frac{\partial f}{\partial y}(0,0), \quad L_{02}f = f(0,1), \quad L_{03}f = \frac{\partial f}{\partial y}(0,1),$$
$$L_{10}f = \frac{\partial f}{\partial x}(0,0),$$
$$L_{20}f = \frac{\partial^2 f}{\partial x \partial y}(0,0), \quad L_{21}f = \frac{\partial f}{\partial y}(1,0),$$
$$L_{30}f = f(1,1),$$

and the linear forms L_{04} and L_{31} are

$$L_{04}f = \begin{cases} \dfrac{\partial^2 f}{\partial y^2}(0,1), & \text{if } w_{04} = 1, \\ \lim\limits_{|\tau|\to\infty} \dfrac{f(0,\tau)}{\tau^4}, & \text{if } w_{04} = 0, \end{cases}$$

$$L_{31}f = \begin{cases} \dfrac{\partial f}{\partial x}(1,1) + \dfrac{\partial f}{\partial y}(1,1), & \text{if } w_{31} = 1, \\ \lim\limits_{|\tau|\to\infty} \dfrac{f(\tau,\tau)}{\tau^4}, & \text{if } w_{31} = 0. \end{cases}$$

Let us now analyse the existence of solutions of our interpolation problem $P(S)$.

Theorem 1. *Let S be an interpolation system satisfying conditions 1,2,3. Let $P(S)$ be the interpolation problem defined by the linear forms L_{ij} of (12) and the space $V(S)$ spanned by the polynomials ϕ_{ij} of (10). For each set of real numbers z_{ij}, $(i,j) \in I$, there exists a unique polynomial $p \in V(S)$ such that*

$$L_{ij}p = z_{ij}, \quad \text{for all } (i,j) \in I.$$

Moreover the matrix

$$(L_{ij}\phi_{hk})_{(i,j),(h,k)\in I} \tag{13}$$

is lower triangular, for the row indices (i,j) and column indices (h,k) ordered by (7).

Proof: Any polynomial $p \in V(S)$ can be written in the form

$$p = \sum_{(h,k)\in I} a_{hk}\phi_{hk} \tag{14}$$

with ϕ_{ij} given by (9). Then the existence and uniqueness of solution of $P(S)$ is equivalent to the nonsingularity of the matrix (13). In order to show that this matrix is non-singular, we shall prove that it is lower triangular with nonzero diagonal elements:

$$L_{ij}\phi_{hk} = 0 \text{ for all } (i,j) < (h,k), \quad L_{ij}\phi_{ij} \neq 0 \text{ for all } (i,j) \in I. \tag{15}$$

If $w_{ij} = 1$ for all $(i,j) \in I$ then the problem $P(S)$ is identical to the one considered in [4]. In fact, in Theorem 1 of [4] it was shown that (15) holds in that case. The same arguments allows to prove (15) in the general case, for any (i,j) with $w_{ij} = 1$. In order to complete the proof we only need to verify that it holds too for all $(i,j) \in I$ with $w_{ij} = 0$. By conditions 2, 3 we must have that $j = M - i$.

Take now $i \in \{0, \dots, n\}$ such that $w_{i,M-i} = 0$. Observe that for a polynomial $\phi \in \Pi_M$, $L_{i,M-i}\phi = \phi^{*M}(\rho_i)$ is precisely the value of the homogeneous part of degree M of ϕ evaluated at the vector ρ_i.

If $(i, M-i) < (h,k)$ then, by (10), $h > i$ and r_i is a factor of ϕ_{hk}. Since

$$\phi_{hk}^{*M} = \prod_{\mu=0}^{h-1} r_\mu^{*1} \prod_{\nu=0}^{k-1} r_{h\nu}^{*1}, \tag{16}$$

and $r_i^{*1}(\rho_i) = 0$, we have that $L_{i,M-i}\phi_{h,k} = 0$.

Finally, let us check that $L_{i,M-i}\phi_{i,M-i} \neq 0$. Clearly

$$\phi_{i,M-i}^{*M} = \prod_{\mu=0}^{i-1} r_\mu^{*1} \prod_{\nu=0}^{M-i-1} r_{i\nu}^{*1}. \tag{17}$$

By Condition 3, the direction ρ_i of r_i is not the same as the direction of the lines r_μ with $\mu < i$ and so $r_\mu^{*1}(\rho_i) = D_{\rho_i} r_\mu \neq 0$ for all $\mu > i$. By Condition 1, we also have that the direction ρ_i of r_i is not the same as the direction of the lines $r_{i\nu}$, $\nu = 0, \dots, M-i-1$, and, analogously, we have $r_{i\nu}^{*1}(\rho_i) = D_{\rho_i} r_{i\nu} \neq 0$ for all $\nu < M - i$. Therefore

$$L_{i,M-i}\phi_{i,M-i} = \prod_{\mu=0}^{i-1} r_\mu^{*1}(\rho_i) \prod_{\nu=0}^{M-i-1} r_{i\nu}^{*1}(\rho_i) \neq 0,$$

and so we have proved that (15) holds. ■

Remark 1. *Observe that, for any $(i, m(i))$ with $w_{i,m(i)} = 1$, the role of $r_{im(i)}$ is only to indicate the point $u_{im(i)}$ used in $L_{im(i)}$, but $r_{im(i)}$ does not appear as a factor in any of the polynomials ϕ_{hk}. When $w_{i,m(i)} = 0$, $r_{im(i)}$ is neither used for the construction of the functional $L_{im(i)}$ nor for the construction of the basis $B(S)$ and can be taken arbitrarily.*

§3. Construction of the solution

As a consequence of Theorem 1, we derive that the solution p written in the form (14) of the interpolation problem $P(S)$ can be constructed by recurrence. The fact that the matrix (13) is lower triangular implies that

$$L_{ij}p = \sum_{(h,k)<(i,j)} a_{hk} L_{ij}\phi_{hk} \tag{18}$$

and, therefore, the coefficients a_{hk} can be computed using the following recurrence relations:

$$\begin{aligned} a_{00} &:= \frac{z_{00}}{L_{00}\phi_{00}}, \\ a_{ij} &:= \frac{z_{ij} - \sum_{(h,k)<(i,j)} a_{hk} L_{ij}\phi_{hk}}{L_{ij}\phi_{ij}}. \end{aligned} \tag{19}$$

The similarity between this recurrence relation and the one for defining univariate divided differencies justifies the name Newton basis which we give to $B(S)$. Analogously formula (14) can be called *Newton formula* and the coefficients a_{ij} of (14) defined by the recursion (19) can be seen as *divided differences* associated to the problem $P(S)$, playing a similar role to that of univariate divided differences.

Usually the numbers z_{ij} are the values of $L_{ij}f$ for a given function f and we say that p interpolates f. In our case, this means that p matches f at some points, some of the derivatives of p match the corresponding ones of f and that the asympotic behaviour along certain prescribed directions of f is reproduced by p. Then we can rewrite (19) as

$$\begin{aligned} a_{00} &:= \frac{L_{00}f}{L_{00}\phi_{00}}, \\ a_{ij} &:= \frac{L_{ij}f - \sum_{(h,k)<(i,j)} a_{hk} L_{ij}\phi_{hk}}{L_{ij}\phi_{ij}}. \end{aligned} \tag{20}$$

Observe that if $w_{i,M-i} = 0$ for some i then $L_{i,M-i}\phi_{hk} = 0$ for all (h,k) with $h+k < M$. Therefore the row indexed with $(i, M-i)$, namely $(L_{i,M-i}\phi_{hk})_{(h,k)\in I}$ has zeros on all columns indexed by (h,k) with $h+k < M$ and also on all columns with $(h,k) > (i,j)$. So, we can write

$$a_{i,M-i} := \frac{L_{ij}f - \sum_{h<i;(h,M-h)\in I} a_{h,M-h} L_{i,M-i}\phi_{h,M-h}}{L_{i,M-i}\phi_{i,M-i}}. \tag{21}$$

Note that the sum includes only terms with $h < i$ satisfying $(h, M-h) \in I$.

In particular if

$$w_{h,M-h} = 0, \quad \text{for all } h \text{ such that } (h, M-h) \in I, \tag{22}$$

then the coefficients $a_{h,M-h}$ (corresponding to the terms of highest degree of the polynomial p) can be computed independently of the other ones. More precisely, if $i_0 < \cdots < i_k$ are just the numbers such that $(i_l, M-i_l) \in I$, $l = 0, 1, \ldots, k$, we may compute (21) in the form

$$
\begin{aligned}
a_{i_0,M-i_0} &= \frac{L_{i_0,M-i_0} f}{L_{i_0,M-i_0}\phi_{i_0,M-i_0}}, \\
a_{i_l,M-i_l} &= \frac{L_{i_l,M-i_l} f - \sum_{h=0}^{l-1} a_{i_h,M-i_h} L_{i_l,M-i_l}\phi_{i_h,M-i_h}}{L_{i_l,M-i_l}\phi_{i_l,M-i_l}}, \quad l = 1, \ldots, k.
\end{aligned} \tag{23}
$$

Once we have computed the coefficients $a_{i_l,M-i_l}$ of the polynomials $\phi_{i_l,M-i_l}$ of the highest degree we may define

$$p_M := \sum_{l=0}^{k} a_{i_l,M-i_l}\phi_{i_l,M-i_l}.$$

Now we may regard the polynomial $\hat{p} := p - p_M$ of degree less than or equal to $M-1$ as the solution of an ordinary Hermite–Birkhoff interpolation problem (in the sense of [4]) interpolating the function $f - p_M$ instead of f at the data $L_{i,j}$ corresponding to $(i,j) \in I$ with $i + j < M$.

Remark 2. *The recurrence relation (23) holds even if (22) does not hold because $w_{h,M-h} = 1$ for some h, provided that the corresponding $L_{i,M-i}\phi_{h,M-h}$ $(h < i)$ vanishes. This happens if and only if at least one of the lines $r_{h0}, \ldots, r_{h,M-h-1}$ has the direction of r_i. Recall that Condition 3 implies that no line r_j with $j < i$ can have the direction of r_i.*

Another consequence of Theorem 1 is that the polynomials ϕ_{ij} of $B(S)$ are linearly independent. Furthermore, if

$$n = M, \qquad m(i) = M - i, \quad i = 0, \ldots, M, \tag{24}$$

or equivalently if $I = \{(i,j) \mid i + j \le M\}$, then one has $\#I = \binom{M+2}{2}$, which is exactly the dimension of Π_M. Hence, in this case, $V(S) = \Pi_M$. Conversely, it is straightforward to see that the unique choice of I which produces $V(S) = \Pi_M$ *in our approach* is (24). So, if we have an interpolation problem $P(S)$ such that (23) holds and $w_{i,M-i} = 0$, $i = 0, \ldots, M$, then we know the asymptotic behaviour of p along $M+1$ different directions. From this information we can compute by (23)

$$p_M = \sum_{i=0}^{M} a_{i,M-i}\phi_{i,M-i}$$

and then solve an ordinary interpolation problem in Π_{M-1} for $f - p_M$.

§4. Interpolation spaces and vanishing asymptotic conditions.

Let S be an interpolation system (9) indexed by a set I. Let $I^* := \{(i,j) \in I \mid w_{ij} = 0\}$ and define $\hat{I} := I \setminus I^*$ and

$$\hat{V}(S) := \{p \in V(S) \mid L_{ij}p = 0, \forall (i,j) \in I^*\}. \tag{25}$$

From Theorem 1, we know that $P(S)$ has always a unique solution. In particular there exists a unique $p \in V(S)$ such that

$$L_{ij}p = z_{ij}, \quad \forall (i,j) \in \hat{I},$$
$$L_{ij}p = 0, \quad \forall (i,j) \in I^*.$$

This is equivalent to saying that the problem $\hat{P}(S)$ of finding a function $p \in \hat{V}(S)$ such that $L_{ij}p = L_{ij}f$, for all $(i,j) \in \hat{I}$ has always a unique solution.

Let us now consider a problem $P(S)$ determined by an interpolation system with $m(i) = M - i$ for all i, that is, with Π_M as interpolation space. In this case, one has

$$\hat{V}(S) = \{p \in \Pi_M \mid p^{*M}(\rho_i) = 0 \text{ for all } i \text{ with } w_{i,M-i} = 0\}. \tag{26}$$

Taking into account that $p^{*M}(\rho_i)$ is the coefficient of τ^M in $p(u_0 + \tau\rho_i)$, we see that $\hat{V}(S)$ can be described as the subspace of polynomials of degree not greater than M such that the restriction of the polynomial to any line parallel to r_i, with $(i, M-i) \in I^*$, has degree less than or equal to $M - 1$. Using the simple fact that $p^{*M}(\rho) = D^M_\rho p/M!$ for any polynomial p in Π_M, we may write

$$\hat{V}(S) = \{p \in \Pi_M \mid D^M_{\rho_i} p = 0 \text{ for all } i \text{ with } w_{i,M-i} = 0\}.$$

Let us try to derive a Newton formula (14) for the problem $\hat{P}(S)$ with coefficients computable by a recurrence similar to (20). Since $\hat{V}(S) \subseteq V(S)$ (in our case $V(S) = \Pi_M$), we may write any polynomial of $\hat{V}(S)$ as a linear combination of the polynomials ϕ_{ij}, $(i,j) \in I$. However we observe that if $(i,j) \in I^*$, then

$$L_{ij}\phi_{ij} = \phi^{*M}_{ij}(\rho_{ij}) \neq 0,$$

i.e. ϕ_{ij}, $(i,j) \in I^*$, has degree M along the line r_i and therefore $\phi_{ij} \notin \hat{V}(S)$. A natural approach is to discard the basic functions corresponding to asymptotic conditions and try to write a polynomial in $\hat{V}(S)$ as

$$p = \sum_{(i,j) \in \hat{I}} a_{ij}\phi_{ij}, \tag{27}$$

hoping that $\{\phi_{ij} \mid (i,j) \in \hat{I}\}$ generates $\hat{V}(S)$. However there might be still indices $(i,j) \in \hat{I}$ with $i + j = M$ and so we can find functions ϕ_{ij}, $(i,j) \in \hat{I}$ having the maximal degree M and we cannot conclude that ϕ_{ij} has less degree along the prescribed directions r_h, $(h, M-h) \in I^*$. In other words, the vector space

$$W(S) = \text{span}\{\phi_{ij} \mid (i,j) \in \hat{I}\}$$

can be different from the space $\hat{V}(S)$, although both are subspaces of $V(S)$ with the same dimension $\#\hat{I}$. So, in the general case (27) will not represent a polynomial in $\hat{V}(S)$.

Let us denote by $i_0 < \cdots < i_k$ the indices such that $I^* = \{(i_0, M - i_0), \ldots, (i_k, M - i_k)\}$ and additionally assume, as in Remark 2,

$$L_{i_l, M-i_l} \phi_{h,M-h} = 0, \quad \forall h < i_l, \quad l = 0, 1, \ldots, k, \tag{28}$$

so that (23) holds. Recall that the precedent assumption is equivalent to saying that some of the lines $r_{h0}, \ldots, r_{h,M-h-1}$, have the same direction as r_{i_l}. In this case, if the asymptotic data are all zero, then (23) implies that

$$a_{i_l, M-i_l} = 0, \quad l = 1, \ldots, k.$$

Therefore (27) holds, we have $W(S) = \hat{V}(S)$ and the problem $\hat{P}(S)$ can be solved by a Newton formula, simply by removing the terms $a_{ij}\phi_{ij}$, $(i,j) \in I^*$ from the Newton formula (20) of the problem $P(S)$. Summarizing , under condition (28), the interpolation problem $\hat{P}(S)$ can be seen as the interpolation problem associated with the reduced system

$$\hat{S} := \{(r_i, r_{ij}, w_{ij}) \mid (i,j) \in \hat{I}\},$$

that is $P(\hat{S}) = \hat{P}(S)$.

The spaces of polynomials whose degree diminishes along some prescribed directions have been used by Dyn and Ron [3] in connection to the problem of obtaining Lagrange formulae similar to those obtained by Chung and Yao [2]. Dyn and Ron consider configurations of points which are the intersections of a given set of lines, where parallel or coincident lines are allowed. If the lines are coincident then we have Hermite data instead of Lagrange data. If the lines are parallel the effect is a change of the interpolation space, allowing only polynomials whose degree is less than the maximal degree along the direction of those parallel lines.

Similar problems were also considered by Bojanov, Hakopian and Sahakian in [1], [6]. Their idea can be explained as follows: each polynomial, when restricted to an affine submanifold of $\mathbb{R}^s$ (a trace of a polynomial), can be interpreted as a polynomial in less than s variables. If we consider improper submanifolds as intersection of proper affine submanifolds when they tend to be parallel, a limiting polynomial trace also arises in this case. For example, the homogeneous part of highest degree of a polynomial can be seen as its trace, when restricted to the improper (ideal) hyperplane. The traces on linear manifolds are an algebraic counterpart of the asymptotic data introduced here. In some sense, Bojanov, Hakopian and Sahakian extend the problems analyzed in [3] allowing asymptotic conditions. However their approach is algebraic in contrast with our analytic (and simpler) approach.

In this paper the existence and uniqueness of solution of the problem has been derived, as in [4], from the triangularity of the linear system. As a natural consequence a Newton formula has been obtained, but it is difficult to describe the interpolation space in a compact form, except for some particular cases, for example when $V(S) = \Pi_M$. So both approaches are complementary and we think that this framework is specially well suited to deal with these problems, due to its simplicity and to the fact that Newton formula (contrarily to Lagrange formula) is valid also for Hermite cases with straightforward changes.

Let us recall some of the results of [3] on Lagrange and Hermite interpolation problems in $\mathbb{R}^s$. Since our paper is devoted to the bivariate case we shall only discuss the case $s = 2$.

Dyn and Ron start with a set of lines

$$\Gamma = \{r_0, \ldots, r_{M+1}\} \tag{29}$$

and U is the set of points in the plane obtained as intersection of any two transversal lines in Γ.

If more than two lines meet at the same point $u \in U$, the interpolation problem will be of Hermite type. Following the terminology of [3] the set Γ is called *simple* if no more than two lines meet at the same point. The *simple case* corresponds to a Lagrange interpolation problem. Let us remark that in this case, coincident lines in Γ are not allowed. We shall analyse only the simple case.

For each point in U we may define

$$l_u = \frac{\prod_{r\in\Gamma; r(u)\neq 0} r}{\prod_{r\in\Gamma; r(u)\neq 0} r(u)}, \tag{30}$$

and then we may express the solution of the Lagrange interpolation problem

$$p(u) = f(u), \quad u \in U, \tag{31}$$

by the Lagrange formula

$$p(x) = \sum_{u\in U} f(u) l_u. \tag{32}$$

In [3] it is proved, using techniques involving exponential box-splines, that the interpolation space $\mathrm{span}\{l_u \mid u \in U\}$ can be alternatively described by

$$H = \mathrm{span}\left\{ \prod_{j\in J} r_j^{*1} \mid J \subset \{0, \ldots, M+1\} \text{ such that } \mathrm{span}\{\rho_j \mid j \notin J\} = \mathbb{R}^2 \right\}. \tag{33}$$

The same space H is also used in that paper for the Hermite problem, when considering the nonsimple case.

Dyn and Ron also show that H contains the set of polynomials of degree not greater than M which are of degree not greater than $M - \mu_i$ along all lines with direction ρ_i, where μ_i denotes the number of lines r_j, $j \neq i$, parallel to r_i. They also say that H coincides with that set.

Let us show that the case $\mu_i \leq 1$ for all $i \in \{0, \ldots, M+1\}$ (no more than two lines in r_i are parallel) can be analysed with our techniques.

Let us take the lines in such an ordering that the pairs (r_0, r_1), $\ldots$, (r_{2h-2}, r_{2h-1}) correspond to parallel lines and $r_{2h}, \ldots, r_{M+1}$ are not parallel to any other line in Γ. Then we have $\mu_0 = \mu_1 = \cdots = \mu_{2h-1} = 1$ and $\mu_{2h} = \cdots \mu_{M+1} = 0$. Now, we may define a system (9) in the following form:

$$I = \{(i,j) \mid i + j \leq M\},\ r_{ij} = r_{M+1-j},\ w_{ij} = \begin{cases} 0 & \text{if } r_i,\ r_{ij} \text{ are parallel,} \\ 1 & \text{otherwise.} \end{cases} \tag{34}$$

Let us recall that the interpolation space corresponding to vanishing asymptotic conditions associated with this system, $\hat{V}(S)$, can be described by (26). Any l_u of (29) belongs to $\hat{V}(S)$ because there exists always a line in Γ not containing the point u parallel to ρ_i, $i = 0, \ldots, 2h-1$. On the other hand $\dim H = \#U = \#\hat{I} = \dim \hat{V}(S)$. Therefore the space H coincides with $\hat{V}(S)$ and they have the same description, as subspaces of Π_M whose degree is less than $M-1$ along lines parallel to r_i with $w_{i,M-i} = 0$. Hence the interpolation problem posed by Dyn and Ron coincides with $\hat{P}(S)$.

As a consequence, the systems defined by (34) admit a Lagrange formula (32) with the basis functions

$$\begin{aligned} l_{ij}(x,y) &= \frac{\prod_{\alpha<i} r_\alpha(x,y) \prod_{\beta \neq j} r_{i\beta}(x,y)}{\prod_{\alpha<i} r_\alpha(u_{ij}) \prod_{\beta \neq j} r_{i\beta}(u_{ij})} \\ &= \prod_{\alpha \neq i, M+1-j} \frac{r_\alpha(x,y)}{r_\alpha(u_{ij})}, \quad (i,j) \in \hat{I}, \end{aligned}$$

where $u_{ij} = r_i \cap r_{M+1-j}$.

On the other hand, it can be easily shown that the system (34) satisfies (28), which means that the space $\hat{V}(S)$ has the following Newton basis:

$$\phi_{ij} = \prod_{h=0}^{i-1} r_h \prod_{k=0}^{j-1} r_{M+1-k}, \quad (i,j) \in \hat{I}$$

to be used in the Newton formula (27) with the recurrence relation (20).

Let us see some examples.

Example 2. This example concerns interpolation data on the intersection points of any two transversal lines of Figure 1.

Figure 1. An interpolation problem for cuartic polynomials.

Following [4] we can solve this problem constructing an interpolation system (the one called $\hat{S}$ above) as follows: the index set is

$$\hat{I} = \{(i,j) \mid 0 \le i \le 1, 0 \le j \le 3\} \cup \{(i,j) \mid 2 \le i \le 3, 0 \le j \le 1\},$$

the lines r_i, $0 \le i \le 3$, are those of Figure 1 and $r_{ij} = r_{5-j}$, $(i,j) \in \hat{I}$. Here $w_{ij} = 1 \quad \forall (i,j)$. From this system, we get the Newton formula (20) to solve the Lagrange interpolation problem, but it is not easy to express in a compact form which is the space generated by the Newton basis

$$\{1, r_5, r_5 r_4, r_5 r_4 r_3, r_0, r_0 r_5, r_0 r_5 r_4, r_0 r_5 r_4 r_3, r_0 r_1, r_0 r_1 r_5, r_0 r_1 r_2, r_0 r_1 r_2 r_5\}.$$

As explained above, we can enlarge $\hat{S}$ to get the system S, with

$$I = \{(i,j) \mid i+j \le 4\},$$

taking the lines r_i, $0 \le i \le 4$ and $r_{ij} = r_{5-j}$, $(i,j) \in I$, $w_{04} = w_{22} = w_{40} = 0$ and all the other $w_{ij} = 1$. The interpolation space $V(S)$ is Π_4 and has dimension 15. The interpolation data associated to S are the twelve data of the previous system and

$$L_{04} f = f^{*4}(\rho_0), \quad L_{22} f = f^{*4}(\rho_2), \quad L_{40} f = f^{*4}(\rho_4).$$

According to our above results, the coefficients a_{04}, a_{22}, a_{40} of the Newton formula can be obtained first (see (23)):

$$\begin{aligned}
a_{04} &= \frac{L_{04} f}{L_{04} \phi_{04}}, \\
a_{22} &= \frac{L_{22} f - a_{04} L_{22} \phi_{04}}{L_{22} \phi_{22}}, \\
a_{40} &= \frac{L_{40} f - a_{04} L_{40} \phi_{04} - a_{22} L_{40} \phi_{22}}{L_{22} \phi_{22}},
\end{aligned}$$

and since

$$L_{22}\phi_{04} = [r_4^{*1}(\rho_2)]^2[r_2^{*1}(\rho_2)]^2 = 0,$$
$$L_{40}\phi_{04} = [r_4^{*1}(\rho_4)]^2[r_2^{*1}(\rho_4)]^2 = 0,$$
$$L_{40}\phi_{22} = [r_0^{*1}(\rho_4)]^2[r_4^{*1}(\rho_4)]^2 = 0,$$

we finally obtain

$$a_{04} = \frac{f^{*4}(\rho_0)}{r_3^{*1}(\rho_0)r_5^{*1}(\rho_0)r_2^{*1}(\rho_0)r_4^{*1}(\rho_0)} = \frac{f^{*4}(\rho_0)}{[r_2^{*1}(\rho_0)]^2[r_4^{*1}(\rho_0)]^2},$$
$$a_{22} = \frac{f^{*4}(\rho_2)}{[r_0^{*1}(\rho_2)]^2[r_4^{*1}(\rho_2)]^2},$$
$$a_{40} = \frac{f^{*4}(\rho_4)}{[r_0^{*1}(\rho_4)]^2[r_2^{*1}(\rho_4)]^2}.$$

Therefore, if we consider the problem $P(S)$ with vanishing asymptotic conditions (i.e. $\hat{P}(S)$), then $a_{04} = a_{22} = a_{40} = 0$, and the problem is reduced to the precedent one $P(\hat{S})$. Now it is clear that the interpolation space is that of polynomials of degee not greater than 4 with degree less than 4 when restricted to any line parallel to r_0, r_2 or r_4. At the same time, we have got a simple Lagrange formula for the solution of the problem:

$$p = \sum_{(i,j)\in\hat{I}} f(u_{ij}) \prod_{\alpha\neq i,5-j} \frac{r_\alpha(x,y)}{r_\alpha(u_{ij})}.$$

Observe that the case of one or several of these 3 couples of parallel lines replaced, respectively, by one or several couples of coincident lines can be considered as a limit case of the example above and the essential result still holds. For example, if $r_0 = r_1$ (and similarly for the others, one by one or simultaneously), the bases and spaces remained unchanged, with the same formulas, but some of the interpolation data change from values of f at some points to directional derivatives, according to (12). In our example, $L_{10}, L_{11}, L_{12}, L_{13}$ change, respectively, to

$$D_{\rho_5} f(u_{00}),\ D_{\rho_4} f(u_{01}),\ D_{\rho_3} f(u_{02}),\ D_{\rho_2} f(u_{03}),$$

where $u_{00} = r_0 \cap r_5, u_{01} = r_0 \cap r_4, u_{02} = r_0 \cap r_3, u_{03} = r_0 \cap r_2$.

If there are more than one couple of coincident lines, the number of data associated with directional derivatives and their order increases. For example, if there are two couples of coincident lines then the intersection point has “multiplicity” four. The four corresponding data are the value of the function at that point, two first order directional derivatives, one in each of the directions of the coincident lines and one mixed derivative of second order in both directions. When the three pairs are coincident we have three points of multiplicity four (see Figure 2).

Figure 2. The limit case produces a Hermite problem.

Example 3. A simpler example arises when we choose two pairs of lines, r_0 parallel to r_1 and r_2 parallel to r_3.

Take a system $\hat{S}$ with $n = 1$, $m(0) = m(1) = 1$, $r_{ij} = r_{3-j}$ and $w_{ij} = 1$, $i, j \in \{0, 1\}$. A Newton basis for the corresponding Lagrange interpolation problem is

$$B(\hat{S}) = \{1, r_3, r_0, r_0 r_3\},$$

generating a subspace $V(\hat{S})$ which contains Π_1.

As before we can complete $\hat{S}$ to get a system S with asymptotic data on r_0 and r_2. Take $n = 2$, $m(0) = 2$, $m(1) = 1$, $m(2) = 0$, $r_{ij} = r_{3-j}$, $i + j \le 2$, $w_{02} = w_{20} = 0$ and $w_{ij} = 1$, $i, j \in \{0, 1\}$.

In this case, the Newton basis is

$$B(S) = \{1, r_3, r_3 r_2, r_0, r_0 r_3, r_0 r_1\},$$

the space generated by $B(S)$ is Π_2 and the data are

$$L_{02} f = f^{*2}(\rho_0) = \frac{1}{2} D^2_{\rho_0} f, \quad L_{20} f = f^{*2}(\rho_2) = \frac{1}{2} D^2_{\rho_2} f,$$

and the values of f at the four intersection points $u_{ij} = r_i \cap r_{3-j}$, $i, j \in \{0, 1\}$.

As in the previous example, we conclude that $\hat{P}(S) = P(\hat{S})$ and the space spanned by $\{1, r_3, r_0, r_0 r_3\}$ can be described in the form

$$V(\hat{S}) = \{p \in \Pi_2 \mid D^2_{\rho_0} p = D^2_{\rho_2} p = 0\}.$$

In order to solve this interpolation problem we can use the Newton formula or the following Lagrange formula

$$p = \sum_{i=0}^{1} \sum_{j=0}^{1} f(u_{ij}) l_{ij}, \quad l_{ij} = \frac{r_{1-i} r_{2+j}}{r_{1-i}(u_{ij}) r_{2+j}(u_{ij})}, \ i, j \in \{0, 1\}.$$

Figure 3. Bilinear interpolation.

The lines r_0, r_1 are often parallel to the x-axis and r_2, r_3 to the y-axis

$$r_0(x,y) = x - x_0, \quad r_1(x,y) = x - x_1,$$
$$r_2(x,y) = y - y_1, \quad r_3(x,y) = y - y_0,$$

describing a rectangle with vertices (x_0, y_0), (x_0, y_1), (x_1, y_0), (x_1, y_1). Then $V(\hat{S})$ is the clasical bilinear space

$$V(\hat{S}) = \operatorname{span}\{1, x, y, xy\} = \left\{ p \in \Pi_2 \mid \frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p}{\partial y^2} = 0 \right\}$$

of functions such that for any fixed value of x the function is linear in y and for any fixed value of y, the resulting function is linear in x.

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