RACSAM
Revista de la Real Academia de Ciencias. Serie A. Matemáticas
VOLUMEN **16,** NÚM **1, 2001, p.51-56**
Matemática Aplicada
Comunicación Preliminar/Preliminary Communication

A Newton approach to bivariate Hermite interpolation on generalized natural lattices*

J. M. Carnicer and M. Gasca

Abstract.— A natural lattice is the set of all the intersections points of a set of lines in general position in the plane. The Lagrange interpolation problem on a natural lattice with $n+2$ lines has a unique solution in the space of bivariate polynomials of degree not greater than n. A generalized natural lattice is the set of all intersection points of a set of distinct lines, allowing parallelism and multiple concurrences. A Hermite interpolation problem is posed on a generalized natural lattice in a space of polynomials whose degree decreases along the directions corresponding to parallel lines. In this paper, we study the unisolvence of this problem and suggest a Newton approach for solving it.

Una aproximación a la interpolación de Hermite bivariada sobre retículos naturales generalizados usando fórmulas de Newton

Resumen.— Un retículo natural es el conjunto de todas las intersecciones de un conjunto de rectas del plano en posición general. El problema de interpolación de Lagrange sobre un retículo natural de $n+2$ rectas tiene solución única en el espacio de los polinomios bivariados de grado menor o igual que n. Un retículo natural generalizado está formado por todas las intersecciones de un conjunto de rectas distintas, sin excluir paralelismos o concurrencias múltiples. A un retículo natural generalizado le asociamos un problema de interpolación de Hermite en un espacio de polinomios cuyo grado disminuye a lo largo de las direcciones correspondientes a las rectas paralelas del retículo. En este trabajo estudiamos la existencia y unicidad de solución del problema y el uso de fórmulas de Newton para su resolución.

1 Introduction

An interpolation problem is determined by the space of interpolating functions and a set of interpolation data. In multivariate polynomial interpolation, the existence and uniqueness of solution of a problem with a set of interpolation data in a polynomial space always depends on the geometrical distribution of the set of interpolation points, also called nodes (see [10]). One of the most frequent problems in this framework is the identification of simple distributions of points such that the unisolvence of the problem on a given space is guaranteed.

In this paper, we study a particularly simple distribution of nodes in the plane. The interpolation points are the intersections of $n+2$ straight lines $r_0, \dots, r_{n+1}$ and the interpolation space is a subspace of the space of polynomials of degree not greater than n, $\Pi_n(\mathbf{R}^2)$. The case of $n+2$ lines in general position, giving rise to $\binom{n+2}{2}$ different intersection points, was studied among other authors by Chung and Yao [7] who introduced the term *natural lattice*. If we denote by X the set of $\binom{n+2}{2}$ intersection points and by $r_i(x)$ an affine polynomial such that $r_i(x) = 0$ is an equation of the line r_i, the polynomial

$$q(x) = \sum_{P \in X} f(P) \prod_{\substack{i=0 \\ r_i(P) \neq 0}}^{n+1} \frac{r_i(x)}{r_i(P)}. \tag{1.1}$$

matches the function f at all points $P \in X$.

*Presentado por Alberto Dou.
Mathematics Subject Classifications: 41A05, 65D05, 41A63.
Keywords: bivariate polynomial interpolation, Newton formula

Submitted May 2, 2001.Revised May 25, 2001.Published September 29, 2001

Natural lattices are the simplest cases of sets satisfying what Chung and Yao called the *geometric characterization*. The geometric characterization is important because the solution of the corresponding interpolation problem can be found by a very simple Lagrange formula, which is a generalization of (1.1). Busch [1] extended Chung and Yao's geometric characterization allowing multiple concurrences but not parallel lines. He provided a recursive procedure for the construction of Lagrange formulae. These formulae become very complicated and we think that a Newton approach is more appropriate for solving the problem.

Our aim in this paper is to extend natural lattices allowing parallel lines and multiple concurrences of lines. This problem was studied by Dyn and Ron in [8]. They analyzed completely the *simple case* where no multiple concurrences of lines occur although parallel lines are allowed. They observed that the Lagrange interpolation formula (1.1) for the interpolating polynomial still holds. This formula describes the unique solution of the problem in a subspace of polynomials in $\Pi_n(\mathbf{R}^2)$ whose degree diminishes when we restrict the variables to lines of some directions. Dyn and Ron also described the interpolation problem for the *general case* but a constructive method to obtain the solution was not provided for this case. On the other hand, their motivation for studying this problem came from some results on box spline spaces. As a consequence, their approach requires a larger background than here. Further results can be found in [2].

For the sake of completeness and in order to offer a simpler approach, we study again in Section 2 the simple case, introduce the interpolation spaces and provide the Lagrange interpolation formula. In Section 3, we study the general case (without coincidences of lines) from a Newton approach, giving a constructive method to find the solution of the problem by a Newton formula. In these sections, we use the spaces of polynomials introduced by Dyn and Ron [8] whose degree diminishes along prescribed directions. These spaces arise in a natural way in the interpolation problems with asymptotic conditions that we have studied in some recent papers [4, 5, 6]. In Section 4, we see the relationship between both approaches. Interpolation problems with asymptotic conditions have also been considered in [11] as traces of usual data when two manifolds tend to be parallel. Finally, some examples are given in Section 5.

In summary, for a set of any $n+2$ different lines, we provide a set of interpolation conditions on the intersection points, a subspace of $\Pi_n(\mathbf{R}^2)$ as the interpolation space and a construction of the unique solution of the interpolation problem.

2 Lagrange formulae for generalized natural lattices in the simple case

Let $r_0, r_1, \dots r_{n+1}$ be $n+2$ different straight lines and assume that any 3 of them do not intersect at the same point. Let X be the set of intersection points

$$X := \{r_i \cap r_j \mid i < j,\ r_i \text{ is not parallel to } r_j\}. \tag{2.1}$$

For each i let us define

$$k_i := \big|\{j > i \mid r_j \text{ is parallel to } r_i\}\big|. \tag{2.2}$$

Observe that each point in X is the intersection of a line r_i with a transversal (not parallel) line r_j, $i < j$. The number of lines r_j, $j > i$, transversal to r_i is $n+1-i-k_i$ and then we have

$$|X| = \sum_{i=0}^{n}(n+1-i-k_i) = \binom{n+2}{2} - \sum_{i=0}^{n} k_i. \tag{2.3}$$

The number of nodes $|X|$ is less than or equal to $\binom{n+2}{2}$ and depends on the number of parallel lines. Formula (1.1) still holds as a Lagrange formula to solve the Lagrange interpolation problem on the set of nodes X. The interpolation space is not $\Pi_n(\mathbf{R}^2)$ in general. How to describe the space generated by the Lagrange polynomials

$$\ell_P(x) := \prod_{\substack{i=0\\ r_i(P)\neq 0}}^{n+1} \frac{r_i(x)}{r_i(P)} \tag{2.4}$$

in terms of the given lines $r_0, \dots, r_{n+1}$? In order to answer to this question, let us associate to each line r_i a directional vector $\rho_i \neq 0$. The direction of r_i can be regarded as the 1-dimensional subspace $\langle \rho_i \rangle$ of $\mathbf{R}^2$. Then r_i is parallel to r_j if and only if they have the same direction $\langle \rho_i \rangle = \langle \rho_j \rangle$. Let $D := \{\langle \rho_0 \rangle, \dots, \langle \rho_{n+1} \rangle\}$ be the set of directions of the lines $r_0, \dots, r_{n+1}$. To each direction $\langle \rho \rangle \in D$ we may associate the number

$$\kappa\langle \rho \rangle := \big|\{i \in \{0, \dots, n+1\} \mid r_i \text{ has direction } \langle \rho \rangle\}\big| - 1. \tag{2.5}$$

If we denote

$$D_1 := \{\langle\rho\rangle \mid \langle\rho\rangle \in D, \kappa\langle\rho\rangle \geq 1\}, \tag{2.6}$$

then we have

$$\sum_{i=0}^{n} k_i = \sum_{\langle\rho\rangle\in D_1} \binom{\kappa\langle\rho\rangle + 1}{2} \tag{2.7}$$

and we may write (2.3) in the form

$$|X| = \binom{n+2}{2} - \sum_{\langle\rho\rangle\in D_1} \binom{\kappa\langle\rho\rangle + 1}{2}.$$

Let us now observe that all Lagrange polynomials $\ell_P(x)$ in (2.4) are products of n linear factors. More precisely, if $P = r_i \cap r_j$, then $\ell_P(x) = \prod_{h\neq i,j}(r_h(x)/r_h(P))$. For any direction $\langle\rho\rangle \in D$, then, at least $\kappa\langle\rho\rangle$ factors of $\ell_p(x)$ correspond to lines with direction $\langle\rho\rangle$. This means that $\ell_P(x)$ belongs to the subspace of polynomials of degree n whose degree decreases to $n - \kappa\langle\rho\rangle$ when restricted to lines with direction $\langle\rho\rangle$, for each $\langle\rho\rangle \in D_1$.

Let us introduce the idea of directional degree.

Definition 1 *Given a bivariate polynomial p and a direction of the plane $\langle v\rangle$, $v \in \mathbf{R}^2$, we define the directional degree of p along $\langle v\rangle$ as*

$$\partial_{\langle v\rangle} p := \max_{a\in\mathbf{R}^2} \deg_t p(a + tv),$$

where $\deg_t p(a + tv)$ denotes the degree of the univariate polynomial $p(a + tv)$ in the indeterminate t.

Let us observe that $\partial_{\langle v\rangle} p$ depends on the direction but not on the particular choice of a vector v in this direction. The partial degree of a polynomial p in each of the indeterminates can be seen as the directional degree along the directions $\langle(1, 0)\rangle$, $\langle(1, 0)\rangle$. Conversely the directional degree can be seen as a partial degree.

Given $v \in \mathbf{R}^2$, take a vector $w \in \mathbf{R}^2$ such that $\{v, w\}$ form a basis of $\mathbf{R}^2$ and define the bivariate polynomial $P(t, s) := p(tv + sw)$. Each $a \in \mathbf{R}^2$ can be writen as $a = t_0 v + s_0 w$, and so $p(a + tv) = P(t_0 + t, s_0)$. Then

$$\partial_{\langle v\rangle} p = \max_{a\in\mathbf{R}^2} \deg_t p(a + tv) = \max_{t_0,s_0\in\mathbf{R}} \deg_t P(t_0 + t, s_0) = \max_{s_0\in\mathbf{R}} \deg_t P(t, s_0).$$

So we have seen that $\partial_{\langle v\rangle} p$ is the partial degree in the indeterminate t of $P(t, s)$, which is independent on the choice of w.

The fact that the directional degree coincides with a partial degree under a change of variables implies that many properties of the partial degree can be extended to directional degrees. In fact, we have

$$\partial_{\langle v\rangle}(pq) = \partial_{\langle v\rangle} p + \partial_{\langle v\rangle} q. \tag{2.8}$$

Let us now introduce a notation for spaces of polynomials whose degree diminishes along prescribed directions.

Definition 2 *Given a set V of different directions and a mapping $\mu : V \to \mathbf{N} \cup \{0\}$, we define $\Pi_n(V, \mu)$ as the polynomial space*

$$\Pi_n(V, \mu) := \Big\{p \in \Pi_n(\mathbf{R}^2) \mid \partial_{\langle v\rangle} p \leq n - \mu\langle v\rangle, \quad \forall\langle v\rangle \in V\Big\}.$$

Remark 1 *The directions $\langle v\rangle$ such that $\mu\langle v\rangle = 0$ are redundant in the previous definition. So, if $V_1 := \{\langle v\rangle \in V \mid \mu\langle v\rangle \geq 1\}$, we have $\Pi_n(V, \mu) = \Pi_n(V_1, \mu)$.*

The following proposition shows that if a product of polynomials belongs to the space $\Pi_n(V, \mu)$, then the factors belong to that space.

Proposition 1 *Let V be a set of directions and $\mu : V \to \mathbf{N} \cup \{0\}$. If $q \in \Pi_n(V, \mu)$, $q \neq 0$, can be factored as a product of two polynomials, $q = q_1 q_2$, then $q_1, q_2 \in \Pi_n(V, \mu)$.*

PROOF.
Clearly $q_1, q_2 \in \Pi_n(\mathbf{R}^2)$, because

$$\deg q_1 + \deg q_2 = \deg q \leq n.$$

Let $v \in \mathbf{R}^2$ with $\langle v \rangle \in V$. By (2.8) we may write

$$\partial_{\langle v \rangle} q_1 + \partial_{\langle v \rangle} q_2 = \partial_{\langle v \rangle} q \leq n - \mu \langle v \rangle$$

and then $\partial_{\langle v \rangle} q_i \leq n - \mu \langle v \rangle$, $i = 1, 2$. ■

The next result shows that the space generated by the Lagrange functions is just $\Pi_n(D, \kappa) = \Pi_n(D_1, \kappa)$.

We discuss first the case in which all lines r_i are parallel, that is, D consists only of one direction $\langle v \rangle$, with $\kappa \langle v \rangle = n + 1$. Then X is the empty set and $\Pi_n(D, \kappa) = 0$, the null space. Since $\dim \Pi_n(D, \kappa) = 0 = |X|$, in a trivial sense $\Pi_n(D, \kappa)$ coincides with the space generated by the empty set of Lagrange polynomials. In order to avoid this trivial case, we shall require that $|D| \geq 2$ or equivalently $X \neq \emptyset$.

Theorem 1 *Let $r_0, \ldots, r_{n+1}$ be a set of $n + 2$ different lines not all of them parallel. Assume that no more than two of these lines intersect at the same point and let*

$$X := \{r_i \cap r_j \mid i < j,\ r_i \text{ is not parallel to } r_j\}.$$

Let D be the set of directions of the lines $r_0, \ldots, r_{n+1}$. For each $\langle \rho \rangle \in D$, let $\kappa \langle \rho \rangle$ be defined by (2.5) *and the set of directions D_1 by* (2.6)*. For any function f defined on a set containing X, the Lagrange interpolation problem: find a polynomial $q \in \Pi_n(D, \kappa)$ such that*

$$q(P) = f(P), \quad P \in X,$$

has a unique solution q, which can be expressed by (1.1)*. Furthermore*

$$\dim \Pi_n(D, \kappa) = \binom{n+2}{2} - \sum_{\rho \in D_1} \binom{\kappa(\rho) + 1}{2} \tag{2.9}$$

and $\Pi_n(D, \kappa) = \Pi_n(D_1, \kappa) = \langle \ell_P \mid P \in X \rangle$, where ℓ_P are the Lagrange polynomials (2.4)*.*

PROOF.

Let $W := \langle \ell_P \mid P \in X \rangle$ be the space generated by the polynomials (2.4). Since the polynomials ℓ_P are linearly independent, we have

$$\dim W = |X| = \binom{n+2}{2} - \sum_{\rho \in D_1} \binom{\kappa \langle \rho \rangle + 1}{2}.$$

Clearly $\ell_P(a + tv)$ is the product of constants and polynomials of first degree in t. In fact, $r_i(a + tv)$ is a constant polynomial for all a if and only if v is a directional vector of r_i. Therefore $\partial_{\langle v \rangle} \ell_P$ is equal to the number of lines not containing the point P nor the direction v, that is, $\partial_{\langle v \rangle} \ell_P = n - \kappa \langle v \rangle$. Then we have $\ell_P \in \Pi_n(D, \kappa)$ for all $P \in X$ and therefore $W \subseteq \Pi_n(D, \kappa) = \Pi_n(D_1, \kappa)$. So we have shown the existence of an interpolant $q \in \Pi_n(D, \kappa)$ given by (1.1). Let us show that there exists a unique $q \in \Pi_n(D, \kappa)$ satisfying the given interpolation conditions. This is equivalent to show that the only polynomial in $\Pi_n(D, \kappa)$ vanishing on X is the zero polynomial. The proof will be done by induction on the number of lines.

If $n = 0$, we have two lines r_0, r_1 determining two different directions and a single intersection point $X = \{r_0 \cap r_1\}$. The space $\Pi_0(D, \kappa)$ coincides with the space Π_0 of all constant polynomials. If a constant vanishes at a point, then it must be the zero polynomial.

Let us now show the result for $n+2$ lines, assuming that it holds for $n+1$ lines. Since neither the interpolation points nor the space depend on the order of the lines, we may assume without loss of generality that the last two lines r_n, r_{n+1} are not parallel. Let ρ_i be a directional vector for the line r_i, $i = 0, \ldots, n+1$. Let q be a polynomial vanishing on X and let $a_0 + t\rho_0$ be a parameterization of the line r_0. Then, $q(a_0 + t\rho_0)$ is a polynomial of degree less than or equal to $n - \kappa \langle \rho_0 \rangle = n - k_0$, where k_0 is given by (2.2) vanishing on the $n+1-k_0$ points of $X \cap r_0$. Therefore $q(a_0 + t\rho_0) = 0$ for all t and then r_0 is a factor of the polynomial q. So, we can write $q = r_0 \hat{q}$, where $\hat{q} \in \Pi_{n-1}(\mathbf{R}^2)$ is a polynomial with $\partial_{\langle \rho \rangle} \hat{q} \leq n - 1 - \kappa \langle \rho \rangle$, if $\langle \rho \rangle \in D$, $\langle \rho \rangle \neq \langle \rho_0 \rangle$, and $\partial_{\langle \rho_0 \rangle} \hat{q} \leq n - \kappa \langle \rho_0 \rangle$. Furthermore q vanishes on $X \setminus r_0$. For the set of lines $r_1, \ldots, r_{n+1}$, we define the set of directions $\hat{D} := \{\langle \rho_1 \rangle, \ldots, \langle \rho_n \rangle, \langle \rho_{n+1} \rangle\}$ and $\hat{\kappa} \langle \rho \rangle := |\{i > 0 \mid r_i \text{ has direction } \langle \rho \rangle\}| - 1$. Then $\hat{q} \in \Pi_{n-1}(\hat{D}, \hat{\kappa})$ and vanishes on $\hat{X} := \{r_i \cap r_j \mid 0 < i < j, r_i \text{ not parallel to } r_j\} = X \setminus r_0$. By the induction hypothesis $\hat{q}$ is the zero polynomial and then q is identically zero.

We have shown that the interpolation problem is unisolvent and this allows us to compute the dimension of the space $\Pi_n(D_1, \kappa)$, $\dim \Pi_n(D_1, \kappa) = |X|$, and conclude that $W = \Pi_n(D_1, \kappa)$. So, the result follows. ■

As a consequence of Theorem 1, we can deduce a formula for $\dim \Pi_n(V, \mu)$ if the space $\Pi_n(V, \mu)$ can be interpreted as the solution space of an interpolation problem described in the theorem.

Corollary 1 *Let V be a set of directions of the plane and $\mu : V \to \mathbf{N} \cup \{0\}$. Let us define $V_1 := \{\langle v\rangle \in V \mid \mu\langle v\rangle \geq 1\}$. If*

$$n \geq \sum_{\langle v\rangle \in V_1} \mu\langle v\rangle + |V_1| - 2, \tag{2.10}$$

then

$$\dim \Pi_n(V,\mu) = \binom{n+2}{2} - \sum_{\langle v\rangle \in V_1} \binom{\mu\langle v\rangle + 1}{2}. \tag{2.11}$$

PROOF.
First, for each different direction of V_1 we take $\mu\langle v\rangle + 1$ lines of direction $\langle v\rangle$. If (2.10) holds, then we can take other lines with any different directions not belonging to V_1 until obtaining $n+2$ lines $r_0, \ldots, r_{n+1}$. Let D be the set of directions of the lines r_i, $i = 0, \ldots, n+1$, $\kappa\langle\rho\rangle$ defined by (2.5) and D_1 defined by (2.6). Then, the condition imposed on the lines $r_0, \ldots, r_{n+1}$ implies that $V_1 = D_1$, $\mu|_{V_1} = \kappa|_{D_1}$. By Remark 1, $\Pi_n(V,\mu) = \Pi_n(D,\kappa)$.

However there might exist more than two lines among $r_0, \ldots, r_{n+1}$ meeting at the same point. In this case, we can always choose a line $\tilde r_i$ parallel to r_i so that at most two lines intersect at the same point. Since the lines $\tilde r_0, \ldots, \tilde r_{n+1}$ have simple intersections, the interpolation problem associated to these lines is the Lagrange problem described in Theorem 1. The space $\Pi_n(D,\kappa)$ does not depend on the lines but rather on the directions of these lines. Then the space associated to the Lagrange problem based on the lines $\tilde r_0, \ldots, \tilde r_{n+1}$ is still $\Pi_n(D,\kappa)$. From (2.9), we obtain (2.11). ■

3 Newton formulae for generalized natural lattices

A *generalized natural lattice* is any set of $n+2$ different straight lines. In the preceding section we have dealt with sets of lines such that no 3 lines intersect at the same point. Now we consider the *general case*: multiple concurrences and also parallel lines are allowed. However, we do not consider coincidences of lines. Multiple concurrences can be interpreted as limit cases of simple intersections and so, directional derivatives appear in a natural way, leading to Hermite interpolation problems. In this case Lagrange formulae become very complicated (see [1]) and so we prefer a Newton approach.

In order to apply the results of [9], we introduce some notations. Let us consider $n+2$ different lines $r_0, r_1, \ldots, r_{n+1}$. Without loss of generality, we may assume that all parallel lines corresponding to any direction have consecutive indices. This assumption allows us to simplify our index notation. Let us define

$$r_{ij} := r_{n+1-j}, \quad j = 0, \ldots, m(i) := n - i - k_i, \tag{3.1}$$

where k_i is given by formula (2.2), and an index set

$$\hat I := \{(i,j) \mid 0 \leq i \leq n; 0 \leq j \leq m(i)\}, \tag{3.2}$$

lexicographically ordered. For notational convenience we introduce the sets

$$R_{ij} := \{r_h \mid h < i\} \cup \{r_{ip} \mid p < j\} = \{r_h \mid h < i \text{ or } h > n+1-j\}, \quad (i,j) \in \hat I. \tag{3.3}$$

Now we introduce the set of polynomials

$$\phi_{ij} := \prod_{h<i} r_h \prod_{p<j} r_{ip} = \prod_{r \in R_{ij}} r, \quad (i,j) \in \hat I. \tag{3.4}$$

Let X be the set of all points determined as intersection of at least two lines among $r_0, \ldots, r_{n+1}$. For any point $P \in X$, let

$$\nu(P) := \left|\left\{i \in \{0, \ldots, n+1\} \mid r_i(P) = 0\right\}\right| - 2. \tag{3.5}$$

In order to state the main result in this section we use multiindex notation. If $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| := \alpha_1 + \alpha_2$ and $D^\alpha := D_{e_1}^{\alpha_1} D_{e_2}^{\alpha_2}$, where $e_1 = (1,0)$, $e_2 = (0,1)$.

Theorem 2 *Let $r_0, \ldots, r_{n+1}$ be $n+2$ different lines not all of them parallel and let X be the set of points lying on at least two of those lines*

$$X := \{r_i \cap r_j \mid i < j,\ r_i \text{ is not parallel to } r_j\} \neq \emptyset.$$

Let D be the set of directions of the lines r_i, $i = 0, \dots, n+1$. For each $\langle\rho\rangle \in D$, let $\kappa\langle\rho\rangle$ be defined by (2.5)*. For any sufficiently differentiable function f defined on an open set containing X, the Hermite interpolation problem: find a polynomial $q \in \Pi_n(D,\kappa)$ such that*

$$D^\alpha q(P) = D^\alpha f(P), \quad |\alpha| \le \nu(P), \quad P \in X, \tag{3.6}$$

where $\nu(P)$ is given by (3.5)*, has a unique solution. Furthermore, the set of polynomials ϕ_{ij}, $(i,j) \in \hat{I}$, defined by* (3.4) *is a basis of $\Pi_n(D,\kappa)$ and q can therefore be expressed by*

$$q := \sum_{i=0}^{n} \sum_{j=0}^{m(i)} a_{ij}\phi_{ij}. \tag{3.7}$$

PROOF.

Let us denote by $u_{ij} \in X$ the intersection of the lines r_i, r_{ij},

$$u_{ij} := r_i \cap r_{ij} = r_i \cap r_{n+1-j}, \quad (i,j) \in \hat{I}. \tag{3.8}$$

Observe that the concurrence of 3 or more lines at the same point give rise to repetition of points. The number of different indices for a given point $P \in X$ is precisely $\binom{\nu(P)+2}{2}$. In order to use the results of [9], we need some definitions. For $(i,j) \in \hat{I}$, we define a functional L_{ij} in the form

$$L_{ij} f := D_{\rho_i}^{t_{ij}} f(u_{ij}), \tag{3.9}$$

where ρ_i is a directional vector of the line r_i and

$$t_{ij} := |\{r \in R_{ij} \mid r(u_{ij}) = 0\}|. \tag{3.10}$$

According to [9], there exists a unique polynomial q in the space

$$W := \langle \phi_{ij} \mid (i,j) \in \hat{I} \rangle \tag{3.11}$$

such that

$$L_{ij} q = L_{ij} f, \quad (i,j) \in \hat{I}. \tag{3.12}$$

The polynomial q can be expressed in the form (3.7) and its coefficients can be computed from (3.12). Indeed, in [9], it is shown that for $(i,j) \in \hat{I}$

$$\begin{gathered} L_{ij}\phi_{hk} = 0, \quad (h,k) \in \hat{I}, \quad (h,k) > (i,j), \\ L_{ij}\phi_{ij} \neq 0, \end{gathered}$$

hence the matrix $(L_{ij}\phi_{hk})_{(i,j),(h,k)\in\hat{I}}$ with the indices lexicographically ordered is lower triangular and the coefficients a_{ij} can be computed by the recurrence

$$a_{00} := L_{00} f, \qquad a_{ij} := \Big(L_{ij} f - \sum_{(h,k)<(i,j)} a_{hk} L_{ij}\phi_{hk}\Big) / L_{ij}\phi_{ij}, \ (i,j) > (0,0). \tag{3.13}$$

In this sense, the functions (3.4) can be considered a Newton basis for the interpolation problem (3.12) in the space W and (3.7) can be seen as a Newton formula.

In order to prove the theorem, we shall show that the space W defined in (3.11) is $\Pi_n(D,\kappa)$ and that the set of interpolation conditions (3.12) is equivalent to the set of conditions (3.6).

First, we show that $W = \Pi_n(D,\kappa)$. Observe that $\phi_{ij} = \prod_{h<i} r_h \prod_{h>n+1-j} r_h$ divides $\psi_{ij} := \prod_{h\neq i,n+1-j} r_h$. For any $\langle\rho\rangle \in D$, $\partial_{\langle\rho\rangle}\psi_{ij}$ is equal to the number of lines r_h, $h \neq i,j$ whose direction is not $\langle\rho\rangle$. So $\partial_{\langle\rho\rangle}\psi_{ij} = n - \kappa\langle\rho\rangle$ for each $\langle\rho\rangle \in D$, and then $\psi_{ij} \in \Pi_n(D,\kappa)$, where D is the set of directions of the lines $r_0, \dots, r_{n+1}$ and κ is defined in (2.5). Since ϕ_{ij} divides ψ_{ij} we deduce from Proposition 1 that $\phi_{ij} \in \Pi_n(D,\kappa)$. So we have seen that $W \subseteq \Pi_n(D,\kappa)$. On the other hand, since the interpolation problem in W is unisolvent, $\dim W$ is equal to the number of interpolation data, that is

$$\dim W = |\hat{I}| = \sum_{i=0}^{n} (m(i)+1) = \sum_{i=0}^{n} (n+1-i-k_i) = \binom{n+2}{2} - \sum_{i=0}^{n} k_i.$$

By (2.7) and Corollary 1, we have

$$\dim W = \binom{n+2}{2} - \sum_{\langle\rho\rangle\in D_1} \binom{\kappa\langle\rho\rangle+1}{2} = \dim \Pi_n(D,\kappa)$$

and so the space W generated by the Newton basis is $\Pi_n(D,\kappa)$.

It only remains to show that the interpolation conditions (3.12) given by the linear functionals L_{ij} in (3.9) are equivalent to the Hermite conditions (3.6). If P is any point of X and $r_{i_0},\ldots,r_{i_{\nu(P)}}, r_{i_{\nu(P)+1}}$, $i_0 < \cdots < i_{\nu(P)} < i_{\nu(P)+1}$, are the lines containing P, then the $\binom{\nu(P)+2}{2}$ interpolation data are

$$D^k_{\rho_{i_j}} f(P), \quad 0 \le j \le k \le \nu(P).$$

The corresponding linear forms $f \mapsto D^k_{\rho_{i_j}} f(P)$ are linearly independent because the interpolation problem (3.12) is unisolvent. This set of linear forms is generated by the $\binom{\nu(P)+2}{2}$ linear functionals $f \mapsto D^\alpha f(P)$, $|\alpha| \le n$. Therefore, the space generated by both sets of linear functionals is the same. So we have shown that the interpolation problems (3.6) and (3.12) are equivalent in the sense that both lead to the same solution q for an arbitrary f. ■

Remark 2 *The proof of Theorem 2 suggests a construction of the solution q of the interpolation problem* (3.6)*. First, we write the problem in the equivalent form* (3.12)*. The definition of the linear forms L_{ij} of* (3.9) *require an ordering of the lines such that parallel lines have consecutive indices and the definitions* (3.1), (3.2), (3.3), (3.8) *and* (3.10)*. Taking into account that*

$$D^k_v = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} v^\alpha D^\alpha$$

we compute $L_{ij}f$ as a linear combination of $D^\alpha f(u_{ij})$ which are data of the problem (3.6)*. Then,* (3.7) *is a Newton formula for the solution of the problem (3.12), because the coefficients a_{ij} can be computed by* (3.13)*. For the efficient computation of each $L_{ij}\phi_{hk}$ in* (3.13) *one can take into account [3]. An alternative is suggested by the formula*

$$D^{t_{ij}}_{\rho_i} \phi_{hk}(u_{ij}) = \frac{d^{t_{ij}}}{ds^{t_{ij}}}\Big|_{s=0} \phi_{hk}(u_{ij} + \rho_i s).$$

We first compute the product of univariate polynomials $\prod_{r\in R_{hk}} r(u_{ij}+\rho_i s) = \phi_{hk}(u_{ij}+\rho_i s)$ and then the coefficient of $s^{t_{ij}}$ of this polynomial is $(1/t_{ij}!)L_{ij}\phi_{hk}$.

4 Generalized natural lattices and asymptotic conditions

Spaces of bivariate polynomials of a certain degree which decreases when the variables are restricted to lines with prescribed directions have appeared in [8] and in some recent papers [4, 5, 6] dealing with interpolation problems with asymptotic conditions. In fact, the results of Section 3 can also be obtained using those asymptotic conditions.

Let p be a bivariate polynomial of degree n and $p(a+tv)$, $t \in \mathbf{R}$, be the restriction of p to the line of parametric equation $x = a+tv$, $a, v \in \mathbf{R}^2$. In [4, 5] asymptotic conditions on p along the line r were introduced as certain conditions on the coefficients of degree $n, n-1, \ldots$ of $p(a+tv)$.

As seen in [9], an interpolation problem with $n+1-i$ data on a straight line r_i, $0 \le i \le n$, is unisolvent in $\Pi_n(\mathbf{R}^2)$. In [5] a construction similar to that of [9] was provided to deal with interpolation problems including asymptotic conditions. In this framework, a problem with $n+1-i$ data (asymptotic or not) on r_i, $0 \le i \le n$, gave rise to $\Pi_n(\mathbf{R}^2)$ as a suitable space in order to have unisolvence.

The problem that we have considered in the proof of Theorem 2 can be studied in this form. Let us denote by I the set of indices

$$I := \{(i,j) \mid 0 \le i \le n; 0 \le j \le n-i\} \tag{4.1}$$

lexicographically ordered. A set of linear forms L_{ij}, $(i,j) \in I$, is used for posing the problem. The set $\hat{I}$ defined in (3.2) is contained in I and the linear forms L_{ij}, $(i,j) \in \hat{I}$, are those defined in (3.9). For $(i,j) \in I \setminus \hat{I}$, $L_{ij}p$ prescribe the values of the coefficients of highest degrees of the polynomial solution of the problem when the variables are restricted to certain lines. In [5], we show that the interpolation problem

$$L_{ij}p = L_{ij}f, \quad (i,j) \in I,$$

has a unique solution $p \in \Pi_n(\mathbf{R}^2)$ and that p can be constructed from a Newton formula in the sense that the the basis constructed there $\{\phi_{ij} \mid (i,j) \in I\}$ satisfies

$$L_{ij}\phi_{hk} = 0, \quad (h,k),(i,j) \in I, \quad (h,k) > (i,j),$$
$$L_{ij}\phi_{ij} \neq 0, \quad (i,j) \in I.$$

Choose a direction $\langle\rho\rangle$. Under our indexing assumptions, the interpolation data of the form (3.9) are given by the indices $s \leq i \leq s + \kappa\langle\rho\rangle$, $0 \leq j \leq n + 1 - s - \kappa\langle\rho\rangle$, that is, indices belonging to $\hat{I}$. In each line r_{s+h} there are $\kappa\langle\rho\rangle - h$ asymptotic conditions, $h = 0, \ldots, \kappa\langle\rho\rangle - 1$. In Theorem 3.2 of [6]. we showed that a polynomial $q \in \Pi_n(\mathbf{R}^2)$ satisfies the vanishing asymptotic conditions on the lines $r_s, r_{s+1}, \ldots, r_{s+\kappa\langle\rho\rangle-1}$ if and only if it has degree $n - \kappa\langle\rho\rangle$ when restricted to any line with direction $\langle\rho\rangle$. The same reasoning works when we consider several groups of parallel lines. So, the subspace of polynomials in $\Pi_n(\mathbf{R}^2)$ satisfying the vanishing asymptotic conditions for $(i,j) \in I \setminus \hat{I}$ is $\Pi_n(D,\kappa)$.

The interpolation problem of finding $p \in \Pi_n(\mathbf{R}^2)$ such that $L_{ij}p$ has prescribed values for all $(i,j) \in \hat{I}$ and satisfying vanishing asymptotic conditions for all indices $(i,j) \in I \setminus \hat{I}$ has a unique solution. In other words, there is a unique polynomial $p \in \Pi_n(D,\kappa)$ such that $L_{ij}p$ takes prescribed values for all $(i,j) \in \hat{I}$. So, we have derived the same conclusions as in Section 3 from another point of view.

5 Examples

Figures 1 and 2 show different natural lattices. We have marked with a black circle the simple intersection which give rise to Lagrange interpolation data (value of the function at that point). We have surrounded a black circle by a concentric circle to indicate the intersection of three lines giving rise to first order Hermite data (value of the function and two partial derivatives of first order).

The interpolation space is $\Pi_4(\mathbf{R}^2)$ in Figure 1 left. In the case of Figure 1 right, the space is that of quartic polynomials which become cubic along two directions, namely the directions of the axes. The dimension of this space is 13.

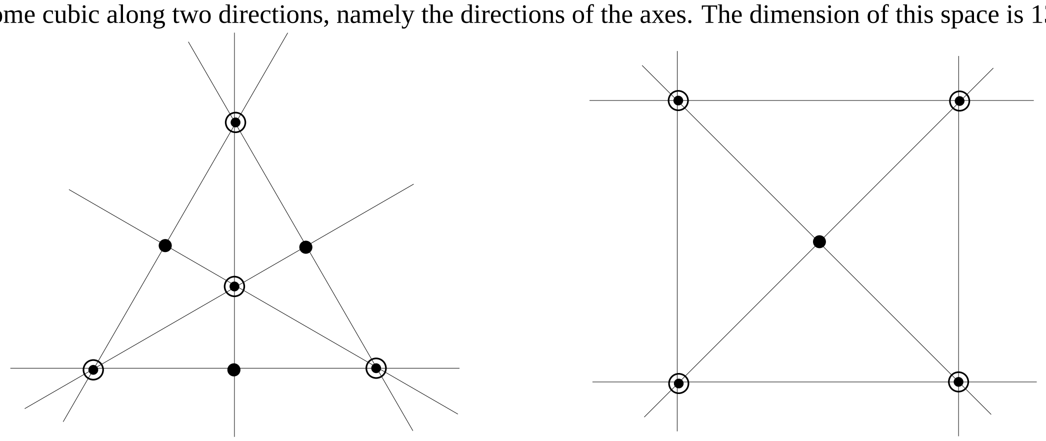

Figure 1. Two generalized natural lattices with 6 lines

In Figure 2 left, the interpolation space is the subspace of polynomials of degree not greater than seven, whose directional degree is not greater than five along the directions of the axes OX, OY and the bisector of the quadrant XOY. This space has dimension 27. Finally, in Figure 2 right, the interpolation space is that of sextic polynomials of degree not greater than five along the directions of the axes and their bisectors. This space has dimension 24.

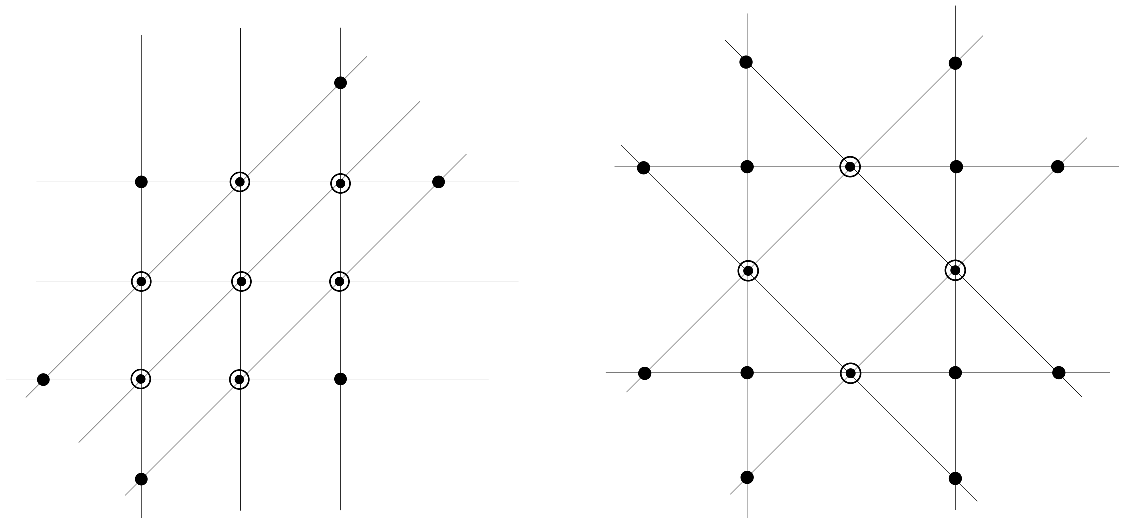

Figure 2. Two generalized natural lattices with 8 and 9 lines

Let us describe the construction of the solution of the problem corresponding to Figure 1 right. Denote by r_0, r_1 the lines of direction OX, r_2, r_3 the lines of direction OY and r_4, r_5 the diagonals of the rectangle determined by r_0, r_1, r_2, r_3. The index set (3.2) is

$$\hat{I} := \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3),(2,0),(2,1),(3,0),(3,1),(4,0)\}.$$

The interpolation points are

$$u_{ij} = r_i \cap r_{5-j}, \quad (i,j) \in \hat{I}.$$

Observe that the four vertices of the rectangle appear three times in the above list

$$u_{00} = u_{02} = u_{30}, \qquad u_{01} = u_{03} = u_{21}, \qquad u_{10} = u_{13} = u_{20}, \qquad u_{11} = u_{12} = u_{31}.$$

The Newton basis (3.4) is given by

$$\phi_{ij} = \prod_{h<i} r_h \prod_{p<j} r_{5-p}, \quad (i,j) \in \hat{I}.$$

Finally the linear forms $L_{ij}f$ defined in (3.9) are given by

$$\begin{aligned} f(u_{ij}), &\quad \text{for } (i,j) \in \{(0,0),(0,1),(1,0),(1,1),(4,0)\}, \\ \frac{\partial f}{\partial x}(u_{ij}), &\quad \text{for } (i,j) \in \{(0,2),(0,3),(1,2),(1,3)\}, \\ \frac{\partial f}{\partial y}(u_{ij}), &\quad \text{for } (i,j) \in \{(2,0),(2,1),(3,0),(3,1)\}. \end{aligned}$$

The problem can be solved using the recurrence (3.13). For the computation see Remark 2.

Acknowledgement.— Research partially supported by the Spanish Research Grant BFM2000–1253

References

[1] Busch, J. R. (1985). Osculatory interpolation in $\mathbf{R}^n$, *SIAM Journal of Numerical Analysis* **22**, 107–113.

[2] Boor, C. de, Dyn, N., and Ron, A. (2000). Polynomial interpolation to data on flats in $\mathbf{R}^d$, *J. Approx. Theory* **105**, 313–343.

[3] Carnicer, J. M. and Gasca, M. (1990). Evaluation of multivariate polynomials and their derivatives, *Math. Comp.* **54**, 231–243.

[4] Carnicer J. M. and Gasca, M. (1998). Bivariate polynomial interpolation with asymptotic conditions, in *Advances in Computational Mathematics*, Z. Chen, Y. Li, C.A. Micchelli and Y. Xu (eds.), 1–18. Marcel Dekker, New York.

[5] Carnicer, J. M. and Gasca, M. (2000). Bivariate polynomial Hermite-Birkhoff interpolation with asymptotic conditions, *J. of Comput. Appl. Math.* **119**, 69–79.

[6] Carnicer, J. M. and Gasca, M. (2002). Asymptotic conditions for degree diminution along prescribed lines, *preprint.*

[7] Chung, K. C. and Yao, T. H. (1977). On lattices admitting unique lagrange interpolation, *SIAM J. Numer. Anal.* **14**, 735–743.

[8] Dyn, N. and Ron, A. (1990). On multivariate polynomial interpolation, in *Algorithms for Approximation II*, J. C. Mason and M. E. Cox (eds.), 177–184. Chapman and Hall, London.

[9] Gasca, M. and Maeztu, J. I. (1982). On Lagrange and Hermite interpolation in $\mathbf{R}^k$, *Numer. Math.* **39**, 1–14.

[10] Gasca, M. and Sauer, T. (2000). Polynomial interpolation in several variables, *Adv. Comp. Math.*, **12**, 377–410.

[11] Hakopian, H. A. and Sahakian, A. A. (1995). Multivariate Polynomial Interpolation to Traces on Manifolds, *J. Approx. Theory* **80**, 50–75.

J. M. Carnicer and M. Gasca
Depto. Matemática Aplicada, Universidad de Zaragoza
Edificio de Matemáticas, Planta 1a. 50009 Zaragoza, Spain