On Chung and Yao's geometric characterization for bivariate polynomial interpolation

J. M. Carnicer and M. Gasca

Abstract. Interpolation problems on sets of points in the plane satisfying Chung and Yao's geometric characterization give rise to Lagrange interpolation formulae in the space of polynomials of degree not greater than n. A conjecture on these sets states that *there exists a line containing* $n+1$ *nodes*. It has only been proved for degree ≤ 4. In this paper, we analyze some consequences of assuming the validity of the conjecture up to a certain degree, which lead to a better knowledge of Chung and Yao's geometric characterization. One of the main results of this paper is the equivalence of the above mentioned conjecture with the existence of at least 3 lines containing $n+1$ nodes.

§1. Introduction

In multivariate polynomial interpolation the existence and uniqueness of solution of a Lagrange problem always depend on the geometrical distribution of the interpolation points [7]. Identification of simple distributions of points giving rise to unisolvent problems in a given polynomial space is a subject of permanent interest.

For the space $\Pi_n(\mathbb{R}^2)$ of bivariate polynomials of degree not greater than n, a well-known unisolvent distribution is that of $\binom{n+2}{2}$ points satisfying the geometric characterization (GC) introduced by Chung and Yao [5]. However, a description of all configurations of $\binom{n+2}{2}$ points satisfying this characterization is yet to be done. It has been conjectured [6] that each of these sets has $n+1$ collinear points, but this has been only proved for $n \leq 4$ [1,3]. If the conjecture is true, GC sets would be a particular case of another well-known distribution of points which gives rise to unisolvent interpolation problems in $\Pi_n(\mathbb{R}^2)$ [6]: *there exist lines* $L_0, \ldots, L_n$ *such that* $L_i \setminus (L_0 \cup \cdots \cup L_{i-1})$ *contains exactly* $n+1-i$ *nodes.*

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In Section 2 we remark the special role of lines containing $n+1$ points of a GC set X of $\binom{n+2}{2}$ points. Then in Section 3 some properties of these lines with a maximal number of nodes are derived. Finally, in Section 4, we prove that if for all $n \le \nu$ each GC set X with $\binom{n+2}{2}$ points has $n+1$ collinear points, then for each of those sets X there exist 3 lines containing $n+1$ points of X. Some new properties which can lead to a better knowledge of the geometric characterization are also obtained.

§2. A Conjecture on the Geometric Characterization

In the bivariate case, the GC condition can be stated as follows.

Definition 2.1. Let $X \subseteq \mathbb{R}^2$, $|X| = (n+2)(n+1)/2$. The set X satisfies the geometric characterization GC_n if for each $x \in X$, there exist lines $L_1^x, \dots, L_n^x$ such that

$$X \setminus (L_1^x \cup \cdots \cup L_n^x) = \{x\}.$$

We say that $\Gamma_{x,X} := \{L_1^x, \dots, L_n^x\}$ is the set of lines used by the node x. The set of lines used by some node is $\Gamma_X := \bigcup_{x \in X} \Gamma_{x,X}$. Observe that $\Gamma_{x,X}$ is uniquely determined due to the unisolvence of the problem.

The geometric characterization is neither constructive nor descriptive. In fact, there exist many different GC sets with different combinatorial features, for instance, natural and principal lattices (see [5]). In order to describe all the sets of points satisfying the GC_n condition, it is essential to derive further collinearity properties of the nodes. A key for reducing the problem to simpler considerations is the existence of lines containing $n+1$ nodes. These lines contain the maximal number of nodes because, by Proposition 2.1 (iv) of [2], no line contains more than $n+1$ nodes.

By Proposition 2.1 (viii) of [2] the number of lines containing $n+1$ nodes cannot be greater than $n+2$. In [2], the *defect* (called *default* in that paper) of a GC_n set was introduced in order to classify the GC configurations.

Definition 2.2. Let $X \subseteq \mathbb{R}^2$ be a set satisfying the GC_n condition. We say that X has defect d if the number of lines containing $n+1$ nodes is $n+2-d$, that is,

$$d = n+2 - |\{K \mid K \text{ is a line}, |K \cap X| = n+1\}|.$$

Observe that the defect of a GC_n set satisfies $0 \le d \le n+2$. We shall say that X is a $\mathrm{GC}_{n,d}$ set to indicate that X is a set of nodes satisfying the geometric characterization for degree n and defect d.

Proposition 2.3 shows how some properties of a GC_n set of nodes with $n+1$ collinear nodes can be related to the properties of GC configurations of strictly lower degree. Part of these results were obtained in Proposition 3 (a) of [3] but we need to review them in order to offer a complete proof.

Proposition 2.3. *Let X be a $\mathrm{GC}_{n,d}$ set and L be a line containing $n+1$ nodes (hence, $d \le n+1$). Then $Y := X \setminus L$ is a $\mathrm{GC}_{n-1,d'}$ set with $d' \le d$ and $\Gamma_{y,Y} = \Gamma_{y,X} \setminus \{L\}$, for each $y \in Y$.*

Proof: Let $Y := X \setminus L$. Since X is $\mathrm{GC}_{n,d}$, then we have for any $y \in Y$ a set of n lines $\Gamma_{y,X}$ containing all points of X except y. By Proposition 2.1 (v) of [2], $L \in \Gamma_{y,X}$ and we can write $\Gamma_{y,X} = \{L_1, \dots, L_{n-1}, L\}$. Since $(L \cup \bigcup_{i=1}^{n-1} L_i) \cap X = X \setminus \{y\}$, we deduce that $\bigcup_{i=1}^{n-1} L_i \cap Y = Y \setminus \{y\}$. So, Y satisfies GC_{n-1} and $\Gamma_{y,Y} = \Gamma_{y,X} \setminus \{L\}$ for each $y \in Y$.

Let $\mathcal{K}$ be the set of lines containing $n+1$ nodes of X and $\mathcal{K}'$ be the set of lines containing n nodes of Y

$$\mathcal{K} := \{K \in \Gamma_X \colon |K \cap X| = n+1\}, \quad \mathcal{K}' := \{K \in \Gamma_Y \colon |K \cap Y| = n\}.$$

Since X is $\mathrm{GC}_{n,d}$ and Y is $\mathrm{GC}_{n-1,d'}$, we have $|\mathcal{K}| = n+2-d$, $|\mathcal{K}'| = n+1-d'$. By Proposition 2.1 (vi) of [2], each of the lines in $\mathcal{K} \setminus \{L\}$ intersects L at a node. Then each line of the set $\mathcal{K} \setminus \{L\}$ contains n nodes of $Y = X \setminus L$. Therefore, $\mathcal{K} \setminus \{L\} \subseteq \mathcal{K}'$ and then

$$n+1-d' = |\mathcal{K}'| \ge |\mathcal{K} \setminus \{L\}| = |\mathcal{K}| - 1 = n+1-d.$$

So, we have $d' \le d$. $\square$

In all particular instances of GC_n sets described until now, the existence of lines containing $n+1$ nodes has always been confirmed. In [6], the following conjecture was launched.

Conjecture 2.4. *Given a GC_n set, there exists at least one line containing $n+1$ nodes.*

The conjecture has only been proved for all GC_n sets with $n = 1, 2, 3, 4$ (see [1,3]). As shown in Proposition 2.3, the verification of the conjecture simplifies the problem of describing all possible GC sets. In fact, in [4] a complete classification of all GC configurations up to degree 4 has been obtained. However, for degrees higher than 4, the conjecture remains unsolved. In [6], it is also mentioned that if the conjecture held for arbitrary degree then all GC sets would consist of a line L_1 with $n+1$ nodes, another line L_2 containing n nodes not in L_1, a third line L_3 containing $n-1$ nodes not in $L_1 \cup L_2$, and so on. The interpolation problem on these sets of nodes is unisolvent in $\Pi_n(\mathbb{R}^2)$ and the solution can be expressed by a simple Newton formula (see [6]).

§3. Some Properties of the Lines in a GC Configuration

Let us start with an auxiliary result

Lemma 3.1. *Let L_1, L_2, L_3 be three lines in the plane intersecting k lines $M_1, \ldots, M_k$ in $3k$ distinct points*

$$x_{ij} := L_i \cap M_j, \quad i \in \{1,2,3\}, \quad j \in \{1,2,\ldots,k\},$$
$$X := \{x_{ij} \mid i \in \{1,2,3\}, \quad j \in \{1,2,\ldots,k\}\}, \quad |X| = 3k.$$

Then each polynomial of total degree less than or equal to k vanishing on $3k-1$ points of X must also vanish at the remaining point of X.

Proof: Without loss of generality we may assume that $p \in \Pi_k(\mathbb{R}^2)$ vanishes on $X \setminus \{x_{3k}\}$. Let M_{k+1} be a line such that $x_{1,k+1} := L_1 \cap M_{k+1}$ is a point in $L_1 \setminus X$. For $h = 4, \ldots, k+1$, let L_h be a line intersecting $M_1, \ldots, M_{k+2-h}$ at $k+2-h$ distinct points, different from x_{ij}, $1 \le i \le h-1$, $1 \le j \le k+2-i$ and define $x_{hj} := L_h \cap M_j$, $j = 1, \ldots, k-2$. Then

$$r_i = L_{i+1}, \quad r_{ij} = M_{j+1}, \quad j = 0, \ldots, k-i, \qquad i = 0, \ldots, k,$$

defines a system of order $k+1$ in the sense of [6], leading to a set

$$\tilde{X} := \{x_{ij} \mid 2 \le i+j \le k+2\}$$

unisolvent in $\Pi_k(\mathbb{R}^2)$ by [6]. Moreover $\tilde{X} \supseteq X \setminus \{x_{3k}\}$. Let ℓ_{ij} be the Lagrange polynomial associated to the point x_{ij}, that is, $\ell_{ij}(x_{i'j'}) = \delta_{ii'}\delta_{jj'}$. We may use the Lagrange formula to write

$$p(x) = \sum_{2 \le i+j \le k+2} p(x_{ij})\ell_{ij}(x),$$

for each $p \in \Pi_n(\mathbb{R}^2)$. If p is a polynomial vanishing on $X \setminus \{x_{3k}\}$, then the above formula reduces to

$$p(x) = p(x_{1,k+1})\ell_{1,k+1}(x) + \sum_{i \ge 4, i+j \le k+2} p(x_{ij})\ell_{ij}(x). \tag{3.1}$$

Clearly $\ell_{1,k+1}(x) = \prod_{i=1}^{k}(M_i(x)/M_i(x_{1,k+1}))$ and, by Bézout's Theorem, $L_1(x)L_2(x)L_3(x)$ divides ℓ_{ij}, $i \ge 4$. Therefore each of the terms in (3.1) vanishes not only on $X \setminus \{x_{3k}\}$ but also in x_{3k}. $\square$

Lemma 3.2. *Let X be a GC_n set and let L be a line with $n+1$ nodes. Let us assume that M_1, M_2 are lines with n nodes of $X \setminus L$ such that $L \cap M_1 \cap X = L \cap M_2 \cap X = \emptyset$, that is, M_1 and M_2 do not contain nodes of $L \cap X$. For each $x \in L \cap X$, let $\alpha(x)$ be the node in $L \cap X$ which is the intersection of L with the line in $\Gamma_{x,X}$ containing $M_1 \cap M_2 \in X$. Then $\alpha : L \cap X \to L \cap X$ is an involution with no fixed points, that is*

$$\alpha(x) \ne x, \quad \alpha(\alpha(x)) = x, \qquad \forall x \in L \cap X. \tag{3.2}$$

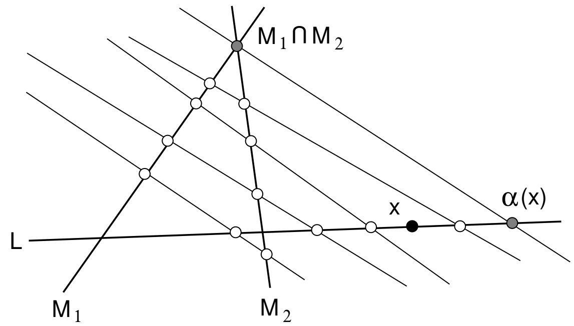

Fig. 2. Definition of $\alpha(x)$.

Proof: Let us see first that $M_1 \cap M_2 \in X$. By Proposition 2.3, $X \setminus L$ satisfies GC_{n-1} and then M_1 and M_2 are lines containing n nodes of $X \setminus L$. By Proposition 2.1 (vi) of [2], $M_1 \cap M_2$ is a point in $X \setminus L$.

Let x be a point in $L \cap X$. Each line of $\Gamma_{x,X}$ must contain a point of $L \cap X \setminus \{x\}$. Since $L \cap M_1 \cap X = L \cap M_2 \cap X = \emptyset$, then $M_1, M_2 \notin \Gamma_{x,X}$. Taking into account that $L \setminus \{x\}, M_1$ and M_2 contain n nodes in $X \setminus \{x\}$ and that they are lines not in $\Gamma_{x,X}$, we deduce that each line in $\Gamma_{x,X}$ passes through a node in $L \setminus \{x\}$, a node in M_1 and a node in M_2. One of the lines in $\Gamma_{x,X}$ passes through $M_1 \cap M_2$ and intersects L at a node $\alpha(x) \neq x$. Let $N_1, \ldots, N_{n-1}$ be the lines in $\Gamma_{x,X}$ not passing through $M_1 \cap M_2$ and let N_n be the line joining $M_1 \cap M_2$ and $\alpha(x)$. The lines $N_1, \ldots, N_{n-1}$ intersect M_1, M_2, L in $3n-3$ nodes

$$(N_1 \cup \cdots \cup N_{n-1}) \cap (M_1 \cup M_2 \cup L) = \\ (M_1 \cup M_2 \cup L) \cap X \setminus \{M_1 \cap M_2, x, \alpha(x)\}. \qquad (3.3)$$

Now, let $K_1, \ldots, K_{n-1}$ be the lines in $\Gamma_{\alpha(x),X}$ not passing through $M_1 \cap M_2$ and let K_n be the line joining $M_1 \cap M_2$ and $\alpha(\alpha(x))$. The polynomial $K_1(x) \cdots K_{n-1}(x)$ vanishes on the $3n-3$ nodes of the set

$$(K_1 \cup \cdots \cup K_{n-1}) \cap (M_1 \cup M_2 \cup L) = \\ (M_1 \cup M_2 \cup L) \cap X \setminus \{M_1 \cap M_2, \alpha(x), \alpha(\alpha(x))\}.$$

Among these $3n-3$ nodes, at least $3n-4$ belong to the set (3.3). If $\alpha(\alpha(x)) \neq x$, then $K_1(x) \cdots K_{n-1}(x)$ vanishes on $3n-4$ points in (3.3) but not on the remaining one, contradicting Lemma 3.1. Therefore $\alpha(\alpha(x)) = x$. □

Lemma 3.3. *Let X be a* GC_n *set, let L be a line with $n+1$ nodes of the set X and M_1, M_2 two lines containing at least n nodes of $X \setminus L$ each. If n is even, then M_1 or M_2 contain $n+1$ nodes of X.*

Proof: By assumption, M_1 and M_2 do not contain nodes of $L \cap X$. By Lemma 3.2, the mapping $\alpha : L \cap X \to L \cap X$ is an involution. If

$|L \cap X| = n+1$ is odd, α must have a fixed point, but $\alpha(x) = x$ contradicts the definition of α. Then at least one of the lines M_i contains a node of $L \cap X$, and so, that line M_i contains $n+1$ nodes. □

Theorem 3.4. *Let X be a GC_n set and let L be a line with $n+1$ nodes of the set X. Let M_1, M_2, M_3 be lines containing at least n nodes of the set $X \setminus L$ each. Then, at least two of the lines M_1, M_2, M_3 intersect L at a node and contain $n+1$ nodes.*

Proof: If n is even, we may apply Lemma 3.3 to the pair of lines M_1, M_2, then to M_2, M_3 and finally to M_1, M_3.

Let us now analyze the case where n is odd and assume that M_i do not contain points of $L \cap X$, $i = 1, 2, 3$. According to Lemma 3.2, $M_1 \cap M_2$ is a node. For each $x \in L \cap X$, let N_x be the line in $\Gamma_{x,X}$ through $M_1 \cap M_2$ and $\alpha(x) := N_x \cap L$, $\beta(x) := N_x \cap M_3$. Since $L \notin \Gamma_{x,X}$, each line of $\Gamma_{x,X}$ contains one node of $L \cap X \setminus \{x\}$ and then $M_3 \notin \Gamma_{x,X}$ because $M_3 \cap L \cap X = \emptyset$. Taking into account that L and M_3 contain n points of $X \setminus \{x\}$ and that $L, M_3 \notin \Gamma_{x,X}$, we see that $\alpha(x), \beta(x) \in X$. By Lemma 3.2, α is an involution and so, it is a bijection from $L \cap X$ onto $L \cap X$. This means that different points $x \in L \cap X$ correspond different lines N_x. Since $M_1 \cap M_2, \alpha(x), \beta(x)$ are collinear we see that $\beta : L \cap X \to M_3 \cap X$ is injective. Therefore $|M_3 \cap X| \geq |L \cap X| = n+1$, contradicting the fact that M_1, M_2, M_3 do not contain points of $L \cap X$.

Therefore, at least one of the lines M_1, M_2, M_3 contain $n+1$ nodes. Without loss of generality we may assume that it is M_3. Then $X \setminus M_3$ is a GC_{n-1} set with $n-1$ even and M_1, M_2 contain $n-1$ nodes of $X \setminus M_3$. By Lemma 3.3, M_i intersects L at a node for some $i \in \{1, 2\}$, that is, M_i has n nodes of the set $X \setminus M_3$. Since $M_i \cap M_3 \in X$, we conclude that M_i contains $n+1$ nodes of X. □

Corollary 3.5. *Let X be a GC_n set and let L be a line with $n+1$ nodes of the set X. Let $M_1, \ldots, M_k$, $k \geq 3$, be lines containing at least n nodes of the set $X \setminus L$ each. Then, at most one of the lines M_i does not contain a node of $L \cap X$ and all the other lines M_j intersect L at a node and contain $n+1$ nodes.*

Proof: Consider all possible sets of 3 lines and apply Theorem 3.4. □

Remark 3.6. In Proposition 2.3, we have proved that if X is a $\mathrm{GC}_{n,d}$ set, $d < n+1$, and L is a line with $|L \cap X| = n+1$, then $Y := X \setminus L$ is a $\mathrm{GC}_{n-1,d'}$ set with $d' \leq d$. If there exist more than two lines containing n nodes of Y, then, by Corollary 3.5, we deduce that at most one line containing n nodes of Y does not intersect L at a node and all the rest contain $n+1$ nodes of X. This can be stated in the form: *if $d' \leq n-2$, then $d \leq d'+1$.*

§4. Some Consequences of the Verification of the Conjecture

In this section, we derive some properties of GC_n sets with the assumption that Conjecture 2.4 holds for degrees $1, 2, \ldots, \nu$ using some inductive arguments. First we state a direct consequence of Theorem 3.4.

Theorem 4.1. *Assume that Conjecture 2.4 holds for all degrees up to* ν. *Then for any given* GC_n *set,* $n \leq \nu$, *there exist at least 3 lines containing* $n+1$ *nodes.*

Proof: We use induction on n. The case $n = 1$ is trivial. Assume that any GC_{n-1} set contains 3 lines with n nodes and that $n \leq \nu$. Let X be a GC_n set. Since $n \leq \nu$, Conjecture 2.4 holds and there exists a line L containing $n+1$ nodes of X. Moreover $X \setminus L$ is GC_{n-1} and, by hypothesis of induction, there exist 3 lines M_1, M_2, M_3 containing n nodes of $X \setminus L$. By Theorem 3.4, two lines among M_1, M_2, M_3 contain $n+1$ nodes. □

It has been shown in [1, 3] that Conjecture 2.4 holds for degrees $n \leq 4$, that is the hypothesis of Theorem 4.1 holds for $\nu = 4$. So, as a consequence of Theorem 4.1, we deduce that, for any GC_n set $n \leq 4$, there exist 3 lines containing $n+1$ nodes. This fact was derived independently in [4].

Now, we use Theorem 4.1 to know how many nodes use a given line in a GC_n set. If L is a line of the plane, then we denote by

$$X_L := \{x \in X | L \in \Gamma_{x,X}\} \tag{3.1}$$

the set of nodes using the line L. Observe that $L \in \Gamma_X$ if and only if X_L is nonempty.

By Proposition 2.1 (v) of [2], if a line $M \in \Gamma_X$ contains exactly $n+1$ nodes, then $|X_L| = \binom{n+1}{2}$. The following proposition states that if a line $M \in \Gamma_X$ contains exactly 2 nodes $|M \cap X| = 2$, then $|X_M| = 1$.

Proposition 4.2. *Let* X *be a* GC_n *set and* $M \in \Gamma_X$. *If* $|X_M| \geq 2$, *then the line* M *contains at least 3 nodes,* $|M \cap X| \geq 3$.

Proof: By hypothesis, at least two points x_1, x_2 use the line M, that is $x_1, x_2 \in X_M$. This means that $M \in \Gamma_{x_i,X}$ for $i \in \{1, 2\}$. By Proposition 2.1 (i) of [2], $|M \cap X| \geq 2$. Let us assume that the line M contains exactly two nodes, $M \cap X = \{y_1, y_2\}$, and denote by N_{ij} the line joining x_i and y_j, $i, j \in \{1, 2\}$. The union U of the lines in $\Gamma_{x_i,X} \setminus \{M\}$ contains all nodes of $X \setminus \{x_i, y_1, y_2\}$. By Proposition 2.1 (iv) of [2], U cannot contain more than $\binom{n+2}{2} - 3$ nodes and so $U \cap X = X \setminus \{x_i, y_1, y_2\}$. In particular, no line among $N_{11}, N_{12}, N_{21}, N_{22}$ is in the set of lines $\Gamma_{x_i,X} \setminus \{M\}$. On the other hand, the set of n lines $\Gamma_{x_i,X} \setminus \{M\} \cup \{N_{i2}\}$ contains all nodes in $X \setminus \{y_1\}$ and so $\Gamma_{y_1,X} = \Gamma_{x_i,X} \setminus \{M\} \cup \{N_{i2}\}$, $i \in \{1, 2\}$. Analogously, $\Gamma_{y_2,X} = \Gamma_{x_i,X} \setminus \{M\} \cup \{N_{i1}\}$, $i \in \{1, 2\}$. Hence

$$\Gamma_{x_1,X} \setminus \{M\} \cup \{N_{12}\} = \Gamma_{y_1,X} = \Gamma_{x_2,X} \setminus \{M\} \cup \{N_{22}\},$$
$$\Gamma_{x_1,X} \setminus \{M\} \cup \{N_{11}\} = \Gamma_{y_2,X} = \Gamma_{x_2,X} \setminus \{M\} \cup \{N_{21}\}.$$

So we have seen that $N_{1j} = N_{2j}$, that is the points x_1, x_2, y_j are collinear, $j \in \{1, 2\}$. Then $M = N_1 = N_2$ contains the four nodes x_1, x_2, y_1, y_2, which is a contradiction. $\square$

For typical GC sets such as natural lattices or principal lattices, a line $M \in \Gamma_X$ containing k nodes is used by exactly $\binom{k}{2}$ nodes. However this is not true for a general GC set, as shown in the following example.

Example 4.3. *Let X be the set of points in the plane:*

$$X = \{(-2,0), (-1,1), (0,-2), (0,0), (0,1), (0,2), (0,4), (2/3, 8/3),\\ (1,0), (1,3), (3/2,1), (2,0), (2,2), (3,1), (4,0)\}.$$

It is a GC_4 *set and the lines in* Γ_X *are depicted in Figure 4. The line* $y = 4 - 2x$ *contains four nodes. However only 3 nodes* $(-2,0)$, $(-1,1)$, $(1,0)$ *use this line. Observe that all nodes using* $y = 4 - 2x$ *also use the lines* $x = 0$ *and* $y = 4 - x$ *and these three lines pass through the node* $(0,4)$.

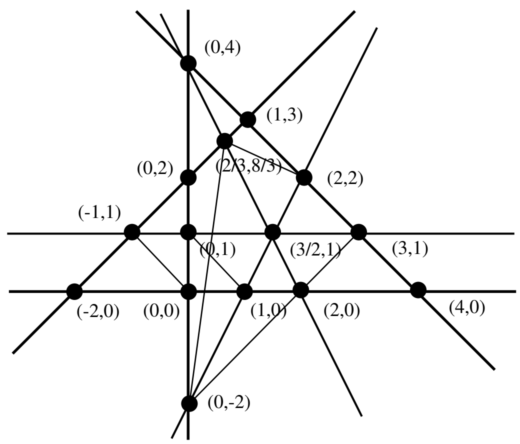

Fig. 4. Only three nodes use a line containing four nodes.

The following lemma relates the set X_M of nodes in a GC_n set X using a given line M with similar sets Y_M obtained from GC_{n-1} subsets Y of X.

Lemma 4.4. *Let X be a* GC_n *set, L be a line containing $n+1$ nodes and $Y := X \setminus L$, which is a* GC_{n-1} *set.*

(i) *If $\Gamma_{x,X}$ does not contain three concurrent lines for each $x \in X$, then $\Gamma_{y,Y}$ does not contain three concurrent lines for each $y \in Y$.*

(ii) *For any line $M \neq L$, $Y_M = X_M \setminus L$. If L does not intersect M at a node, then $Y_M = X_M$.*

Proof: By Proposition 2.3, Y is a GC_{n-1} set and the lines associated to y are $\Gamma_{y,Y} = \Gamma_{y,X} \setminus \{L\}$. Then (i) trivially follows. We also deduce that

$Y_M = X_M \cap Y = X_M \setminus L$. If $L \cap M \cap X = \emptyset$, then no node lying on the line L uses the line M, that is, $X_M \cap L = \emptyset$ and so, $X_M = Y_M$. $\square$

Theorem 4.5. *Assume that Conjecture 2.4 holds for all degrees up to ν. Let X be a GC_n set, $n \le \nu$ and $M \in \Gamma_X$ be a line containing $k := |M \cap X|$ nodes. Then*
(i) *X_M contains at most $k-1$ collinear nodes,*
(ii) $|X_M| \le \binom{k}{2}$.
(iii) *If for each $x \in X$, $\Gamma_{x,X}$ does not contain more than two lines intersecting at the same node, then we have $|X_M| = \binom{k}{2}$.*

Proof: We shall use induction on the degree n. The result is trivial for GC_1 sets. Assume now that X is a GC_n configuration, $2 \le n \le \nu$ and that (i), (ii) and (iii) have been proved for all GC configurations of degree less than n.

If there exists $L \in \Gamma_X$, $|L \cap X| = n+1$ with $L \cap M \cap X = \emptyset$, then, by Lemma 4.4 (ii), $Y := X \setminus L$ is a GC_{n-1} set and $Y_M = X_M$. By Lemma 4.4 (i), the hypotesis of (iii) is inherited by Y. Taking into account that $|M \cap Y| = k$, (i), (ii) and (iii) follow from the induction hypothesis.

If there exist $L_1, L_2 \in \Gamma_X$, $|L_i \cap X| = n+1$, $L_i \ne M$, $i = 1, 2$, with $L_1 \cap L_2 \cap M \cap X \ne \emptyset$, then $X_M \cap L_1 = \emptyset$. By Lemma 4.4 (ii), $Y_1 := X \setminus L_1$ is a GC_{n-1} set and $(Y_1)_M = X_M \setminus L_1 = X_M$. So, $M \in \Gamma_{Y_1}$, $|M \cap Y_1| = k-1$ and, by the induction hypothesis, $X_M = (Y_1)_M$ contains at most $k-2$ collinear nodes and

$$|X_M| = |(Y_1)_M| \le \binom{k-1}{2} < \binom{k}{2}.$$

Hence (i) and (ii) hold. Observe that, in this case, equality does not hold. In fact, the hypothesis of (iii) fails, because each point of X_M uses three concurrent lines M, L_1, L_2.

It remains to deal with the case where M intersects each line containing $n+1$ nodes at distinct nodes. If $k = n+1$, then we deduce from Proposition 2.1 (v) of [2] that $X_M = X \setminus M$, which is a GC_{n-1} set and (i), (ii), (iii) follow. Otherwise, by Theorem 4.1, there exist at least three lines L_1, L_2, L_3 containing each $n+1$ nodes. Since M intersects each line containing $n+1$ nodes at different nodes we have $L_i \cap M \cap X \ne \emptyset$, $i = 1, 2, 3$.

Let us apply Lemma 4.4, to the GC_{n-1} sets $Y_i := X \setminus L_i$, $i = 1, 2, 3$, the GC_{n-2} sets $Y_{ij} := X \setminus (L_i \cup L_j)$, $i < j$ in $\{1, 2, 3\}$, and the GC_{n-3} set $Y_{123} := X \setminus (L_1 \cup L_2 \cup L_3)$.

In order to show (i), let N be any line and let us see that $|X_M \cap N| < |M \cap X|$. By the induction hypothesis $|(Y_1)_M \cap N| < |M \cap X|$. Taking into account that M intersects L_1 at a node, we have $|M \cap Y_1| + 1 = |M \cap X|$

$$|X_M \cap N| \le |(Y_1)_M \cap N| + 1 < |M \cap Y_1| + 1 = |M \cap X|.$$

So (i) holds. By (i), $|X_M \cap L_1| \le k-1$,

$$|X_M| = |X_M \setminus L_1| + |X_M \cap L_1| \le \binom{k-1}{2} + k - 1 = \binom{k}{2}$$

and (ii) follows. Finally, in order to show (iii), we remark that the hypothesis of (iii) is inherited by all the sets Y_i, Y_{ij}, Y_{123} and by the induction hypothesis we have $|X_M \setminus L_i| = |(Y_i)_M| = \binom{k-1}{2}$, $i \in \{1,2,3\}$, $|X_M \setminus (L_i \cup L_j)| = |(Y_{ij})_M| = \binom{k-2}{2}$, $i < j$ in $\{1,2,3\}$, $|X_M \setminus (L_1 \cup L_2 \cup L_3)| = |(Y_{123})_M| = \binom{k-3}{2}$ and $|X_M| = \sum_{i=1}^{3} |X_M \setminus L_i| - \sum_{i=1}^{3} \sum_{j>i} |X_M \setminus (L_i \cup L_j)| + |X_M \setminus (L_1 \cup L_2 \cup L_3)| = 3\binom{k-1}{2} - 3\binom{k-2}{2} + \binom{k-3}{2} = \binom{k}{2}$. So, (iii) has been proved. □

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J. M. Carnicer and M. Gasca
University of Zaragoza
Departamento de Matemática Aplicada
Edificio de Matemáticas, 50009 Zaragoza, Spain
`carnicer@posta.unizar.es` and `gasca@posta.unizar.es`