Asymptotic conditions for degree diminution along prescribed lines

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December 4, 2002

Abstract

One of the problems in bivariate polynomial interpolation is the choice of a space of polynomials suitable for interpolating a given set of data. Depending on the number of data, a usual space is that of polynomials in 2 variables of total degree not greater than k. However, these spaces are not enough to cover many interpolation problems. Here, we are interested in spaces of polynomials of total degree not greater than k whose degree diminishes along some prescribed directions. These spaces arise naturally in some interpolation problems and we describe them in terms of polynomials satisfying some asymptotic interpolation conditions. This provides a general frame to the interpolation problems studied in some of our recent papers.

Keywords: polynomials, multivariate interpolation
AMS Classification: 41A05, 41A63, 65D05

1 Introduction

One of the first problems which arise in multivariate polynomial interpolation is the choice of a subspace of polynomials suitable for interpolating a given set of data. Even assuming that the dimension of the space is equal to the number of interpolation data, the unisolvence of the problem depends on the distribution of the data points (see [9] for more information on these problems). Taking into account the number of data, a natural space is that of polynomials in d variables of total degree not greater than k, $\Pi_k(\mathbf{R}^d)$. Another natural choice in some contexts is the tensor product space of the univariate spaces $\Pi_{k_i}(\mathbf{R}), i = 1, \ldots, d$, which can be denoted by $\Pi_{k_1,\ldots,k_d}(\mathbf{R}^d)$. A generalization

*Partially supported by the Spanish Research Grant CICYT BFM2000–1253.

of the above mentioned spaces is the space of polynomials of total degree not greater than k whose degree diminishes along some prescribed directions. This kind of spaces arises naturally in many cases as shown in [7]. See also [2].

On the other hand, a Newton-like approach to bivariate interpolation problems was introduced by Gasca and Maeztu in [8]. The interpolation space associated with a given problem is described as the space spanned by a Newton basis. A question which arises is how to characterize these spaces in a simple form and this was done in [8] only for a few cases. In [3, 4] we generalized the Newton approach including asymptotic interpolation conditions among the interpolation data.

Asymptotic conditions have a precedent in the work of Bojanov, Hakopian and Sahakian [1], [10]. Each polynomial, when restricted to an affine submanifold of $\mathbf{R}^s$ (a trace of a polynomial), can be interpreted as a polynomial in less than s variables. Improper submanifolds can be seen as intersections of proper affine submanifolds when they tend to be parallel and a limiting polynomial trace can be defined for improper manifolds.

In [3, 4], we showed that an interpolation problem with some vanishing asymptotic interpolation data and some standard Hermite–Birkhoff interpolation data on a polynomial interpolation space V can be seen as an interpolation problem with only the standard interpolation data on the interpolation space formed by the polynomials of V which satisfy the 0 asymptotic conditions. Then the problem can be solved in the most convenient form of both. Several examples were given and we saw that the resulting subspace of V was usually a space of polynomials whose degree decreases along some directions. However in [3, 4] we did not state this property in general. The aim of this paper is to describe which kind of vanishing asymptotic conditions, when satisfied by a polynomial p of a certain degree, make that the (univariate) restriction of p to any line of a given direction has lower degree. These techniques have been used in [5] to generalize the results in [6].

In Section 2 we recall the definition of asymptotic interpolation condition and some notations. In Section 3 we state a necessary and sufficient condition for a polynomial of $\Pi_n(\mathbf{R}^2)$ to become a univariate polynomial of degree less than or equal to $n-k-1$ $(0 \leq k \leq n-1)$ when restricted to any line of $\mathbf{R}^2$ with a given direction v. Observe that the same polynomial can have this property along several directions with prescribed corresponding degrees. Finally, in Section 4 we provide some examples.

2 Asymptotic conditions

Let p be a d-variate polynomial of total degree not greater than n, $\mathbf{u}_0$ a point of $\mathbf{R}^d$ and $\mathbf{v} = (v_1, \ldots, v_d)$ a vector of $\mathbf{R}^d$. As usual we shall denote by $D_{\mathbf{v}} f$

the directional derivative operator

$$D_{\mathbf{v}}f = \sum_{i=1}^{d} v_i \frac{\partial f}{\partial x_i}$$

and then, for a parameter $\lambda \neq 0$, we have

$$\lambda^n p(\mathbf{u}_0 + \lambda^{-1}\mathbf{v}) = \sum_{i=0}^{n} \frac{1}{i!} D_{\mathbf{v}}^i p(\mathbf{u}_0)\lambda^{n-i} = q(\lambda),$$

where $q(\lambda)$ is a univariate polynomial of degree not greater than n.

For $0 \leq k \leq n$, we define

$$\bar{D}^{k,n}_{\mathbf{u}_0,\mathbf{v}} p := D^k q(0) = \frac{k!}{(n-k)!} D_{\mathbf{v}}^{n-k} p(\mathbf{u}_0). \tag{2.1}$$

Equivalently, one has

$$\bar{D}^{k,n}_{\mathbf{u}_0,\mathbf{v}} p = k! \lim_{\mu\to\infty} \frac{p(\mathbf{u}_0 + \mu\mathbf{v}) - \sum_{i=0}^{k-1} \mu^{n-i} \bar{D}^{i,n}_{\mathbf{u}_0,\mathbf{v}} p}{\mu^{n-k}},$$

with

$$\bar{D}^{0,n}_{\mathbf{u}_0,\mathbf{v}} p = \lim_{\mu\to\infty} \frac{p(\mathbf{u}_0 + \mu\mathbf{v})}{\mu^n}.$$

In other words, $\bar{D}^{k,n}_{\mathbf{u}_0,\mathbf{v}} p$ is the result of multiplying by $k!$ the coefficient of μ^{n-k} in the univariate polynomial $p(\mathbf{u}_0 + \mu\mathbf{v})$. Let us observe that $\bar{D}^{0,n}_{\mathbf{u}_0,\mathbf{v}} p$ can be obtained replacing $\mathbf{x}$ by $\mathbf{v}$ in $p_n(\mathbf{x})$, the homogeneous part of degree n of

$$p(\mathbf{x}) = p_0(\mathbf{x}) + p_1(\mathbf{x}) + \cdots + p_n(\mathbf{x}),$$

($p_i(\mathbf{x})$ is a homogeneous polynomial of degree i), that is, $\bar{D}^{0,n}_{\mathbf{u}_0,\mathbf{v}} p = p_n(\mathbf{v})$.

By prescribing the value of $\bar{D}^{k,n}_{\mathbf{u}_0,\mathbf{v}} p$ we say we have an *asymptotic condition* for p. So we can consider the problem of determining a polynomial p in $\Pi_n(\mathbf{R}^d)$ which satisfies some usual interpolation conditions and some additional asymptotic conditions. Here, by usual interpolation conditions we understand values of p and/or some of its directional derivatives at given points and by asymptotic conditions we mean that the values of some operators of the type $\bar{D}^{k,n}_{\mathbf{u}_0,\mathbf{v}}$ applied to *p and/or some of its derivatives* are also given. See [3, 4].

The value of $p(\mathbf{x})$ along the straight line of parametric equation $\mathbf{u}_1 + \mu\mathbf{v}$, $\mu \in \mathbf{R}$, where $\mathbf{u}_1$ is a given point in $\mathbf{R}^d$ and $\mathbf{v}$ is a directional vector of the line, is given by $p(\mathbf{u}_1 + \mu\mathbf{v})$ and, if we write $\mathbf{u}_1 = \mathbf{u}_0 + \nu_1\mathbf{w}$, with $\mathbf{w}$ a directional vector of the line joining $\mathbf{u}_0$ and $\mathbf{u}_1$, we have $p(\mathbf{u}_1 + \mu\mathbf{v}) = p(\mathbf{u}_0 + \nu_1\mathbf{w} + \mu\mathbf{v})$. By using the Taylor expansion, we can write

$$p(\mathbf{u}_0 + \nu\mathbf{w} + \mu\mathbf{v}) = \sum_{i=0}^{n} \frac{1}{i!} D_{\mathbf{v}}^i p(\mathbf{u}_0 + \nu\mathbf{w})\mu^i = \sum_{i+j\leq n} \frac{1}{i!j!} D_{\mathbf{v}}^i D_{\mathbf{w}}^j p(\mathbf{u}_0)\mu^i\nu^j$$

and identifying the coefficients of μ^{n-k} at both sides we have

$$\frac{1}{(n-k)!} D_{\mathbf{v}}^{n-k} p(\mathbf{u}_0 + \nu \mathbf{w}) = \frac{1}{(n-k)!} \sum_{j=0}^{k} \frac{1}{j!} D_{\mathbf{v}}^{n-k} D_{\mathbf{w}}^{j} p(\mathbf{u}_0) \nu^j. \tag{2.2}$$

If we set $\nu = \nu_1$, we get

$$\frac{1}{(n-k)!} D_{\mathbf{v}}^{n-k} p(\mathbf{u}_1) = \frac{1}{(n-k)!} \sum_{j=0}^{k} \frac{1}{j!} D_{\mathbf{v}}^{n-k} D_{\mathbf{w}}^{j} p(\mathbf{u}_0) \nu^j,$$

that is, by (2.1),

$$\bar{D}_{\mathbf{u}_1,\mathbf{v}}^{k,n} = \frac{k!}{(n-k)!} \sum_{j=0}^{k} \frac{1}{j!} D_{\mathbf{v}}^{n-k} D_{\mathbf{w}}^{j} p(\mathbf{u}_0) \nu_1^j. \tag{2.3}$$

Observe that, for $k = 0$, (2.3) can be written, taking into account (2.1),

$$\bar{D}_{\mathbf{u}_0,\mathbf{v}}^{0,n} p = \bar{D}_{\mathbf{u}_1,\mathbf{v}}^{0,n} p. \tag{2.4}$$

In general, $\bar{D}_{\mathbf{u}_0,\mathbf{v}}^{k,n}$ depends on the direction $\mathbf{v}$ and on $\mathbf{u}_0$. However (2.4) means that it does not depend on $\mathbf{u}_0$ when $k = 0$.

Our aim in this paper is to interpret vanishing asymptotic data as conditions which decrease the degree of a polynomial when restricted to a given direction, that is to any line parallel to a given one.

3 Degree diminution along prescribed directions

In this section we are going to see that the degree of a polynomial decreases when it is restricted to any line of a given direction if and only if the polynomial satisfies some set of asymptotic conditions 0. First the asymptotic conditions will be referred to only one point $\mathbf{u}_0$.

Proposition 3.1. *Let $\mathbf{v}$ be a nonzero vector of $\mathbf{R}^d$. A necessary and sufficient condition for a polynomial p of total degree not greater than n to become a univariate polynomial of degree not greater than $n-k-1$, $(0 \le k \le n-1)$, when restricted to any straight line with the direction of $\mathbf{v}$ is that, for some point $\mathbf{u}_0 \in \mathbf{R}^d$ and a set $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{d-1}$ of vectors such that $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{d-1}, \mathbf{v}\}$ is a basis of $\mathbf{R}^d$, one has*

$$\bar{D}_{\mathbf{u}_0,\mathbf{v}}^{t+|\alpha|,n} D_{\mathbf{w}_1}^{\alpha_1} D_{\mathbf{w}_2}^{\alpha_2} \cdots D_{\mathbf{w}_{d-1}}^{\alpha_{d-1}} p = 0, \quad |\alpha| + t \le k, \tag{3.1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_{d-1}) \in \mathbf{N}_0^{d-1}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_{d-1}$.

Proof: According to (2.1), condition (3.1) can be written

$$D_{\mathbf{v}}^{n-t-|\alpha|} D_{\mathbf{w}_1}^{\alpha_1} D_{\mathbf{w}_2}^{\alpha_2} \cdots D_{\mathbf{w}_{d-1}}^{\alpha_{d-1}} p(\mathbf{u}_0) = 0, \quad |\alpha| + t \le k. \tag{3.2}$$

If we write any point of $\mathbf{R}^d$ in the form $\mathbf{u}_0 + \mu\mathbf{v} + \sum_{j=1}^{d-1} \nu_j \mathbf{w}_j$, from Taylor formula we obtain

$$p\Big(\mathbf{u}_0 + \mu\mathbf{v} + \sum_{j=1}^{d-1} \nu_j \mathbf{w}_j\Big) = \sum_{i+|\alpha|\le n} \frac{1}{i!\alpha!} D^i_{\mathbf{v}} D^{\alpha_1}_{\mathbf{w}_1} D^{\alpha_1}_{\mathbf{w}_2} \cdots D^{\alpha_{d-1}}_{\mathbf{w}_{d-1}} p(\mathbf{u}_0)\mu^i\nu^\alpha$$

$$= \sum_{i=0}^{n} \frac{1}{i!}\Bigg(\sum_{|\alpha|\le n-i} \frac{1}{\alpha!} D^i_{\mathbf{v}} D^{\alpha_1}_{\mathbf{w}_1} D^{\alpha_2}_{\mathbf{w}_2} \cdots D^{\alpha_{d-1}}_{\mathbf{w}_{d-1}} p(\mathbf{u}_0)\nu^\alpha \Bigg)\mu^i,$$

with $\nu = (\nu_1, \dots, \nu_{d-1})$, which is a polynomial of degree $\le n-k-1$ in μ if and only if

$$D^i_{\mathbf{v}} D^{\alpha_1}_{\mathbf{w}_1} D^{\alpha_1}_{\mathbf{w}_2} \cdots D^{\alpha_{d-1}}_{\mathbf{w}_{d-1}} p(\mathbf{u}_0) = 0, \quad |\alpha| \le n-i, \quad i = n-k, \dots, n.$$

By denoting $t = n - i - |\alpha|$, we get (3.2). ∎

Our next step is to consider other sets of asymptotic conditions equivalent to (3.1). More precisely, (3.1) is a set of $\binom{k+d}{d}$ conditions, all of them referred to the point $\mathbf{u}_0$. We want to consider sets of conditions referred to points distributed along parallel lines with direction $\mathbf{v}$. Since the bivariate case is the most important and the simplest one, we restrict ourselves to this case. In particular, (3.1) reduces to

$$\bar{D}^{t+r,n}_{\mathbf{u}_0,\mathbf{v}} D^r_{\mathbf{w}} p = 0, \quad r + t \le k, \tag{3.3}$$

where $r \in \mathbf{N}_0$. So, formula (3.2) can be written

$$D^{n-t-r}_{\mathbf{v}} D^r_{\mathbf{w}} p(\mathbf{u}_0) = 0, \quad r + t \le k. \tag{3.4}$$

Now we shall prove the following

Theorem 3.2. *Let $\mathbf{v} \in \mathbf{R}^2$, $\mathbf{v} \ne \mathbf{0}$, and $l_0, l_1, \dots, l_k$ be parallel lines with directional vector $\mathbf{v}$ (coincidences are allowed). Let $\{\mathbf{u}_{rt} \in \mathbf{R}^2 \mid r + t \le k\}$ be a set of $\binom{k+2}{2}$ points not necessarily distinct, distributed along the lines l_r: $\mathbf{u}_{rt} \in l_r$, $t + r \le k$. A necessary and sufficient condition for a polynomial p of total degree $\le n$ to become a univariate polynomial of degree $\le n-k-1$ when restricted to any line of $\mathbf{R}^2$ with direction $\mathbf{v}$ is that*

$$\bar{D}^{t+r,n}_{\mathbf{u}_{rt},\mathbf{v}} D^{h_r}_{\mathbf{w}_{rt}} p = 0, \quad r + t \le k, \tag{3.5}$$

where h_r is the number of lines in the sequence $l_0, \dots, l_{r-1}$ coincident with l_r and $\mathbf{w}_{rt}$ is any nonzero vector with direction different from $\mathbf{v}$.

Proof: According to (2.1), formula (3.5) can be written

$$D^{n-t-r}_{\mathbf{v}} D^{h_r}_{\mathbf{w}_{rt}} p(\mathbf{u}_{rt}) = 0, \quad r + t \le k. \tag{3.6}$$

We are going to prove the equivalence with (3.4) by induction on k. For $k = 0$ both conditions

$$D^n_{\mathbf{v}} p(\mathbf{u}_0) = 0, \qquad D^n_{\mathbf{v}} p(\mathbf{u}_{00}) = 0$$

are equivalent, since, according to (2.4), $D_{\mathbf{v}}^n p$ is a constant when p has degree $\leq n$.

Assume that the equivalence has been proved until k ($< n$) and consider $k+1$. So, we can replace conditions (3.6) by (3.4) until k, and then we have to prove the equivalence of the sets of conditions

$$D_{\mathbf{v}}^{n-k-1} D_{\mathbf{w}}^{r} p(\mathbf{u}_0) = 0, \quad r = 0, \ldots, k+1, \tag{3.7}$$

and

$$D_{\mathbf{v}}^{n-k-1} D_{\mathbf{w}_{r,k+1-r}}^{h_r} p(\mathbf{u}_{r,k+1-r}) = 0, \quad r = 0, \ldots, k+1, \tag{3.8}$$

assuming that

$$D_{\mathbf{v}}^{n-t-r} D_{\mathbf{w}}^{r} p(\mathbf{u}_0) = 0, \quad r + t \leq k. \tag{3.9}$$

holds.

The vector $\mathbf{w}_{r,k+1-r}$ can be written as a linear combination of $\mathbf{v}$ and $\mathbf{w}$, and so $D_{\mathbf{w}_{r,k+1-r}}^{h_r}$ can be put as a linear combination of $D_{\mathbf{v}}^{\alpha} D_{\mathbf{w}}^{\beta}$, $\alpha + \beta = h_r (\leq r)$. Hence

$$D_{\mathbf{v}}^{n-k-1} D_{\mathbf{w}_{r,k+1-r}}^{h_r} p(\mathbf{u}_{r,k+1-r})$$

can be expanded as a sum with each of its summands the product of a real number by

$$D_{\mathbf{v}}^{n-k-1+\alpha} D_{\mathbf{w}}^{\beta} p(\mathbf{u}_{r,k+1-r}), \quad \alpha + \beta = h_r. \tag{3.10}$$

On the other hand, $\mathbf{u}_{r,k+1-r} - \mathbf{u}_0$ can also be written in the form

$$\mathbf{u}_{r,k+1-r} - \mathbf{u}_0 = \delta_r \mathbf{w} + \gamma_{r,k+1-r} \mathbf{v},$$

with $\delta_r, \gamma_{r,k+1-r}$ real numbers. Observe that the coefficient of $\mathbf{w}$ depends only on the line l_r and not on the particular point $\mathbf{u}_{k+1-r,r}$ in this line (let us remember that these lines are parallel with direction $\mathbf{v}$), and that different lines l_r give different numbers δ_r.

Now we can use Taylor expansion to write (3.10) as a sum where each term is the product of a real number by

$$D_{\mathbf{v}}^{n-k-1+\alpha+a} D_{\mathbf{w}}^{\beta+b} p(\mathbf{u}_0), \quad \alpha + \beta = h_r, \quad a, b \geq 0, \quad a + b \leq k + 1 - h_r. \tag{3.11}$$

All terms (3.11) with $\alpha + a \geq 1$ are 0 by (3.9) and so we only need to consider summands with $\alpha = a = 0$, $\beta = h_r$, $b \leq k + 1 - h_r$, that is

$$D_{\mathbf{v}}^{n-k-1} D_{\mathbf{w}}^{h_r+b} p(\mathbf{u}_0), \quad b = 0, 1, \ldots, k + 1 - h_r.$$

In other words, one has

$$D_{\mathbf{v}}^{n-k-1} D_{\mathbf{w}_{r,k+1-r}}^{h_r} p(\mathbf{u}_{r,k+1-r}) = D_{\mathbf{v}}^{n-k-1} D_{\mathbf{w}}^{h_r} p(\mathbf{u}_0) + \delta_r D_{\mathbf{v}}^{n-k-1} D_{\mathbf{w}}^{h_r+1} p(\mathbf{u}_0) +$$
$$\frac{\delta_r^2}{2} D_{\mathbf{v}}^{n-k-1} D_{\mathbf{w}}^{h_r+2} p(\mathbf{u}_0) + \cdots + \frac{\delta_r^{k+1-h_r}}{(k+1-h_r)!} D_{\mathbf{v}}^{n-k-1} D_{\mathbf{w}}^{k+1} p(\mathbf{u}_0),$$

and in particular, if $h_r = 0$,

$$D_{\mathbf{v}}^{n-k-1} p(\mathbf{u}_{r,k+1-r}) = D_{\mathbf{v}}^{n-k-1} p(\mathbf{u}_0) + \delta_r D_{\mathbf{v}}^{n-k-1} D_{\mathbf{w}}^1 p(\mathbf{u}_0) + \frac{\delta_r^2}{2} D_{\mathbf{v}}^{n-k-1} D_{\mathbf{w}}^2 p(\mathbf{u}_0) + \cdots + \frac{\delta_r^{k+1}}{(k+1)!} D_{\mathbf{v}}^{n-k-1} D_{\mathbf{w}}^{k+1} p(\mathbf{u}_0).$$

Hence the vector formed with (3.8)

$$\widehat{\mathbf{d}}_{k+1} = [D_{\mathbf{v}}^{n-k-1} D_{\mathbf{w}_{r,k+1-r}}^{h_r} p(\mathbf{u}_{r,k+1-r})]_{0\le r\le k+1} \tag{3.12}$$

is related to the vector formed with (3.7)

$$\mathbf{d}_{k+1} = [D_{\mathbf{v}}^{n-k-1} D_{\mathbf{w}}^{r} p(\mathbf{u}_0)]_{0\le r\le k+1},$$

by

$$\widehat{\mathbf{d}}_{k+1} = V[\delta_0, \delta_1, \ldots, \delta_{k+1}]\mathbf{d}_k, \tag{3.13}$$

where $V[\delta_0, \delta_1, \ldots, \delta_{k+1}]$ is the (confluent) Vandermonde matrix of order $k+2$ whose row corresponding to the index r, $0 \le r \le k+1$, is the derivative of order h_r of $[\delta_r^j/j!]_{0\le j\le k+1}$ with respect to δ_r, that is

$$\begin{aligned}
&[1, \delta_r, \delta_r^2/2!, \ldots, \delta_r^{k+1}/(k+1)!]^T, \quad \text{if } h_r = 0,\\
&[0, 1, \delta_r, \delta_r^2/2!, \ldots, \delta_r^{k}/k!]^T, \quad \text{if } h_r = 1,\\
&[0, 0, 1, \delta_r, \delta_r^2/2!, \ldots, \delta_r^{k-1}/(k-1)!]^T, \quad \text{if } h_r = 2,
\end{aligned}$$

and so on.

The confluent Vandermonde matrix is the coefficient matrix of a Hermite univariate polynomial interpolation problem and it is well known that its determinant is nonzero. Therefore $\widehat{\mathbf{d}}_{k+1} = \mathbf{0}$ if and only if $\mathbf{d}_{k+1} = \mathbf{0}$, and the equivalence of (3.7) and (3.8) under condition (3.9) has been proved. ∎

From the previous Theorem, it follows:

Corollary 3.3. *Let $\mathbf{v} \in \mathbf{R}^2$, $\mathbf{v} \neq \mathbf{0}$, and $l_0, l_1, \ldots, l_k$ be distinct parallel lines with directional vector $\mathbf{v}$ (coincidences not allowed). Let $\{\mathbf{u}_{rt} \in \mathbf{R}^2 \mid r+t \le k\}$ be a set of $\binom{k+2}{2}$ points not necessarily distinct, distributed along the lines l_r: $\mathbf{u}_{rt} \in l_r$, $t + r \le k$. A necessary and sufficient condition for a polynomial p of total degree $\le n$ to become a univariate polynomial of degree $\le n-k-1$ when restricted to any line of $\mathbf{R}^2$ with direction $\mathbf{v}$ is that*

$$\bar{D}_{\mathbf{u}_{rt},\mathbf{v}}^{t+r,n} p = 0, \quad r + t \le k. \tag{3.14}$$

Remark 3.4. Several sets of conditions (3.5) with different directions $\mathbf{v}$ can be prescribed to a polynomial, always taking into account that the dimension of $\Pi_n(\mathbf{R}^2)$ is $\binom{n+2}{2}$. On the other hand, observe that (3.5) produces a polynomial whose degree decreases to $n-k-1$ along any line with direction $\mathbf{v}$. If some of these conditions are not satisfied, it can happen that the degree of the polynomial decreases to different numbers along different lines of the same direction $\mathbf{v}$. For example, in the case $k = 1$, if (3.14) holds only for $(r, t) = (0, 0)$ and $(r, t) = (1, 0)$, it can be seen [4] that the degree of the polynomial is $n - 1$ on any line of direction v and in particular $n - 2$ on the line l_1.

4 Interpolation problems and some examples

The techniques of [3, 4] allow us to construct unisolvent interpolation problems on spaces of polynomials with degree diminution along some directions and to find their solution.

In [8], for a given set of interpolation data a Newton basis was constructed which spans the interpolation space. In particular, in that paper it was shown that when there are $n + 1$ data on a line r_0, n data on another line r_1, and so on until r_n, with a total of $\binom{n+2}{2}$ interpolation data, the resulting interpolation space is $\Pi_n(\mathbf{R}^2)$. Coincidences of lines and/or data points were allowed giving rise to derivatives among the interpolation data.

In [3, 4] we showed that asymptotic conditions can be combined with usual interpolation conditions in some form so that a similar technique to the one of [8] still works. The results of Section 3 allow us to be more precise. Consider a usual interpolation problem with $\binom{n+2}{2}$ data on the lines $r_0, r_1, \ldots, r_n$ as above and suppose that, for example, k of these lines $r_{i_1}, r_{i_2}, \ldots, r_{i_k}$, with $i_1 < i_2 < \cdots < i_k$ are parallel. Suppose that we replace $\binom{k+2}{2}$ interpolation data by $\binom{k+2}{2}$ vanishing asymptotic conditions as in Theorem 3.2: k on the line r_{i_1}, $k - 1$ on r_{i_2} and so on until 1 on r_{i_k}. So we have an interpolation problem with $\binom{n+2}{2}$ conditions ($\binom{k+2}{2}$ of them asymptotic) on $\Pi_n(\mathbf{R}^2)$. According to Theorem 3.2 it can be seen as an interpolation problem with $\binom{n+2}{2} - \binom{k+2}{2}$ usual interpolation data on the space of polynomials of $\Pi_n(\mathbf{R}^2)$ which become univariate polynomials of degree $n - k - 1$ when restricted to any line parallel to r_{i_1}. For example, in Figure 1 we have 15 data, 9 of them ordinary data (black points) and 6 of them vanishing asymptotic conditions (white points). The interpolation problem only with ordinary data is unisolvent in the space of polynomials of $\Pi_4(\mathbf{R}^2)$ which have degree 1 along the lines parallel to r_0.

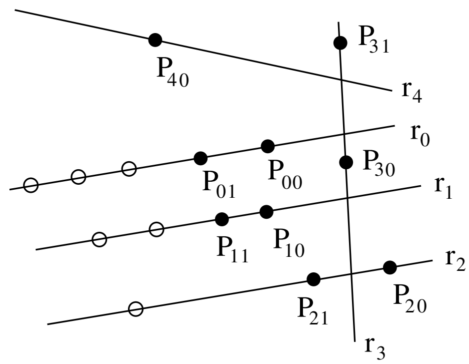

Figure 1. An interpolation problem with 9 ordinary data
and 6 vanishing asymptotic conditions

It must be said that the techniques of [3, 4] allow us to solve these problems very easily by a Newton formula. More precisely, let us consider the interpolation problem with data $f(0,0)$, $f(0,1)$ and $f(1,3)$ (see Figure 2, left) and suppose we want to interpolate with the space of polynomials of $\Pi_2(\mathbf{R}^2)$ which become constant along any line of direction (1,1). According to [3, 4] this problem is unisolvent (see Figure 2, right) and a Newton basis to solve it is $1, x-y, (x-y)(x-y+1)$. So the solution can be written $p = a_0 + a_1(x-y) + a_2(x-y)(x-y+1)$ and the coefficients can be found recursively as in Newton formulae from the data.

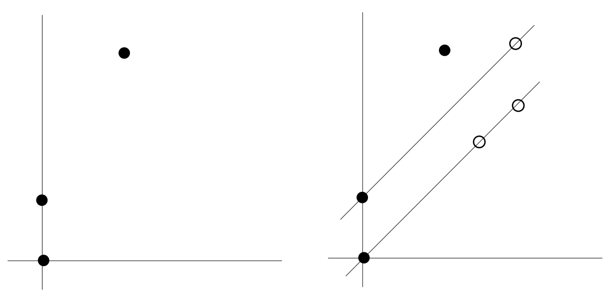

Figure 2. An interpolation problem with quadratic polynomials

We can also consider, for example, polynomials of $\Pi_4(\mathbf{R}^2)$ which have degree 2 on lines of one direction v and on lines of another direction w. So we have 3 vanishing asymptotic conditions for each direction and 9 remaining (usual) interpolation conditions. The problem can be solved as above assuming that the data are distributed in such a form that [3, 4] can be applied. See those papers for more examples.

We shall finish mentioning that the tensor product space $\Pi_{m,n}(\mathbf{R}^2)$ can be interpreted as the subspace of $\Pi_{m+n}(\mathbf{R}^2)$ formed by polynomials which have degree m on lines parallel to OX and degree n on lines parallel to OY. The space $\Pi_{m,n}(\mathbf{R}^2)$ arises when we consider $\binom{n+1}{2}$ vanishing asymptotic conditions of the type of Theorem 3.2 on n lines parallel to OX and analogous $\binom{m+1}{2}$ vanishing asymptotic conditions on m lines parallel to OY. Observe that the dimension of $\Pi_{m,n}(\mathbf{R}^2)$ is $(m+1)(n+1)$, that of $\Pi_{m+n}(\mathbf{R}^2)$ is $\binom{m+n+2}{2}$ and that

$$\binom{m+n+2}{2} = (m+1)(n+1) + \binom{m+1}{2} + \binom{n+1}{2}.$$

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