Planar Configurations with Simple Lagrange Interpolation Formulae

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Abstract. The geometric condition (GC) for multivariate interpolation is equivalent to the existence of a Lagrange formula whose terms are products of linear factors. In 1982, Gasca and Maeztu conjectured that any set of $(n+2)(n+1)/2$ points in the plane satisfying the GC condition must contain $n+1$ collinear points. The conjecture has only been proved for degrees $n \leq 4$. In this paper we classify some configurations of points in the plane satisfying the GC condition.

§1. Introduction and Auxiliary Results

Let $\Pi_n(\mathbb{R}^k)$ be the space of all polynomials in k variables of degree less than or equal to n, whose dimension is $\binom{n+k}{k}$. For any finite set $X \subseteq \mathbb{R}^k$ we may pose the

Lagrange interpolation problem. *Given* $X \subseteq \mathbb{R}^k$ *and* $f \in \mathbb{R}^X$, *find* $p \in \Pi_n(\mathbb{R}^k)$ *such that*

$$p(x) = f(x), \quad \forall x \in X. \tag{1.1}$$

Every polynomial p of degree not greater than n can be written in the form $p(x) = \sum_{|\alpha| \leq n} c_\alpha x^\alpha$, and the interpolation conditions give rise to the system of $|X|$ equations and $\binom{n+k}{k}$ unknowns

$$\sum_{|\alpha| \leq n} c_\alpha x^\alpha = f(x), \quad x \in X. \tag{1.2}$$

An interesting problem in multivariate interpolation is to infer the existence and uniqueness of the solution of the Lagrange interpolation problem from the distribution of the points in X. This leads to the following

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Definition 1.1. We say that a set $X \subseteq \mathbb{R}^k$ is unisolvent in $\Pi_n(\mathbb{R}^k)$ if the Lagrange interpolation problem for X has a unique solution for any $f \in \mathbb{R}^X$.

Equation (1.2) confirms that a necessary condition for a set X to be unisolvent in $\Pi_n(\mathbb{R}^k)$ is that $|X| = \binom{n+k}{k}$. If $|X| = \binom{n+k}{k}$, the linear system (1.2) has the same number of equations and unknowns. Then any set X of $\binom{n+k}{k}$ points, X is unisolvent in $\Pi_n(\mathbb{R}^k)$ if and only if there exists no $p \in \Pi_n(\mathbb{R}^k)$ vanishing at all the points of X. This condition can be geometrically expressed by saying that not all points of X lie on the same algebraic hypersurface of degree less than or equal to n. The question of easily recognizing and generating unisolvent sets for posing Lagrange interpolation problems can be analyzed from several points of view. The Newton approach consists of finding a basis of functions of $\Pi_n(\mathbb{R}^k)$ vanishing on bigger and bigger subsets of X (see [3]). The Lagrange approach analyzes the existence and construction of certain functions called Lagrange polynomials.

Definition 1.2. For a set $X \subseteq \mathbb{R}^k$, we say that $l \in \Pi_n(\mathbb{R}^k)$ is a Lagrange polynomial associated to $x \in X$ if $l(x) = 1$ and $l(y) = 0$, for all $y \in X \setminus \{x\}$.

In view of this definition we can deduce the following proposition as a direct consequence of well-known results of Linear Algebra.

Proposition 1.3. *Let $X \subseteq \mathbb{R}^k$. Then the following properties are equivalent:*
(i) *X is unisolvent in $\Pi_n(\mathbb{R}^k)$.*
(ii) *For each $x \in X$ there exists a unique Lagrange polynomial $l_x \in \Pi_n(\mathbb{R}^k)$.*
(iii) *$|X| = \binom{n+k}{k}$ and there exists a Lagrange polynomial $l_x \in \Pi_n(\mathbb{R}^k)$ for all $x \in X$.*

Furthermore, the solution p of the Lagrange interpolation problem (1.1) can be expressed by the Lagrange formula

$$p = \sum_{x \in X} f(x) l_x. \tag{1.3}$$

The following properties of Lagrange polynomials will be useful throughout this paper:

Proposition 1.4. *Let $X \subseteq \mathbb{R}^k$ be unisolvent in $\Pi_n(\mathbb{R}^k)$, and let l_x be the Lagrange polynomial associated with $x \in X$. Then*
(i) $\deg l_x = n$.
(ii) *The factorization of l_x into irreducibles cannot have multiple factors.*
(iii) *For any polynomial g with $\deg g = r$, one has $|\{x \in X \mid g(x) = 0\}| \le \binom{n+k}{k} - \binom{n-r+k}{k}$.*

Proof: (i) Let h be a polynomial of degree 1 vanishing on $x \in X$. If $\deg l_x < n$, then hl_x is a polynomial of degree less than or equal to n vanishing on X, contradicting the fact that X is unisolvent. (ii) If the factorization of l_x into irreducibles has repeated factors, then removing all the repeated factors we would be able to construct a Lagrange polynomial of degree less than n which

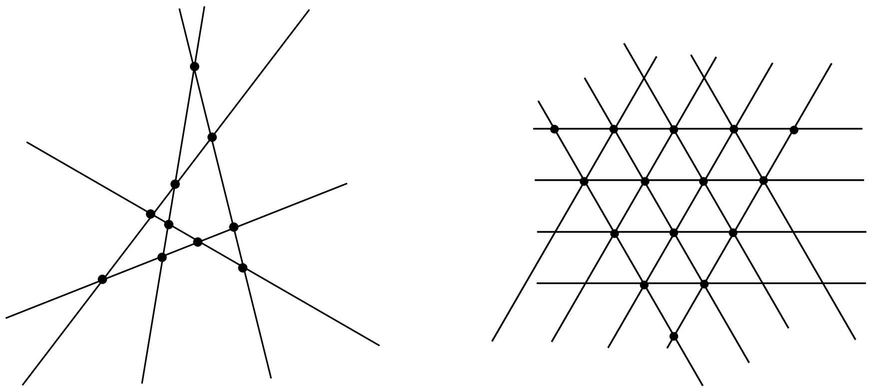

Fig. 1. A natural lattice (left) and a principal lattice (right).

is impossible by (i). (iii) Let $Y := \{x \in X \mid g(x) \neq 0\}$, and assume that $|Y| \leq \binom{n-r+k}{k}$. Then there exists a polynomial $f \in \Pi_{n-r}(\mathbb{R}^k)$ vanishing on Y. Now $fg \in \Pi_n(\mathbb{R}^k)$ vanishes on X, contradicting the unisolvence of X. □

In the case of polynomials of 1 variable, the Lagrange polynomials have a simple expression as a product of linear factors: $l_x(\xi) = \Pi_{y \in X \setminus \{x\}} \frac{\xi - y}{x - y}$. This formula does not have a simple extension to several variables, unless the points of X are structured in a special way.

Definition 1.5. Let $X \subseteq \mathbb{R}^k$ with $|X| = \binom{n+k}{k}$. The set X satisfies the geometric condition (GC_n) if for all $x \in X$, there exist affine functions h_i^x, $i = 1, \ldots, r$, $r \leq n$, such that the union of all hyperplanes $h_i^x = 0$ contains all points of $X \setminus \{x\}$, but not the point x. We say that $\{h_i^x = 0 \mid i = 1, \ldots, r\}$ is *the set of hyperplanes associated* with the point x. The set of all hyperplanes associated with some point $x \in X$ is denoted by Γ_X.

The GC condition, introduced by Chung and Yao [2], is equivalent to the existence of Lagrange polynomials which are a product of linear factors: $l_x = \Pi_{i=1}^n h_i^x$ (we may assume h_i^x normalized to have $h_i^x(x) = 1$). Therefore, if X satisfies the GC_n condition, then X must be unisolvent in $\Pi_n(\mathbb{R}^k)$. By Proposition 1.3, the set of hyperplanes associated with a point must be unique, and by Proposition 1.4 (i), (ii), it must have exactly n elements.

An interesting question is how to construct sets of points X satisfying the GC condition. Some important examples have been given in [2], such as natural lattices and principal lattices. Natural lattices are the set of intersection points of $n + 2$ lines which are in general position, that is, no two of them are parallel and no three of them are concurrent. Principal lattices can be described as the intersection points of three families of $n+1$ parallel lines such that each point is the intersection of three lines, one of each family.

A generalization of principal lattices (also satisfying the GC condition) was provided in [4]. A pencil of lines is a set of lines intersecting at one point (the center of the pencil) or parallel lines (the center is at the infinity line). A 3-pencil lattice of order n is defined as a set of $\binom{n+2}{2}$ points generated by three pencils of $n + 1$ lines each, in such a form that every point is the intersection of exactly one line of each pencil.

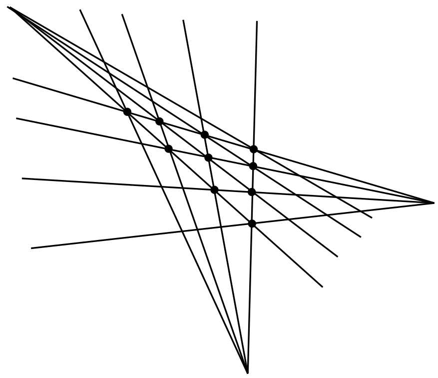

Fig. 2. A 3-pencil lattice.

The distributions of points satisfying the GC condition have not been completely described even in the two-dimensional case. The combinatorics of the GC condition are so difficult to study that it still has not been possible to solve the conjecture on the GC_n condition on the plane made in [3]:

Conjecture 1.6. *Let $X \subseteq \mathbb{R}^2$ satisfy the GC_n condition. Then, there exists a line in Γ_X containing $n+1$ points of X.*

Based on the work of Bush [1], we know that the conjecture has been verified for degrees less than or equal to 4. The purpose of this paper is to offer a classification of some configurations of points satisfying the GC condition in the plane. This analysis could be a starting point for dealing with more complicated cases.

§2. Natural Lattices and Default

Let us summarize some properties of the GC configurations.

Proposition 2.1. *Let $X \subseteq \mathbb{R}^2$ satisfy the GC_n condition. Then*

(i) *Each line in Γ_X has at least 2 points of X.*
(ii) *For each point $x \in X$, there exist at least two lines in Γ_X containing x, associated with different points $y, z \in X$ ($n \geq 1$).*
(iii) *Γ_X contains at least $n+2$ lines.*
(iv) *A set of r lines cannot contain more than $r(2n+3-r)/2$ points of X. In particular, no line contains more than $n+1$ points of X.*
(v) *A line containing $n+1$ points of X must be in Γ_X, and it is associated with every point not lying on it ($n \geq 1$).*
(vi) *Two lines, each containing $n+1$ points of X, cannot be parallel, and meet at a point $x \in X$.*
(vii) *Three lines, each containing $n+1$ points of X, cannot be concurrent.*
(viii) *There are at most $n+2$ lines containing $n+1$ points of X.*

Proof: (i) Let $H \equiv h_j^y = 0$ be a line in Γ_X associated with $y \in X$. Assume that $\{x\} = H \cap X$ and let g be an affine function such that $g = 0$ is the line passing through x and y. Then $g\Pi_{i\neq j} h_i^y$ is a polynomial of degree n vanishing on X, contradicting the fact that X is unisolvent. (ii) Take any $y \neq x \in X$.

Clearly, there exist H in the set of lines associated with y such that x lies in H. By (i), $H \cap X$ must contain a second point z. There must exist a line associated with z passing through x, which will be different from H. (iii) If we take $x \in X$, then there are n lines in Γ_X associated with x. By (ii), there are also 2 lines in Γ_X passing through x. (iv) follows directly from Proposition 1.4 (iii). (v) Let K be a line with $|K \cap X| = n+1$. Since $n \geq 1$, there must be at least one point $x \in X \setminus K$. Let $H_1, \ldots, H_n$ be the lines associated with x, then $K \cap X = \bigcup_{i=1}^{n} (K \cap H_i)$. Since $|K \cap X| = n+1$, at least one of the sets $K \cap H_i$ has more than one point, and so $K = H_i$ is associated with x. (vi) If two lines containing $n+1$ points are parallel or they meet at a point not in the set X, then this set of two lines contains $2n+2$ points of X, a contradiction with (iv). (vii) If there were three concurrent lines, each of them with $n+1$ points of X, then this set of three lines would contain $3n+1$ points of X, which contradicts (iv). (viii) Let m be the number of lines with $n+1$ points of X. These lines cannot be either parallel or concurrent, and X must contain all pairs of intersections of lines. Therefore $\binom{m}{2} \leq |X| = \binom{n+2}{2}$, and $m \leq n+2$. □

In the sequel, the $\binom{n+2}{2}$ points of a set X satisfying the GC_n condition will be denoted by x_{ij}:

$$X = \{x_{ij} \in \mathbb{R}^2 | i < j \in \{1, \ldots, n+2\}\}. \tag{2.1}$$

As we have seen in Proposition 2.1, an important subset of Γ_X is the set of lines $K_1, \ldots, K_m$ containing $n+1$ points of X. From Proposition 2.1 (vi-viii) we deduce that these lines are in *general position* and their number allows us to establish a classification of sets satisfyng the GC_n condition.

Definition 2.2. Let $X \subseteq \mathbb{R}^2$ be a set satisfying the GC_n condition. We say that X has default d or that X is a d-lattice if the number of lines in Γ_X with $n+1$ points is just $n+2-d$.

Let $m = n+2-d$ and let $K_1, \ldots, K_m$ be the lines with $n+1$ points of a set X with default d. From Proposition 2.1, all the intersection points of these lines are points of X. Then we can assume without loss of generality in formula (2.1) that

$$x_{ij} \in K_l \Longleftrightarrow l \in \{i, j\}, \tag{2.2}$$

which means that

$$x_{ij} = K_i \cap K_j,\ i < j \leq m;\ x_{ij} \in K_i,\ i \leq m < j;\ x_{ij} \notin \bigcup_{r=1}^{m} K_r,\ m < i < j.$$

By Proposition 2.1 (viii), $m \leq n+2$. Conjecture 1.6 means that $m \geq 1$. In fact, this number is at least 3 in all known examples. Principal lattices and 3-pencil lattices have exactly three lines with $n+1$ points of X. In other words, Conjecture 1.6 means that the default d of a set X satisfies $d \leq n+1$. We even conjecture that it is less than or equal to $n-1$. In the rest of the paper we completely describe sets with default 0,1 and 2.

Proposition 2.3. *Let X be a set satisfying the GC_n condition. Then the following properties are equivalent:*

(i) *X is a natural lattice.*
(ii) *X is a 0-lattice.*
(iii) *The lines associated with each $x \in X$ are the set of all lines of Γ_X not containing the point x.*
(iv) $|\Gamma_X| = n + 2$.

Proof: (i) $\Longrightarrow$ (iv) and (iii) follows from Proposition 2.1 (iii,v). (iii) $\Longrightarrow$ (iv): Let $n+2+k$ be the number of lines of Γ_X. By Proposition 2.1 (iii), $k \geq 0$. From (iii) we see that for each $x \in X$, there exist exactly n lines in Γ_X not vanishing at x and $k+2$ vanishing on it. Taking into account the intersections of the lines in Γ_X, we obtain that $\binom{k+2}{2}\binom{n+2}{2} \leq \binom{n+2+k}{2}$ which means that $k \leq 0$, and so, $k = 0$. (iv) $\Longrightarrow$ (ii): Let $\Gamma_X = \{K_1, \ldots, K_{n+2}\}$ and denote $r_i := |K_i \cap X|$. By Proposition 2.1 (iv), $r_i \leq n+1$, for all i and by Proposition 2.1 (ii), $r_1 + \cdots + r_{n+2} \geq 2|X| = (n+2)(n+1)$. So $r_i = n+1$ for all i. (ii) $\Longrightarrow$ (i): By Proposition 2.1 (vi)–(vii), the $n+2$ lines with $n+1$ points are in general position and X is formed by the $\binom{n+2}{2}$ intersection points. □

Now we describe all 1-lattices.

Proposition 2.4. *A set X given by (2.1) with $n > 1$ is a 1-lattice if and only if the following properties simultaneously hold:*

(i) *There exist lines $K_1, \ldots, K_{n+1}$ in general position such that (2.2) holds, that is,*

$$x_{ij} = K_i \cap K_j,\ i < j \in \{1, \ldots, n+1\}; \quad x_{i,n+2} \in K_i,\ i < n+2.$$

(ii) *Not all points $x_{i,n+2}$, $i = 1, \ldots, n+1$, lie on the same line.*

Proof: Assume that a set (2.1) satisfies (2.2). Clearly, each K_i has $n+1$ points. Let K_{ij} be the line containing $x_{i,n+2}$ and $x_{j,n+2}$. Then we have that the set of lines associated with x_{ij}, $i < j < n+2$, consists of the line K_{ij} and all the lines K_r, $r \neq i, j$. The set of lines associated with $x_{i,n+2}$ is K_r, $r \neq i$. Therefore the GC_n condition holds, and Γ_X consists of the lines K_i and K_{ij}. Since not all points $x_{i,n+2}$, $i = 1, \ldots, n+1$ lie on the same line, no line K_{ij} can contain $n+1$ points and then X is a 1-lattice. For the converse, if $K_1, \ldots, K_{n+1}$ are the lines with $n+1$ points, then the set X must contain all intersections $K_i \cap K_j$, and each line K_i must have an additional point. So, X satisfies (2.1)-(2.2). Since $K_1, \ldots, K_{n+1}$ must be the lines with $n+1$ points, not all points $x_{i,n+2}$ may lie on the same line. □

Let us observe that 1-lattices with $n = 1$ do not exist because a set satisfying GC_1 is trivially a natural lattice. In Figure 3, we show a lattice with default 1, and satisfying GC_3.

Now we provide a complete description of all 2-lattices.

Proposition 2.5. *A set X given by (2.1) with $n > 2$ is a 2-lattice if and only if the following properties simultaneously hold:*

(i) *There exist lines $K_1, \ldots, K_n$ in general position such that (2.2) holds.*

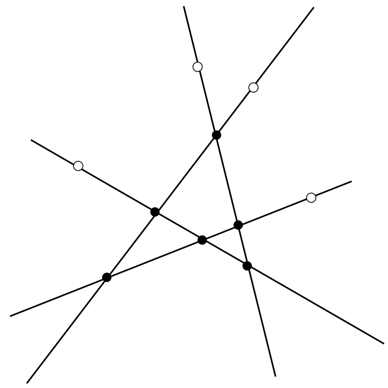

Fig. 3. A lattice with default 1.

(ii) *There exist lines* L_1, L_2, L_3 *such that* $x_{n+1,n+2} = L_1 \cap L_2 \cap L_3$ *and* $\{x_{i,n+1}, x_{i,n+2}\} \subseteq K_i \cap (L_1 \cup L_2 \cup L_3)$ *for all* $i < n+1$.
(iii) *No line* L_r, $r = 1, 2, 3$, *contains* $n+1$ *points of* X.

Furthermore, each of the lines L_r *must have at least 3 points of* X.
Proof: Let us show first that a set (2.1) with (i), (ii), (iii) satisfies the GC_n condition. For $x_{n+1,n+2}$ the associated lines are $K_1, \dots, K_n$. From (ii), there exist indices $j, k \in \{1, 2, 3\}$ such that

$$x_{i,n+1} = K_i \cap L_j, \quad x_{i,n+2} = K_i \cap L_k.$$

Then the set of lines associated with $x_{i,n+1}$ consists of L_k and K_r, $r \neq i$. Analogously, the set of lines associated with $x_{i,n+2}$ consists of L_j and K_r, $r \neq i$. Finally, given $i < j \leq n$, the three lines L_1, L_2, L_3 contain the four points $x_{i,n+1}, x_{i,n+2}, x_{j,n+1}, x_{j,n+2}$, and therefore there exists k such that L_k contains two of them. Let H be the line connecting the other two. So K_r, $r \neq i, j$, L_k and H are the lines associated with x_{ij}. We have thus checked the GC_n condition. On the other hand, X cannot be a natural or a 1-lattice. Indeed, if there exists another line with $n+1$ points, it would contain the point $x_{n+1,n+2}$ and one point of $(L_1 \cup L_2 \cup L_3) \cap K_i$ for each i. Since $n > 2$, this line must be one of the L_1, L_2 or L_3, contradicting (iii).

Conversely, let X be a 2-lattice. There exist lines $K_1, \dots, K_n$ with $n+1$ points and X contains all the points $x_{ij} = K_i \cap K_j$, $i < j \leq n$. Each line K_i contains two additional points $x_{i,n+1}, x_{i,n+2}$, $i \leq n$. The set X must still have a point $x_{n+1,n+2}$ not belonging to any of the lines $K_1, \dots, K_n$. So, (2.2) holds for $m = n$. By Proposition 2.1 (v), all lines K_r, $r \neq i, j$, are associated with x_{ij} ,$i < j \leq n$. The set of lines associated with x_{ij} must contain two more lines with the five points $\{x_{i,n+1}, x_{i,n+2}, x_{j,n+1}, x_{j,n+2}, x_{n+1,n+2}\}$. Therefore three of these points lie on the same line, say H_{ij}: one is $x_{n+1,n+2}$, the second one is in $\{x_{i,n+1}, x_{i,n+2}\}$ and the third one in $\{x_{j,n+1}, x_{j,n+2}\}$. Since the default is 2, H_{ij} cannot contain $n+1$ points of X. So, for each i, j, there exists $k \neq i, j$ such that the lines H_{ij}, H_{ik}, H_{jk} are different, they are concurrent at $x_{n+1,n+2}$ and $x_{r,n+1}, x_{r,n+2} \in (H_{ij} \cup H_{ik} \cup H_{jk}) \cap K_r$, $r = i, j, k$. Let us define $L_1 := H_{ij}, L_2 := H_{ik}, L_3 := H_{jk}$. Now, for $r \neq i, j, k$ one has $H_{ir} \in \{L_1, L_2\}$,

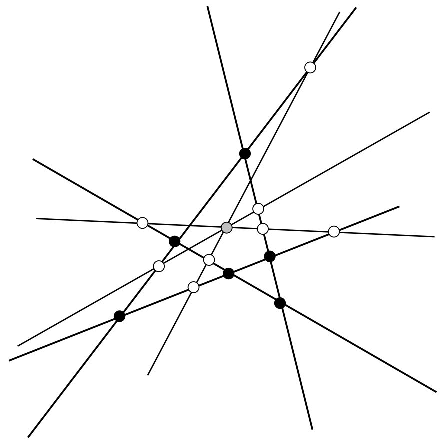

Fig. 4. A lattice with default 2.

$H_{jr} \in \{L_1, L_3\}$, $H_{kr} \in \{L_2, L_3\}$. Two of these lines are different. That means that $x_{r,n+1}, x_{r,n+2}$ lie in two of the three lines L_1, L_2, L_3, and we see that (ii) holds. Since Γ_X has only n lines with $n+1$ points, (iii) follows immediately. Finally, we have also shown that L_1 contains at least three points: one is $x_{n+1,n+2}$, a second one in K_i and a third one in K_j. On the other hand L_1 does not intersect K_k, and so it has at most $n-1$ points of $X \setminus \{x_{n+1,n+2}\}$. Analogously L_2, L_3 also contain at least three and at most n points of X. □

For $n = 2$ there are no GC_2 set X with default 2. Indeed, from Proposition 2.1, it is very easy to deduce that there exist at least 3 lines with 3 points. Figure 4 shows a lattice with default 2, satisfying GC_4.

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