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ON BIVARIATE HERMITE INTERPOLATION WITH MINIMAL DEGREE POLYNOMIALS

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Abstract. A Newton type approach is used to deal with bivariate polynomial Hermite interpolation problems when the data are distributed in the intersections of two families of straight lines, as a generalization of regular grids. The interpolation operator is degree reducing and the interpolation space is a minimal degree space. Integral remainder formulas are given for the Lagrange case, then extended to the Hermite case and finally used to obtain error estimates.

Key words. Polynomial interpolation, finite differences, reversible systems, error representation

AMS subject classifications. 65D05, 65D10, 41A05, 41A10

1. Introduction.

In [3] Gasca and Maeztu introduced a Newton type approach to deal with multivariate polynomial interpolation problems. In the bivariate case, from a given set of points distributed along straight lines, $r_0, r_1, \ldots, r_n$, a Birkhoff interpolation problem is constructed, including as particular cases Lagrange and Hermite interpolation problems. The interpolation space is always polynomial and is spanned by a Newton basis. The problem has a unique solution which is easily obtained by solving a triangular linear system.

A particular distribution of points was studied a little later by Maeztu [4] in order to provide the coefficients of the solution, in that case, with some properties similar to those of univariate divided differences. Since we are going to deal with this distribution, which on the other hand includes many important particular cases, we avoid the general notations of [3] and recall the necessary definitions of [4].

In general an interpolation problem is defined by an *interpolation space* V of dimension N and a set of N linear functionals L_i on V. The problem consists in finding an element p of V such that $L_i p = z_i \; \forall i$, where the z_i's are N given real values. Usually V is a subspace of a space F of functions, the L_i's are linear functionals on F and the problem is stated in the form $L_i p = L_i f \; \forall i$, for a given $f \in F$. The values $L_i f$ are called *interpolation data*. The interpolation problem is said to be *poised* if it has a unique solution for any set $\{z_i\}$, that is, for any f.

In Section 2 we state the problem and prove constructively that it is poised. In fact we prove that the problem is a particular case of the general one considered in [3] and remark some of the special properties of this case. More precisely, due to the special structure of the interpolation data, the linear system to solve this problem is not only lower triangular but also block lower triangular with diagonal blocks in the diagonal. Some well-known examples of this structure are given in Section 3.

In Section 4 we prove that in the problem we are considering, with the terminology of [6], the interpolation operator is degree reducing and the interpolation space is a minimal degree space for the problem. Divided differences and finite differences are introduced in the next section. The first ones are used to construct the solution of

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the problem recursively, while the main interest of finite differences, as introduced in [6,7,8], is to provide a first remainder formula for the problem. The relationship between both concepts is given.

Finally Section 6 is devoted to the development of the remainder formula from the preceeding section. First we study the Lagrange case and see a simplified formula which shows explicitly the role played by the geometry of the data. Then this formula is extended to the Hermite case by an argument of "coalescent lines". In the last subsection some error estimates are given.

2. Statement of the problem. Let $\Gamma = \{r_0, r_1, \dots, r_n,\}, \Gamma' = \{r'_0, r'_1, \dots, r'_m,\}$ be two indexed systems of straight lines in $\mathbb{R}^2$ such that each pair $(r_i, r'_j) \in \Gamma \times \Gamma'$ intersects at exactly one point u_{ij} of $\mathbb{R}^2$. In this form the product $\Gamma \times \Gamma'$ can also be interpreted as a system of $(n+1)(m+1)$ points $u_{ij} \in \mathbb{R}^2$. Observe there is no restriction on each set Γ, Γ' separately. Namely, parallelism and coincidence are allowed in Γ and in Γ', and consequently repetitions (which will be adequately interpreted) can happen in the set of points u_{ij}. Throughout this paper we will always denote points in $\mathbb{R}^2$ by $u = (x, y)$ or $u_{ij} = (x_{ij}, y_{ij})$, respectively.

An interpolation problem P can be constructed following [3] and [4] as soon as we choose some of the points u_{ij}, that is, when we take a set of indices

$$I = \{(i,j) \mid i = 0, 1, \dots, n; j = 0, 1, \dots, m(i)\} \tag{1}$$

under these conditions:

i) $m = m(0) \geq m(1) \geq \dots \geq m(n) \geq 0$.

ii) If $u_{hj} = u_{ij}$ for $(h,j), (i,j)$ both in I, then the lines r_h, r_i are coincident.

iii) If $u_{ik} = u_{ih}$ for $(i,k), (i,h)$ both in I, then the lines r'_k, r'_h are coincident.

Note that these conditions do not prevent, for example, the possibility of $u_{st} = u_{lt}$ (respectively $u_{st} = u_{sl}$) with r_s not coincident with r_l (resp. r'_t not coincident with r'_l), but in that case (l,t) (resp. (s,l)), should not belong to I.

In the sequel, r_i will denote, simultaneously, a straight line and an affine polynomial $a_i x + b_i y + c_i$ which vanishes at the points of that line. In other words, r_i denotes both the polynomial $a_i x + b_i y + c_i$ and the graph of the equation $a_i x + b_i y + c_i = 0$. Since this polynomial is fixed up to a constant factor, we assume in addition that the normal vector (a,b) is normalized with respect to the Euclidean norm and that either $a > 0$ or $a = 0$ and $b > 0$, which is no restriction on the generality of the interpolation problem. The same can be said, obviously, for the straight lines r'_j.

If we denote $S = (\Gamma \times \Gamma', I)$ we define a *Newton basis* B_S associated to S as the set of polynomials

$$B_S = \{\phi_{ij} \mid (i,j) \in I\} \tag{2}$$

with

$$\phi_{ij} = \prod_{h=0}^{i-1} r_h \prod_{k=0}^{j-1} r'_k. \tag{3}$$

As usual, the empty product ($i = 0$ or $j = 0$) is understood as 1.

For a vector $\rho = (a,b)$ (not necessarily unitary), different from zero, the *derivative of f at the point (x_i, y_i) in the direction ρ* will be denoted by

$$\frac{\partial f}{\partial \rho}(x_i, y_i) = a \frac{\partial f}{\partial x}(x_i, y_i) + b \frac{\partial f}{\partial y}(x_i, y_i).$$

Let ρ_i, ρ'_j, defined as $\rho_i = (-b_i, a_i), \rho'_j = (-b'_j, a'_j)$, be the directional vectors of r_i, r'_j respectively. According to our assumptions above, they are unitary with respect to the Euclidean norm, i.e., $\|\rho_i\| = \|\rho'_j\| = 1$.

The interpolation data associated to S are defined by

$$L_{ij} f = \frac{\partial^{s_i+t_j} f}{\partial {\rho'}_j^{s_i} \partial \rho_i^{t_j}} \left(u_{ij}\right), \tag{4}$$

where s_i (resp. t_j) is the number of lines r_h (r'_k) with $h < i$ ($k < j$) which are coincident with r_i (r'_j).

The interpolation problem P to be considered is the following: for a given set $\{z_{ij} \mid (i,j) \in I\}$ of real numbers, find a polynomial p in the space V_S spanned by B_S such that

$$L_{ij} p = z_{ij} \quad \forall (i,j) \in I. \tag{5}$$

Note that, under condition i) for I, this set can also be described as

$$I = \{(i,j) | j = 0, 1, \ldots, m; i = 0, 1, \ldots, n(j)\},$$

with $n = n(0) \geq n(1) \geq \ldots \geq n(m) \geq 0$. This fact and conditions ii) and iii) allow us to interchange the roles of the sets Γ, Γ' and the interpolation problem would be obviously the same. This was the reason to call S in [4] a *reversible system of interpolation*.

Observe also [4, Theorem 1] that if an interpolation datum of the form

$$\frac{\partial^{s+t} f}{\partial {\rho'}^s \partial \rho^t} (u),$$

with $s + t \neq 0$, appears in the set $\{L_{ij} p \mid (i,j) \in I\}$, then all

$$\frac{\partial^{\alpha+\beta} f}{\partial {\rho'}^\alpha \partial \rho^\beta} (u) \quad 0 \leq \alpha \leq s \quad 0 \leq \beta \leq t$$

will appear also in that set. In other words, our problem is a Hermite interpolation problem. In the terminology of [7], it is even a *regular Hermite interpolation problem* and therefore an *ideal interpolation scheme* in the sense of [1]. We remark that there is no "common" notion of a multivariate Hermite interpolation problem.

Let us consider on I the lexicographical order

$$(i,j) < (h,k) \quad \text{if} \quad i < h \quad \text{or} \quad i = h \quad \text{and} \quad j < k.$$

Theorem 1. *Under the above conditions, the interpolation problem P has a unique solution*

$$p = \sum_{(i,j) \in I} a_{ij} \phi_{ij}$$

in the space V_S, and the functions ϕ_{ij} satisfy, for any $(i,j), (h,k) \in I$,

$$L_{ij} \phi_{hk} = 0 \quad \textit{if} \quad (i,j) < (h,k), \tag{6}$$

$$L_{ij} \phi_{ij} \neq 0 \quad \forall (i,j), \tag{7}$$

$$L_{ij} \phi_{hk} = 0 \quad \textit{if} \quad (i,j) > (h,k) \quad \textit{and} \quad j < k. \tag{8}$$

Proof. The existence and uniqueness of p will be a consequence of (6) and (7), since these equations mean that the matrix

$$(L_{ij}\phi_{hk})_{(i,j),(h,k)\in I}$$

is lower triangular for the lexicographical order in I, with the diagonal entries different from zero. The proof of (6)–(8) is based on the following results which use the notations introduced above (see [3]) and are direct consequences of the definitions or of the Leibniz rule for differentiation:

a) If $i,h \in \{0,1,\ldots,n\}$, then

$$\frac{\partial r_h}{\partial \rho_i} = A_{hi} = -\frac{\partial r_i}{\partial \rho_h},$$

with $A_{hi} \neq 0$ except if r_i and r_h are parallel or coincident.

b) If $(i,j) \in I$ then

$$\frac{\partial r_i}{\partial \rho'_j} = -\frac{\partial r'_j}{\partial \rho_i} = B_{ij} \neq 0.$$

c) If w is any polynomial, then for any non-negative integer s and $(i,j) \in I$ one has

$$\frac{\partial^s r_i w}{\partial \rho_i^s} = r_i \frac{\partial^s w}{\partial \rho_i^s}, \tag{9}$$

$$\frac{\partial^s r_i^s w}{\partial {\rho'_j}^s} = s! B_{ij}^s w + r_i w_1, \tag{10}$$

$$\frac{\partial^s r_i^{s+1} w}{\partial {\rho'_j}^s} = r_i w_2. \tag{11}$$

Here w_1, w_2 are polynomials.

d) If $v_1,\ldots,v_k$ are affine polynomials and w is an arbitrary polynomial, then for any non-negative integer s and any vector $\rho \in \mathbb{R}^2$ one has

$$\frac{\partial^s v_1 \cdots v_k w}{\partial \rho^s} = \sum_{t=0}^{k} t! \binom{s}{t} \frac{\partial^{s-t} w}{\partial \rho^{s-t}} \sum_{h_1,\ldots,h_t} \frac{\partial v_{h_1}}{\partial \rho} \cdots \frac{\partial v_{h_t}}{\partial \rho} \frac{v_1 \cdots v_k}{v_{h_1} \cdots v_{h_t}},$$

where the summation $\sum_{h_1,\ldots,h_t}$ ranges over all subsets of $\{1,2,\ldots,k\}$ having t different elements $h_1 < \ldots < h_t$. When $t=0$ this summation reduces to $v_1 \cdots v_k$.

Equations (6), (7) are a particular case of the general situation considered in [3, Theorem 1] and therefore they are proved exactly like there.

On the contrary, (8) does not hold for that general situation. In the present problem, L_{ij} is given by (4), and if $j<k$ then ϕ_{hk} contains at least t_j+1 factors r'_j. Hence, as in (11), we have

$$\frac{\partial^{t_j} \phi_{hk}}{\partial \rho_i^{t_j}} = r'_j w_2,$$

with w_2 a polynomial, and, by (9),

$$L_{ij}\phi_{hk} = \frac{\partial^{s_i} r'_j w_2}{\partial {\rho'_j}^s}(u_{ij}) = r'_j \frac{\partial^{s_i} w_2}{\partial {\rho'_j}^s}(u_{ij}) = 0,$$

that is (8). $\square$

Remark. Equations (6)-(8) can also be written in the form

$$L_{ij}\phi_{hk} = 0 \qquad \text{if} \qquad i < h \text{ and } j > k,$$

$$L_{ij}\phi_{hk} = 0 \qquad \text{if} \qquad i > h \text{ and } j < k,$$

$$L_{ij}\phi_{hk} = \delta_{(i,j),(h,k)} L_{hk}\phi_{hk} \qquad \text{if} \qquad i \le h \text{ and } j \le k,$$

with $L_{hk}\phi_{hk} \neq 0$, $(h,k) \in I$. As usual, $\delta_{(h,k)(i,j)}$ takes the value 1 if $(h,k) = (i,j)$ and 0 elsewhere.

Let us now order the index set I in the form

$$(i,j) \prec (h,k) \quad \text{if} \quad i+j < h+k \quad \text{or} \quad i+j = h+k \quad \text{and} \quad i < h. \tag{12}$$

This ordering is usually called the *graded lexicographical order*. According to that we make the following partition:

$$I = J_1 \cup J_2 \cup \ldots \cup J_M, \tag{13}$$

with

$$J_r = \{(i,j) \in I \mid i+j = r\}$$

and

$$M = \max\{i+j \mid (i,j) \in I\}.$$

Then the collocation matrix of our problem

$$(L_{ij}\phi_{hk})_{(i,j),(h,k)\in I}$$

with the graded lexicographical order $\prec$ in I is not only lower triangular but also *block lower triangular* with diagonal blocks in the diagonal. This implies that the same would happen if each part J_r of I in (13) is ordered separately in any other form, for example $i > h$ in (12) instead of $i < h$.

We remark that the space V_S is a subspace of Π_M, the space of bivariate polynomials of total degree not greater than M.

3. Some examples. A simple and well-known example of the problem P arises when we take the lines r_i, r'_j parallel to the coordinate axes. If we consider the lines

$$\begin{aligned} r_i(u) &:= x - x_i, \qquad 0 \le i \le n, \\ r'_j(u) &:= y - y_j, \qquad 0 \le i \le n, \end{aligned} \tag{14}$$

then the points u_{ij} are

$$u_{ij} = (x_i, y_j), \quad (i,j) \in I, \tag{15}$$

the Newton basis is formed by

$$\phi_{ij}(u) = \prod_{h=0}^{i-1}(x - x_h)\prod_{k=0}^{j-1}(y - y'_k) \tag{16}$$

and the interpolation space V_S is spanned by the monomials

$$W = \{x^\alpha y^\beta, \quad (\alpha,\beta) \in I\}.$$

Observe that due to the conditions of our problem one has

$$x^\alpha y^\beta \in W \Rightarrow x^{\alpha'} y^{\beta'} \in W \quad \forall \alpha' \leq \alpha, \beta' \leq \beta.$$

In particular, if $m(i) = m \quad \forall i$, then V_S is the tensor product of the univariate polynomial spaces $\Pi_n(x)$ and $\Pi_m(y)$, and if $m = n$ and $m(i) = n - i \quad \forall i$, then V_S is the bivariate space Π_n of polynomials of total degree not greater than n.

The linear functionals L_{ij} are, in these problems,

$$L_{ij} f = \frac{\partial^{s_i + t_j} f}{\partial x^{s_i} \partial y^{t_j}} (x_i, y_j), \tag{17}$$

where s_i (respectively t_j) is the number of times that the value x_i (resp. y_j) appears in the list $\{x_0, x_1, \ldots, x_{i-1}\}$ ($\{y_0, y_1, \ldots, y_{j-1}\}$).

In the case that all the x_i's and the y_j's are different, then one has $(x_i, y_j) \neq (x_h, y_k)$ for $(i, j) \neq (h, k)$ and

$$L_{ij} f = f(x_i, y_j) \quad \forall (i, j),$$

that is, a Lagrange interpolation problem.

If we want to interpolate a function f on a set of N given points

$$X = \{(\mu_i, \nu_i) \mid 1 \leq i \leq N\},$$

all them different, we can easily check if they are distributed on lines parallel to the axes according to (14)-(15). First we check how many different ordinates appear among the points of X, say $m + 1$. Then we denote by $y_0, y_1, \ldots, y_m$ these ordinates, ordered by decreasing number of points of X on each of them, say $n(0) \geq n(1) \geq \ldots \geq n(m)$. If several ordinates have the same number of points, then the relative order among them is irrelevant. Afterwards we do the same for the abscissae, denoted by $x_0, x_1, \ldots x_n$, ordered by decreasing number of points $m(0) \geq m(1) \geq \ldots \geq m(n)$.

These orderings can always be done. Now the problem belongs to the class we are considering in this paper if and only if the set X coincides with the set

$$\{(x_i, y_j) \mid 0 \leq i \leq n, 0 \leq j \leq n(i)\} \tag{18}$$

(or equivalently with $\{(x_i, y_j) \mid 0 \leq j \leq m, 0 \leq i \leq m(j)\}$). For example, the set of points $X = \{(0,0), (1,0), (0,1), (2,1)\}$, with $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $y_0 = 0$, $y_1 = 1$, cannot be put in the form (18).

Repetitions of lines in (14) give rise to Hermite interpolation problems (see (17)). Reciprocally, suppose we have a problem such that if

$$\frac{\partial^{s+t} f}{\partial x^s \partial y^t} (a, b)$$

is an interpolation datum, then all

$$\frac{\partial^{h+k} f}{\partial x^h \partial y^k} (a, b) \quad h \leq s, k \leq t \tag{19}$$

are also data. In this case we should check whether or not the problem can be stated in the form (14)-(15). For it we can proceed similarly to the Lagrange case but the partial derivatives with respect to x or y should be interpreted as repetitions of abscissae or ordinates respectively. For example, a problem with the data

$$\left\{ f(u_i), \frac{\partial f}{\partial x}(u_i), \frac{\partial f}{\partial y}(u_i), \quad 0 \le i \le 2 \right\}, \tag{20}$$

with $u_0 = (0,0)$, $u_1 = (1,0)$, $u_2 = (0,1)$, satisfies (19) but cannot be put in the form (14)-(17) because it does not satisfy (18). In fact, it has four data for $y_0 = 0$, $((f(0,0), \frac{\partial f}{\partial x}(0,0), f(1,0), \frac{\partial f}{\partial x}(1,0))$, two data for $y_1 = 0$, $(\frac{\partial f}{\partial y}(0,0), \frac{\partial f}{\partial y}(1,0))$, two data for $y_2 = 1$ $(f(0,1), \frac{\partial f}{\partial x}(0,1))$ and one datum for $y_3 = 1$ $(\frac{\partial f}{\partial y}(0,1))$. Analogously, the abscissae are $x_0 = x_1 = 0$, $x_2 = x_3 = 1$. However, according to (18), since $\frac{\partial f}{\partial y}(1,0)$ is one of the data of the problem, corresponding to (x_2, y_1), then (x_1, y_0) (and correspondingly $\frac{\partial^2 f}{\partial x \partial y}(0,0)$) should also appear among the data, what obviously does not happen. This example corresponds to Figure 1a, where, as usual in finite elements the arrows mean partial derivatives.

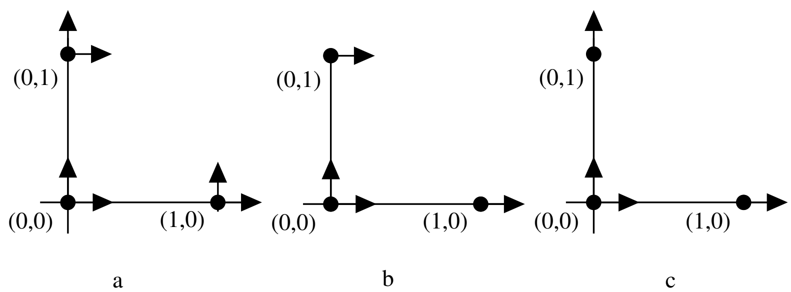

Figure 1

On the contrary, it is easy to see that the problems described in Figures 1b and 1c can be put in the form (14)-(17). In the case of Figure 1b we can choose $y_0 = y_2 = 0, y_1 = 1$, $x_0 = x_1 = 0$, $x_2 = x_3 = 1$, with $m(0) = 4, m(1) = 2, m(2) = 1$, and the interpolation space spanned by

$$\{1, x, x^2, x^3, y, xy, y^2\}.$$

In the case of Figure 1c we can take $y_0 = y_1 = 0$, $y_2 = y_3 = 1$, $x_0 = x_1 = 0$, $x_2 = x_3 = 1$, with $m(0) = 4, m(1) = m(2) = m(3) = 1$, and the interpolation space spanned by

$$\{1, x, x^2, x^3, y, y^2, y^3\}.$$

The case of lines not parallel to the axes can be treated similarly but in practice is, obviously, more complicated.

4. Degree reducing interpolation operators and minimal degree interpolation spaces. Recall that we have defined

$$M = \max\{i + j \mid (i,j) \in I\}$$

and complete Γ, Γ', if necessary, with *arbitrary* lines $r_{n+1}, \ldots, r_M$ and $r'_{m+1}, \ldots, r'_M$, according to the same conditions as in Section 2 for the lines $r_0, \ldots, r_n$, and $r'_0, \ldots, r'_m$. We denote these new sets by

$$\Gamma^* = \{r_0, \ldots, r_M\}, \qquad \Gamma'^* = \{r'_0, \ldots, r'_M\}, \tag{21}$$

and also set $m^*(i) = M - i$, $i = 0, \ldots, M$. Obviously this can be done in many ways and we consider an arbitrary one. Now we can easily prove the following

PROPOSITION 2. *Under the above conditions, the set of functions*

$$\phi_{ij}^* := \prod_{h=0}^{i-1} r_h \prod_{k=0}^{j-1} r'_k, \quad i + j \le M, \tag{22}$$

is a basis of the space Π_M.

Proof. Let us consider a new interpolation problem P^* defined similarly to P in Section 2 in the following form. The problem P^* has

$$\{L_{ij} f \mid i + j \le M\},$$

defined as in (4), and $\{\phi_{ij} \mid i + j \le M\}$ defined as in (3), as interpolation data and Newton basis respectively. Observe that this notation is consistent with that of the problem P, because this one is a "subproblem" of P^* : the polynomial system $\left\{\phi_{ij}^* \mid i + j \le M\right\}$ is an extension of the polynomial system $\{\phi_{ij} \mid (i,j) \in I\}$ because one has $\phi_{ij}^* = \phi_{ij}$, $(i,j) \in I$.

According to [3,4] the problem P^* has a unique solution in the space spanned by $\{\phi_{ij} \mid i + j \le M\}$ and these functions form a Newton basis for that space. Since the cardinality of this basis coincides with the dimension of Π_M, the space spanned by the ϕ_{ij}'s is also Π_M. $\square$

For the remainder of the paper all the functions ϕ_{ij}^* of (22) will be denoted by ϕ_{ij} when this does not cause any confusion. For each $r = 0, 1 \ldots, M$ we denote, as in [6],

$$I_r = \{(i,j) \in I \mid i + j \le r\}$$

and

$$I'_r = \{(i,j) \notin I \mid i + j \le r\}.$$

Observe that for all r

$$I_r \cup I'_r = \{(i,j) \mid i + j \le r\}, \qquad I_r \cap I'_r = \emptyset, \qquad I_M = I,$$

and for J_r introduced in (13)

$$J_r = I_r \setminus I_{r-1}.$$

The interpolation problem P is defined by the linear functionals (4) as interpolation data and the polynomial space V_S spanned by (2) as interpolation space. Consider the space F of functions f such that the linear functionals (4) applied to f are well defined. The interpolation operator $L(\cdot, P)$ associates to each $f \in F$ the solution $p \in V_S$ of the problem

$$L_{ij} p = L_{ij} f \quad \forall (i,j) \in I.$$

So we have $L(f, P) = p$.

According to [6] the interpolation operator $L(\cdot, P)$ is said to be *degree reducing* if for each $q \in \Pi_r$, $r = 0, 1, \ldots M$, the interpolating polynomial $L(q, P)$ also belongs to Π_r. The space V_S, which is included in Π_M, is said to be a *minimal degree* interpolation space if there is no subspace V of Π_{M-1} such that the interpolation problem of finding $p \in V$ satisfying

$$L_{ij} p = L_{ij} f \quad \forall (i,j) \in I$$

is poised.

THEOREM 3. *The interpolation operator $L(\cdot, P)$ defined by the problem P is degree reducing and the space V_S spanned by $\{\phi_{ij} \mid (i,j) \in I\}$ is a minimal degree interpolation space for P.*

Proof. ¿From Proposition 2 we easily deduce that, for each $r = 0, 1, \ldots, M$, the set of functions $\{\phi_{ij} \mid i + j \leq r\}$ is a basis of Π_r. Consequently, any $q \in \Pi_r$ can be written in the form

$$q = \sum_{(i,j)\in I_r} \alpha_{ij}\phi_{ij} + \sum_{(i,j)\in I'_r} \alpha_{ij}\phi_{ij},$$

and therefore, due to the linearity of the problem P,

$$L(q, P) = \sum_{(i,j)\in I_r} \alpha_{ij} L(\phi_{ij}, P) + \sum_{(i,j)\in I'_r} \alpha_{ij} L(\phi_{ij}, P). \tag{23}$$

From the poisedness of P we have that, for any $(i,j) \in I$ (and in particular for $(i,j) \in I_r$),

$$L(\phi_{ij}, P) = \phi_{ij}. \tag{24}$$

On the other hand, let $(i,j) \in I'_r$, $(v,w) \in I$ and consider again the problem P^* of the proof of Proposition 2. If $(v,w) < (i,j)$ then from (6) applied to the problem P^* we get $L_{vw}\phi_{ij} = 0$. If $(v,w) > (i,j)$, then $v > i$, and therefore condition i) of I (see Section 2) and the fact that $(i,j) \in I'_r$ imply that $w < j$. Hence we can also use (8) applied to the problem P^* to get $L_{vw}\phi_{ij} = 0$. In summary, for any $(i,j) \in I'_r$ one has

$$L_{vw}\phi_{ij} = 0, \qquad (v,w) \in I,$$

and consequently

$$L(\phi_{ij}, P) = 0. \tag{25}$$

¿From (24), (25) and (23) we get

$$L(q, P) = \sum_{(i,j)\in I_r} \alpha_{ij}\phi_{ij} \in \Pi_r,$$

that is, $L(\cdot, P)$ is degree reducing.

For the minimal degree property we just have to prove that the collocation matrix formed with $\{L_{ij} \mid (i,j) \in I\}$ and *any* basis of Π_{M-1} has rank less than $\#I$ (the cardinality of I). Let us take the basis

$$\{\phi_{ij} \mid i + j \leq M - 1\} = \{\phi_{ij} \mid (i,j) \in I_{M-1}\} \cup \left\{\phi_{ij} \mid (i,j) \in I'_{M-1}\right\}.$$

As in (25), we easily see that in the matrix $(L_{hk}\phi_{ij})$ (with rows indexed by $(h,k) \in I$ and columns indexed by (i,j) with $i + j \leq M - 1$) all the columns corresponding to $\left\{\phi_{ij} \mid (i,j) \in I'_{M-1}\right\}$ vanish, and therefore the rank R of that matrix is less than or equal to $\#I_{M-1}$. Since at least one of the indices $(i,j) \in I$ belongs to I_M and not to I_{M-1} by definition of M, we have

$$R \leq \#I_{M-1} < \#I_M = \#I.$$

Hence, the interpolation problem defined by

$$\{L_{ij} \mid (i,j) \in I\}$$

and any subspace of Π_{M-1} can not be poised and V_S is a minimal degree space. □

Let us denote by P_r, $r = 0, \dots, M$, the interpolation problem defined by the functionals L_{ij}, $(i,j) \in I_r$, and the polynomial space spanned by $\{\phi_{ij} \mid (i,j) \in I_r\}$. Since all the properties of the interpolation problem P carry over to these subproblems we have the following immediate consequence of Theorem 3.

COROLLARY 4. *For $r = 0, \dots, M$, the interpolation operator $L(\cdot, P_r)$ defined by the problem P_r is degree reducing and the space spanned by $\{\phi_{ij} \mid (i,j) \in I_r\}$ is a minimal degree interpolation space for P_r.*

5. Divided differences, finite differences and the computation of the solution. As we have seen, the solution p (or $L(f,P)$) of an interpolation problem of the type we are considering can be written in the form

$$p = \sum_{(i,j) \in I} a_{ij} \phi_{ij} \tag{26}$$

and $\{\phi_{ij} \mid (i,j) \in I\}$ is a Newton basis. In [4] the coefficients a_{ij} were denoted by $\left[r_0, r_1, \dots, r_i \mid r_0', r_1', \dots, r_j'\right] f$ and called *divided differences associated to the reversible system S*. In the same paper, a complicated recurrence relation was obtained for them. The name "divided differences" was due to the fact that the Newton formula we have for p can be considered as an extension of the univariate Newton interpolation formula. However, no remainder formula was obtained in [4].

On the other hand, in [7] Sauer and Xu introduced the concept of *finite differences* for an interpolation problem with a *blockwise* or graded Newton basis. They used this concept to get some interesting remainder formulas. In this section we adapt both concepts to our problem, show the relationship between them and obtain a remainder formula which will be further developed in Section 6.

The special structure of the collocation matrix

$$\left(L_{ij} \phi_{hk}\right)_{(i,j),(h,k) \in I},$$

which is lower triangular (as seen in Section 2) when I is lexicographically ordered allows us to compute the coefficients a_{ij} recursively. They can be computed in the form

$$a_{ij} = \frac{L_{ij} f - \sum_{(h,k) < (i,j)} a_{hk} L_{ij} \phi_{hk}}{L_{ij} \phi_{ij}}, \tag{27}$$

with $(i,j) \in I$ linearly ordered by $<$, as it can be done in any interpolation problem considered in [3]. However, as we have also seen in Section 2, in the present problem the collocation matrix is block lower triangular with diagonal blocks in the diagonal when we use the *graded* lexicographical order $\prec$. Therefore the coefficients can be computed recursively by

$$a_{ij} = \frac{L_{ij} f - \sum_{(h,k) \in I_{i+j-1}} a_{hk} L_{ij} \phi_{hk}}{L_{ij} \phi_{ij}}, \tag{28}$$

once we have computed all coefficients a_{hk}, $(h,k) \in J_{i+j-1}$, and so on. Note that all coefficients a_{rs} with $(r,s) \in J_{i+j}$ can be computed simultaneously.

In order to see the relationship between these coefficients and the finite differences introduced in [7] the latter order of computation will be more convenient. With this aim, we denote

$$\lambda_{ij}^{i+j}(P) f = a_{ij}, \tag{29}$$

and observe from (28) that the following algorithm provides all the coefficients of the solution p of our problem:

For $(i,j) \in J_0 \cup J_1 \cup \ldots \cup J_M$

$$\lambda^0_{ij}(P)f = \frac{L_{ij}f}{L_{ij}\phi_{ij}}. \tag{30}$$

For $r = 0, 1, \ldots, M-1$:
For $(i,j) \in J_{r+1} \cup J_{r+2} \cup \ldots \cup J_M$,

$$\lambda^{r+1}_{ij}(P)f = \lambda^0_{ij}(P)f - \sum_{(h,k)\in I_r} \lambda^r_{hk}(P)f \frac{L_{ij}\phi_{hk}}{L_{ij}\phi_{ij}}. \tag{31}$$

End

In order to introduce here the finite differences of [7] we need (a basis of) polynomials $\{p_{ij} \mid (i,j) \in I\}$ such that

$$L_{hk}p_{ij} = 0, \quad \text{if} \quad h+k < i+j, \tag{32}$$

$$L_{hk}p_{ij} = \delta_{(h,k)(i,j)}, \quad \text{if} \quad h+k = i+j. \tag{33}$$

Taking into account (6)-(8), the basis $\{p_{ij} \mid (i,j) \in I\}$ can be obtained as a normalization of the basis $\{\phi_{ij} \mid (i,j) \in I\}$ by setting

$$p_{ij} = \frac{\phi_{ij}}{L_{ij}\phi_{ij}}.$$

This is stated more precisely in the following proposition. First we need some more notation: let $\eta_i = (a_i, b_i)$ and $\eta'_j = (a'_j, b'_j)$ be the normal directions of r_i and r'_j, respectively, which, by assumption, are unitary with respect to the Euclidean norm. As usual, we denote by $\langle\cdot,\cdot\rangle$ the Euclidean scalar product. Finally, for any two points $u, v \in \mathbb{R}^2$ we define

$$\delta(u,v) = \|u-v\| + \delta_{u,v},$$

which equals the (Euclidean) distance between the two points u, v except for the case when both points coincide, where we have $\delta(u,u) = 1$. The numbers s_i and t_j which appear in the proposition were defined in Section 2.

Proposition 5. *For $(i,j) \in I$ there exists $\sigma_{ij} \in \{-1, 1\}$, such that*

$$p_{ij} = \frac{\sigma_{ij}}{s_i!t_j!} \prod_{h=0}^{i-1} \frac{r_h}{\langle \eta_h, \rho'_j\rangle\, \delta\left(u_{ij}, u_{hj}\right)} \prod_{k=0}^{j-1} \frac{r'_k}{\langle \rho_i, \eta'_k\rangle\, \delta\left(u_{ij}, u_{ik}\right)}. \tag{34}$$

To verify the proposition, we will have to prove that $L_{ij}\phi_{ij}$ equals the denominator of the right-hand side of (34). For this purpose, we take into account the following lemma.

Lemma 6. *Let $(i,j), (h,k) \in I$ and suppose $r_h \neq r_i$ and $r'_k \neq r'_j$. Then there exist numbers $\sigma_{ijh}, \sigma'_{ijk} \in \{-1, 1\}$ such that*

$$\begin{aligned} r_h\left(u_{ij}\right) &= \sigma_{ijh}\left\langle \eta_h, \rho'_j\right\rangle \left\|u_{ij} - u_{hj}\right\|, \\ r'_k\left(u_{ij}\right) &= \sigma'_{ijk}\left\langle \rho_i, \eta'_k\right\rangle \left\|u_{ij} - u_{ik}\right\|. \end{aligned} \tag{35}$$

Proof. We prove only the first identity, the second one is proved in the same way. Recalling that $r_h(u) = \langle \eta_h, u \rangle + c_h$ and that

$$u_{ij} - u_{hj} = \sigma_{ijh} \|u_{ij} - u_{hj}\| \rho'_j,$$

since both points are on r'_j, we have that

$$\begin{aligned} r_h(u_{ij}) &= \langle \eta_h, u_{ij} + u_{hj} - u_{hj} \rangle + c_h = \langle \eta_h, u_{hj} \rangle + c_h + \langle \eta_h, u_{ij} - u_{hj} \rangle \\ &= r_h(u_{hj}) + \sigma_{ijh} \left\langle \eta_h, \rho'_j \right\rangle \|u_{ij} - u_{ik}\| \\ &= \sigma_{ijh} \left\langle \eta_h, \rho'_j \right\rangle \|u_{ij} - u_{ik}\| . \end{aligned}$$

□

Proof of Proposition 5. Now we can proceed with the proof of the proposition. In the case of simple lines without repetitions (i.e. P is a Lagrange interpolation problem), (34) is a direct consequence of Lemma 6 because $s_i = t_j = 0$.

In the case of repeated lines we have to be a bit more careful. Indeed, suppose that for some $(i,j) \in I$ the lines r_i and r'_j have already appeared s_i and t_j times in the sets $\{r_k \mid k < i\}$ and $\{r'_k \mid k < j\}$, respectively. The interpolation datum is

$$\frac{\partial^{s_i + t_j}}{\partial {\rho'}_j^{s_i} \partial \rho_i^{t_j}} (u_{ij})$$

and the basis function can be written as

$$\phi_{ij} = r_i^{s_i} {r'_j}^{t_j} \prod_{r_h \neq r_i} r_h \prod_{r'_k \neq r'_j} r'_k =: r_i^{s_i} {r'_j}^{t_j} \psi_{jk}.$$

Recalling statement a) in the proof of Theorem 1 we note that for any $i, j \leq M$ and any $s, t \geq 1$

$$\frac{\partial r_i^s}{\partial \rho_i^s} = \frac{\partial {r'_j}^t}{\partial {\rho'_j}^t} = 0.$$

Applying the Leibniz formula we first obtain

$$\frac{\partial^{t_j} \phi_{ij}}{\partial {\rho_i}^{t_j}} = r_i^{s_i} \frac{\partial^{t_j}}{\partial {\rho_i}^{t_j}} \left({r'_j}^{t_j} \psi_{jk} \right) = r_i^{s_i} \sum_{t=0}^{t_j} \binom{t_j}{t} \frac{t_j!}{t!} \left(\frac{\partial r'_j}{\partial \rho_i} \right)^{t_j - t} {r'_j}^t \frac{\partial^t \psi_{ij}}{\partial {\rho_i}^t}$$

and then

$$\begin{aligned} &\frac{\partial^{s_i + t_j} \phi_{ij}}{\partial {\rho'_j}^{s_i} \partial {\rho_i}^{t_j}} \\ &\qquad = \sum_{s=0}^{s_i} \sum_{t=0}^{t_j} \binom{s_i}{s} \binom{t_j}{t} \frac{s_i! t_j!}{s! t!} \left(\frac{\partial r_i}{\partial \rho'_j} \right)^{s_i - s} \left(\frac{\partial r'_j}{\partial \rho_i} \right)^{t_j - t} r_i^s {r'_j}^t \frac{\partial^{s+t} \psi_{ij}}{\partial {\rho'_j}^s \partial {\rho_i}^t}. \end{aligned}$$

Since $r_i(u_{ij}) = r'_j(u_{ij}) = 0$, all terms of the above sum except $s = t = 0$ vanish when evaluated at u_{ij} and, consequently,

$$L_{ij} \phi_{ij} = \frac{\partial^{s_i + t_j} \phi_{ij}}{\partial {\rho'}_j^{s_i} \partial \rho_i^{t_j}} (u_{ij}) = s_i! t_j! \left(\frac{\partial r_i}{\partial \rho'_j} \right)^{s_i} \left(\frac{\partial r'_j}{\partial \rho_i} \right)^{t_j} \psi_{ij}(u_{ij}). \tag{36}$$

Now (34) follows easily from Lemma 6 taking into account that

$$\frac{\partial r_i}{\partial \rho'_j} = \left\langle \eta_i, \rho'_j \right\rangle, \qquad \frac{\partial r'_j}{\partial \rho_i} = \left\langle \rho_i, \eta'_j \right\rangle, \qquad (i,j) \in I.$$

□

We can also define finite differences associated to the minimal degree Hermite interpolation problem P and give recurrence relations for them. For that purpose we let $P_{-1} = \emptyset$ be the (trivial) problem of interpolating no data with the zero polynomial. Then we define the finite differences $\lambda_r \left[P_{r-1}; u\right] f$, $r = 0, \dots, M+1$, as

$$\begin{aligned} \lambda_0 \left[\emptyset; u\right] f &= f(u), \\ \lambda_{r+1} \left[P_r; u\right] f &= \lambda_r \left[P_{r-1}; u\right] f - \sum_{(i,j)\in J_r} L_{ij} \lambda_r \left[P_{r-1}; \cdot\right] f \; p_{ij}(u). \end{aligned} \tag{37}$$

This is, of course, not what one would expect of a difference at first glance: the iteration asks us to consider the finite difference as a function of the additional argument $u = (x,y)$ and (in the case of a Hermite interpolation scheme) requires *directional derivatives* of this function. However, they allow us to get a first formula for the remainder.

THEOREM 7. *For $r = 0, \dots, M$ we have*

$$L\left(f, P_r\right)(u) = \sum_{(i,j)\in I_r} L_{ij} \lambda_{i+j} \left[P_{i+j-1}; u\right] f \; p_{ij}(u) \tag{38}$$

and

$$f(u) - L\left(f, P_r\right)(u) = \lambda_{r+1} \left[P_r; u\right] f. \tag{39}$$

Proof. We first remark that iteration of (37) immediately implies the formula

$$\lambda_r \left[P_{r-1}; \cdot\right] f = f - \sum_{(i,j)\in I_{r-1}} L_{ij} \lambda_{i+j} \left[P_{i+j-1}; \cdot\right] f \; p_{ij}. \tag{40}$$

Because of Corollary 4 and the structure of the polynomials p_{ij}, in order to prove (38), it suffices to show that for any $(h,k) \in I_r$ we have

$$\sum_{(i,j)\in I_r} L_{ij} \lambda_{i+j} \left[P_{i+j-1}; \cdot\right] f \; L_{hk} p_{ij} = L_{hk} f. \tag{41}$$

For that purpose we apply L_{hk} to both sides of (40) and recall that $L_{hk} p_{ij} = \delta_{(h,k),(i,j)}$, $(i,j) \in J_r$, to obtain

$$\begin{aligned} L_{hk} f - \sum_{(i,j)\in I_{r-1}} L_{ij} \lambda_{i+j} \left[P_{i+j-1}; \cdot\right] f \; L_{hk} p_{ij} &= L_{hk} \lambda_r \left[P_{r-1}; \cdot\right] f \\ &= \sum_{(i,j)\in J_r} L_{ij} \lambda_r \left[P_{r-1}; \cdot\right] f \; L_{hk} p_{ij}, \end{aligned}$$

that is (41). Substituting (38) into (40) then gives (39). □

The relationship between divided differences and finite differences becomes clear:

COROLLARY 8. *For all $(i,j) \in I$ one has*

$$\lambda_{ij}^{i+j}(P)f = \frac{L_{ij}\lambda_{i+j}[P_{i+j-1};\cdot]f}{L_{ij}\phi_{ij}}. \tag{42}$$

6. Remainder formulas. In this section we will describe the interpolation error $f - L(f,P)$ in terms of derivatives of the function f, assuming that $f \in C^{M+1}(\Omega)$, where, as before, $M = \max\{i+j \,|\, (i,j) \in I\}$ and $\Omega \subset \mathbb{R}^2$ is any convex and compact set with nonempty interior such that $\{u_{ij} \,|\, (i,j) \in I\} \subset \Omega$. This formula will be obtained from the more general approach in [6] for minimal degree Lagrange interpolation which will be recalled together with the underlying notation in the first subsection. We will then specialize this formula to reversible Lagrange interpolation systems which will reveal a very appealing connection with the underlying geometry. Thereafter we show by a limit argument of coalescent lines this formula remains valid when passing from the Lagrange to the Hermite case, which gives the general result we are heading for. Finally, we will briefly comment on error estimates for reversible systems which follow directly from the error formula.

6.1. General preliminaries. In order to give the remainder formula we need some more terminology. For any set $\{x_0,\ldots,x_N\}$ of (not necessarily distinct) points in $\mathbb{R}^2$ we define the *simplex spline integral* (cf. [5]) as the functional

$$\begin{aligned}\int_{[x_0,\ldots,x_N]} f &= \int_{\Delta_N} f\left(\sigma_0 x_0 + \cdots + \sigma_N x_N\right) d\sigma_1 \cdots d\sigma_N \\ &=: \frac{1}{N!}\int_{\mathbb{R}^2} f(t) M\left(t|x_0,\ldots,x_N\right) dt,\end{aligned} \tag{43}$$

where

$$\Delta_N = \left\{\sigma \in \mathbb{R}^{N+1} \mid \sigma_j \geq 0,\, \sigma_0 + \cdots + \sigma_N = 1\right\}.$$

The (normalized) function $M\left(\cdot|x_0,\ldots,x_N\right)$ is called the *simplex spline* with knots $x_0,\ldots,x_N$. Clearly, the simplex spline integral is symmetric in the knots.

In general we shall use notations and concepts similar to those of [6,7,8] with some slight modifications. In this respect, see also [2].

Remember that in Section 2 we have denoted

$$J_r = \{(i,j) \in I \mid i+j = r\}, \quad J_r' = \{(i,j) \notin I \mid i+j = r\}.$$

A *path μ of length* $r\,(0 \leq r \leq M)$ *in I* is defined as a vector $\mu = (\mu_0,\mu_1,\ldots,\mu_r)$ such that $\mu_s \in J_s$, $s = 0,\ldots,r$. The set of all paths of length r in I will be denoted by $\Lambda_r(I)$. Similarly, a path in I' is a vector $\mu = (\mu_0,\mu_1,\ldots,\mu_r)$ whose first r components form a path of length $r-1$ in I and the last component μ_r belongs to J_r' :

$$\Lambda_r(I') = \{(\mu,\mu_r) \mid \mu \in \Lambda_{r-1}(I),\, \mu_r \in J_r'\}.$$

These notions, which have been introduced in [6,7,8], have turned out to be crucial for remainder formulas for general Lagrange and Hermite interpolation problems.

With any path μ in either $\Lambda_r(I)$ or $\Lambda_r(I')$ and $s \leq r$ we associate the collection of points

$$X_\mu^s = \{u_{\mu_0},\ldots,u_{\mu_s}\}$$

and the number

$$\pi_\mu = \prod_{j=0}^{r-1} L_{\mu_{j+1}} p_{\mu_j},$$

with the convention that for $\mu \in \Lambda_r(I')$ the point u_{μ_r} and the functional L_{μ_r} are taken from the extended interpolation problem P^* introduced in Proposition 2.

6.2. Lagrange interpolation. If there is no repetition of lines, i.e., if all r_i, $i = 0, \ldots, n$ and r'_j, $j = 0, \ldots, m(n)$, respectively, are pairwise different, then the interpolation problem P is a *minimal degree Lagrange interpolation problem with additional points* in the terminology of [6] and we can apply the remainder formula of [6, Corollary 1] to obtain

$$\begin{aligned} f(u) - L(f,P)(u) &= \lambda_{M+1}\,[P,u]\, f \\ &= \sum_{\mu \in \Lambda_M(I)} p_{\mu_M}(u)\pi_\mu \int\limits_{[X_\mu^M, u]} \frac{\partial^{M+1} f}{\partial(u - u_{\mu_M})\partial(u_{\mu_M} - u_{\mu_{M-1}})\cdots\partial(u_{\mu_1} - u_{\mu_0})} \\ &\quad + \sum_{r=0}^{M} \sum_{\mu \in \Lambda_r(I')} p_{\mu_r}(u)\pi_\mu \int\limits_{[X_\mu^{r-1}, u]} \frac{\partial^r f}{\partial(u_{\mu_r} - u_{\mu_{r-1}})\cdots\partial(u_{\mu_1} - u_{\mu_0})}. \end{aligned} \tag{44}$$

In Lagrange interpolation we have

$$L_{ij} f = f(u_{ij}) \quad \forall (i,j) \in I.$$

Hence, for $\mu \in \Lambda_r(I)$ or $\Lambda_r(I')$:

$$\pi_\mu = \prod_{s=0}^{r-1} p_{\mu_s}(u_{\mu_{s+1}}).$$

Let us briefly comment on formula (44): since the polynomials ϕ_{ij}, (or p_{ij}), $(i,j) \in I'_M$, vanish on all the interpolation points and therefore belong to the kernel of the operator $L(P,\cdot)$, they have to be reproduced by $I - L(P,\cdot)$ and that is exactly what the second term in the remainder formula produces. If $P = P^*$, i.e, if we have a full system of interpolation conditions, with Π_M as interpolation space, then that term vanishes. In other cases we assume that the extended problem P^* has been chosen to be a Lagrange problem.

We are going to see that the remainder formula (44) can be simplified due to the special structure of the interpolation data in our problem. This structure allows us to consider significantly smaller subsets of $\Lambda_r(I)$ and $\Lambda_r(I')$. These sets, which will be denoted by $\hat\Lambda_r(I)$ and $\hat\Lambda_r(I')$, are defined as

$$\hat\Lambda_r(I) = \{\mu \in \Lambda_r(I) \,|\, \mu_j \lhd \mu_{j+1},\, j = 0, \ldots, r-1\}, \tag{45}$$

and $\hat\Lambda_r(I')$ analogously. Here $\lhd$ denotes the partial ordering

$$(i,j) \lhd (h,k) \qquad \Leftrightarrow \qquad i \le h,\ j \le k \text{ and } (i,j) \ne (h,k).$$

Observe that the set $\hat\Lambda_r(I)$ is formed by the paths $\mu = (\mu_0, \ldots, \mu_r) \in \Lambda_r(I)$ such that if $\mu_l = (i_l, j_l)$, $l = 0, \ldots, r-1$, then one has either $\mu_{l+1} = (i_l + 1, j_l)$ or

$\mu_{l+1} = (i_l, j_l + 1)$. Therefore, for μ in $\hat\Lambda_r(I)$ or in $\hat\Lambda_r(I')$, the difference $u_{\mu_{l+1}} - u_{\mu_l}$ is either a multiple of ρ'_{j_l} or a multiple of ρ_{i_l}, respectively. In both cases we will denote this direction by $\rho_l(\mu)$, i.e.

$$\rho_l(\mu) = \begin{cases} \rho'_{j_l} & \text{if } \mu_{l+1} = (i_l+1, j_l), \\ \rho_{i_l} & \text{if } \mu_{l+1} = (i_l, j_l+1), \end{cases} \qquad l = 0, \ldots, r-1. \tag{46}$$

In the same fashion, we define

$$\eta_l(\mu) = \begin{cases} \eta_{i_l} & \text{if } \mu_{l+1} = (i_l+1, j_l), \\ \eta'_{j_l} & \text{if } \mu_{l+1} = (i_l, j_l+1), \end{cases} \qquad l = 0, \ldots, r-1. \tag{47}$$

Notice that with this notation $\eta_l(\mu)$ is not normal to $\rho_l(\mu)$, while η'_{j_l} and η_{i_l} are normal to ρ'_{j_l} and ρ_{i_l} respectively. Now we can formulate the main result of this subsection.

THEOREM 9. *Suppose that the lines in Γ and Γ' are pairwise distinct. Let $\Omega \subset \mathbb{R}^2$ be a convex set which contains the interpolation points u_{ij}, $(i,j) \in I$ and let $f \in C^{M+1}(\Omega)$. Then, for any $u \in \Omega$,*

$$f(u) - L(f,P)(u) = \sum_{r=1}^{M+1} \sum_{\mu \in \hat\Lambda_r(I')} \frac{\phi^*_{\mu_r}(u)}{\theta_\mu} \int_{[X_\mu^{r-1}, u]} \frac{\partial^r f}{\partial \rho_{r-1}(\mu) \cdots \partial \rho_0(\mu)}, \tag{48}$$

where

$$\theta_\mu = \prod_{l=0}^{r-1} \langle \eta_l(\mu), \rho_l(\mu) \rangle, \qquad \mu \in \hat\Lambda_r(I'),\ r = 0, \ldots, M+1. \tag{49}$$

Proof. The starting point of our proof is the general remainder formula (44). As an immediate consequence of Theorem 1 we observe that

$$L_{ij} p_{hk} \neq 0 \qquad \Rightarrow \qquad (h,k) \triangleleft (i,j)$$

and therefore, for $\mu \in \Lambda_r(I)$,

$$\pi_\mu \neq 0 \qquad \Rightarrow \qquad \mu \in \hat\Lambda_r(I),$$

and the same for $\mu \in \Lambda_r(I')$. Consequently, the summations in (44) run only over $\hat\Lambda_M(I)$ and $\hat\Lambda_r(I')$, $r = 0, \ldots, M$. Hence,

$$\begin{aligned} f(u) - L(f,P)(u) &= \lambda_{M+1}[P,u] f \\ &= \sum_{\mu \in \hat\Lambda_M(I)} p_{\mu_M}(u) \pi_\mu \int_{[X_\mu^M, u]} \frac{\partial^{M+1} f}{\partial(u - u_{\mu_M}) \partial(u_{\mu_M} - u_{\mu_{M-1}}) \cdots \partial(u_{\mu_1} - u_{\mu_0})} \\ &\quad + \sum_{r=0}^{M} \sum_{\mu \in \hat\Lambda_r(I')} p_{\mu_r}(u) \pi_\mu \int_{[X_\mu^{r-1}, u]} \frac{\partial^r f}{\partial(u_{\mu_r} - u_{\mu_{r-1}}) \cdots \partial(u_{\mu_1} - u_{\mu_0})}. \end{aligned} \tag{50}$$

First, observe that in the present problem I always contains $(0,0)$, hence $\hat\Lambda_0(I') = \emptyset$ and, consequently, the second sum above only runs from $r = 1$ to $r = M$.

Now, let us consider one of the terms that appear in either of the summations in (50). For that purpose we fix any $\mu \in \hat{\Lambda}_r(I)$ or $\mu \in \hat{\Lambda}_r(I')$ and write again $\mu_l = (i_l, j_l)$, $l = 0, \dots, r$. Then, there are the two possibilities

$$\mu_{l+1} = (i_{l+1}, j_{l+1}) = \begin{cases} (i_l + 1, j_l) \\ (i_l, j_l + 1) \end{cases}, \qquad l = 0, \dots, r-1.$$

In either case

$$u_{\mu_{l+1}} - u_{\mu_l} = \begin{cases} u_{i_l+1,j_l} - u_{i_l,j_l} \\ u_{i_l,j_l+1} - u_{i_l,j_l} \end{cases},$$

and therefore, as in the proof of Lemma 6,

$$u_{\mu_{l+1}} - u_{\mu_l} = \sigma_{\mu_l,\mu_{l+1}} \rho_l(\mu)\, \delta\left(u_{\mu_{l+1}}, u_{\mu_l}\right), \tag{51}$$

where $\sigma_{\mu_l,\mu_{l+1}} \in \{-1, 1\}$. We also observe, from the definition of ϕ_{ij} and (35), that

$$\begin{aligned} \frac{\phi_{\mu_{l+1}}\left(u_{\mu_{l+1}}\right)}{\phi_{\mu_l}\left(u_{\mu_{l+1}}\right)} &= \begin{cases} r_{i_l}\left(u_{i_l+1,j_l}\right) \\ r'_{i_l}\left(u_{i_l,j_l+1}\right) \end{cases} \\ &= \begin{cases} \sigma_{\mu_l,\mu_{l+1}} \left\langle \eta_{i_l}, \rho'_{j_l} \right\rangle \delta\left(u_{i_l+1,j_l}, u_{i_l,j_l}\right) \\ \sigma_{\mu_l,\mu_{l+1}} \left\langle \rho_{i_l}, \eta'_{j_l} \right\rangle \delta\left(u_{i_l,j_l+1}, u_{i_l,j_l}\right) \end{cases} \\ &= \sigma_{\mu_l,\mu_{l+1}} \left\langle \eta_l(\mu), \rho_l(\mu) \right\rangle \delta\left(u_{\mu_{l+1}}, u_{\mu_l}\right). \end{aligned}$$

Since, in addition,

$$\begin{aligned} p_{\mu_r}(u)\pi_\mu = p_{\mu_r}(u) \prod_{l=0}^{r-1} p_{\mu_l}\left(u_{\mu_{l+1}}\right) &= \frac{\phi_{\mu_r}(u)}{\phi_{\mu_r}\left(u_{\mu_r}\right)} \prod_{l=0}^{r-1} \frac{\phi_{\mu_l}\left(u_{\mu_{l+1}}\right)}{\phi_{\mu_l}\left(u_{\mu_l}\right)} \\ &= \phi_{\mu_r}(u) \underbrace{\frac{1}{\phi_{\mu_0}\left(u_{\mu_0}\right)}}_{=1} \prod_{l=0}^{r-1} \frac{\phi_{\mu_l}\left(u_{\mu_{l+1}}\right)}{\phi_{\mu_{l+1}}\left(u_{\mu_{l+1}}\right)}, \end{aligned}$$

we get

$$p_{\mu_r}(u)\pi_\mu = \sigma_\mu \frac{\phi_{\mu_r}(u)}{\prod_{l=0}^{r-1} \left\langle \eta_l(\mu), \rho_l(\mu) \right\rangle \delta\left(u_{\mu_{l+1}}, u_{\mu_l}\right)}, \tag{52}$$

with

$$\sigma_\mu = \sigma_{\mu_0,\mu_1} \cdots \sigma_{\mu_{r-1},\mu_r}. \tag{53}$$

To finish the proof, we first consider the sum over $\hat{\Lambda}_M(I)$. Applying (51), we obtain that

$$\begin{aligned} \frac{\partial}{\partial\left(u_{\mu_M} - u_{\mu_{M-1}}\right)} \cdots \frac{\partial}{\partial\left(u_{\mu_1} - u_{\mu_0}\right)} &= \prod_{l=0}^{M-1} \sigma_{\mu_l,\mu_{l+1}} \delta\left(u_{\mu_{l+1}}, u_{\mu_l}\right) \frac{\partial}{\partial \rho_l(\mu)} \\ &= \sigma_\mu \frac{\partial^M}{\partial \rho_{M-1}(\mu) \cdots \partial \rho_0(\mu)} \prod_{l=0}^{M-1} \delta\left(u_{\mu_{l+1}}, u_{\mu_l}\right), \end{aligned}$$

which, with (52), give, for any $\mu \in \hat{\Lambda}_M(I)$,

$$p_{\mu_M}(u)\pi_\mu \int_{[X_\mu^M,u]} \frac{\partial^{M+1} f}{\partial(u-u_{\mu_M})\partial(u_{\mu_M}-u_{\mu_{M-1}})\cdots\partial(u_{\mu_1}-u_{\mu_0})} = \frac{\phi_{\mu_M}(u)}{\theta_\mu} \int_{[X_\mu^M,u]} \frac{\partial^{M+1} f}{\partial(u-u_{\mu_M})\partial\rho_{M-1}(\mu)\cdots\partial\rho_0(\mu)}.$$

Applying an identical argument to the sum over $\hat{\Lambda}_r(I')$, $r=0,\ldots,M$, we obtain the representation

$$\begin{aligned} f(u) - L(f,P)(u) = \sum_{\mu\in\hat{\Lambda}_M(I)} \frac{\phi_{\mu_M}(u)}{\theta_\mu} \int_{[X_\mu^M,u]} \frac{\partial^{M+1} f}{\partial(u-u_{\mu_M})\partial\rho_M(\mu)\cdots\partial\rho_0(\mu)} \\ + \sum_{r=1}^{M} \sum_{\mu\in\hat{\Lambda}_r(I')} \frac{\phi^*_{\mu_r}(u)}{\theta_\mu} \int_{[X_\mu^{r-1},u]} \frac{\partial^r f}{\partial\rho_r(\mu)\cdots\partial\rho_0(\mu)}. \end{aligned} \tag{54}$$

We moreover note that, for any $(i,j) \in I_M$ and $u \in \mathbb{R}^2$,

$$\begin{aligned} \phi_{ij}(u)\,(u-u_{ij}) = (u_{i+1,j}-u_{ij})\,\frac{\phi_{ij}\,(u_{i+1,j})}{\phi^*_{i+1,j}\,(u_{i+1,j})}\phi^*_{i+1,j}(u) \\ + (u_{i,j+1}-u_{ij})\,\frac{\phi_{ij}\,(u_{i,j+1})}{\phi^*_{i,j+1}(u_{i,j+1})}\phi^*_{i,j+1}(u). \end{aligned} \tag{55}$$

This *vector interpolation formula*, which was crucial for the inductive proof of [6, Corollary 1] is easily verified by checking its validity for all points u_{hk}, $h+k \le i+j+1$. Indeed, if $h+k < i+j+1$, then both sides vanish when setting $u = u_{hk}$ and the same happens for $h+k = i+j+1$ as long as $(h,k) \notin \{(i+1,j),(i,j+1)\}$ because then either $h<i$ or $k<j$. Finally, for $(h,k) = (i+1,j)$ then both sides of (55) become $\phi_{ij}\,(u_{i+1,j})\,(u_{i+1,j}-u_{ij})$ and similarly for $(h,k) = (i,j+1)$. Since interpolation at these points is unique in Π_{i+j+1}, (55) follows.

Specializing to $(i,j) \in I_M$ and $u \in \Omega$, there are numbers $\sigma_{ij}, \sigma'_{ij} \in \{-1,1\}$ such that

$$u_{i+1,j} - u_{ij} = \sigma_{ij}\,\rho'_j\,\delta\,(u_{i+1,j},u_{ij})\,, \qquad u_{i,j+1} - u_{ij} = \sigma'_{ij}\,\rho_i\,\delta\,(u_{i,j+1},u_{ij})\,,$$

and, by (35),

$$\frac{\phi_{ij}\,(u_{i+1,j})}{\phi^*_{i+1,j}\,(u_{i+1,j})} = \frac{1}{r_i\,(u_{i+1,j})} = \frac{1}{\sigma_{ij}\,\langle\eta_i,\rho'_j\rangle\,\delta\,(u_{i+1,j},u_{ij})},$$

$$\frac{\phi_{ij}\,(u_{i,j+1})}{\phi^*_{i,j+1}\,(u_{i,j+1})} = \frac{1}{\sigma'_{ij}\,\langle\eta'_j,\rho_i\rangle\,\delta\,(u_{i,j+1},u_{ij})}.$$

Hence, for any $(i,j) \in I_M$ and $u \in \Omega$, we can rewrite (55) as

$$\phi_{ij}(u)\,(u-u_{ij}) = \frac{\phi^*_{i+1,j}(u)}{\langle\eta_i,\rho'_j\rangle}\rho'_j + \frac{\phi^*_{i,j+1}(u)}{\langle\eta'_j,\rho_i\rangle}\rho_i. \tag{56}$$

¿From equation (56) and the linearity of directional derivatives it follows that for $(i,j) \in I_M$, $u \in \Omega$ and $f \in C^1(\Omega)$

$$\phi_{ij}(u)\frac{\partial f}{\partial\left(u-u_{ij}\right)} = \frac{\phi^*_{i+1,j}(u)}{\left\langle \eta_i, \rho'_j \right\rangle}\frac{\partial f}{\partial \rho'_j} + \frac{\phi^*_{i,j+1}(u)}{\left\langle \eta'_j, \rho_i \right\rangle}\frac{\partial f}{\partial \rho_i}.$$

Writing $\mu_M = (i_M, j_M)$, and substituting this into the first sum of (54), we then get

$$\begin{aligned}
&\sum_{\mu\in\hat{\Lambda}_M(I)} \frac{\phi_{\mu_M}(u)}{\theta_\mu} \int\limits_{\left[X^M_\mu, u\right]} \frac{\partial^{M+1} f}{\partial(u-u_{\mu_M})\partial\rho_{M-1}(\mu)\cdots\partial\rho_0(\mu)} \\
&\qquad = \sum_{\mu\in\hat{\Lambda}_M(I)} \frac{\phi_{i_M,j_M}(u)}{\theta_\mu} \int\limits_{\left[X^M_\mu, u\right]} \frac{\partial^{M+1} f}{\partial(u-u_{i_M,j_M})\partial\rho_{M-1}(\mu)\cdots\partial\rho_0(\mu)} \\
&\qquad = \sum_{\mu\in\hat{\Lambda}_M(I)} \frac{\phi^*_{i_M+1,j_M}(u)}{\left\langle \eta_{i_M}, \rho'_{j_M} \right\rangle \theta_\mu} \int\limits_{\left[X^M_\mu, u\right]} \frac{\partial^{M+1} f}{\partial\rho'_{j_M}\partial\rho_{M-1}(\mu)\cdots\partial\rho_0(\mu)} \\
&\qquad\qquad + \frac{\phi^*_{i_M,j_M+1}(u)}{\left\langle \eta'_{j_M}, \rho_{i_M} \right\rangle \theta_\mu} \int\limits_{\left[X^M_\mu, u\right]} \frac{\partial^{M+1} f}{\partial\rho_{i_M}\partial\rho_{M-1}(\mu)\cdots\partial\rho_0(\mu)}.
\end{aligned}$$

Now, we shall see that this sum can be written running over $\hat{\Lambda}_{M+1}(I)$. Writing μ_{M+1} for (i_M+1, j_M) we note that $\rho_M(\mu) = \rho'_{j_M}$ and $\eta_M(\mu) = \eta_{i_M}$. Hence, the first term in the sum becomes

$$\frac{\phi^*_{\mu_{M+1}}(u)}{\theta_\mu} \int\limits_{\left[X^M_\mu, u\right]} \frac{\partial^{M+1} f}{\partial\rho_M(\mu)\cdots\partial\rho_0(\mu)},$$

where θ_μ includes now the factor $\left\langle \eta_{i_M}, \rho'_{j_M} \right\rangle = \langle \eta_M(\mu), \rho_M(\mu) \rangle$. The same happens for the second term in the sum, with $\mu_{M+1} = (i_M, j_M+1)$, and consequently

$$\begin{aligned}
&\sum_{\mu\in\hat{\Lambda}_M(I)} \frac{\phi_{\mu_M}(u)}{\theta_\mu} \int\limits_{\left[X^M_\mu, u\right]} \frac{\partial^{M+1} f}{\partial(u-u_{\mu_M})\partial\rho_{M-1}(\mu)\cdots\partial\rho_0(\mu)} \\
&\qquad = \sum_{\mu\in\hat{\Lambda}_{M+1}(I)} \frac{\phi^*_{\mu_{M+1}}(u)}{\theta_\mu} \int\limits_{\left[X^M_\mu, u\right]} \frac{\partial^{M+1} f}{\partial\rho_M(\mu)\cdots\partial\rho_0(\mu)},
\end{aligned}$$

which, together with (54), gives (48) and completes the proof. ∎

6.3. Hermite interpolation as a limit process. For this subsection, we fix an index $(i,j) \in I$. We have already denoted by (s_i, t_i) the *multiplicity* of u_{ij}, i.e., the number of repetitions of the lines r_i and r'_j in Γ and Γ', respectively. For the sake of brevity we drop the subscripts in the multiplicity and simply write (s,t) instead. Hence, there exist numbers

$$i_0 < i_1 < \cdots < i_s = i \qquad \text{and} \qquad j_0 < j_1 < \cdots < j_t = j$$

such that

$$r_{i_0} = \cdots = r_{i_s} \qquad \text{and} \qquad r_{j_0} = \cdots = r_{j_t}.$$

Also assume that (s,t) is the *maximal multiplicity* of the lines r_i and r'_j, i.e.,

$$r_i \neq r_h, \quad h = i+1, \ldots, n, \qquad \text{and } r'_j \neq r'_k, \quad k = j+1, \ldots, m(i).$$

Now, we choose $\varepsilon > 0$ and define

$$r_{i,\varepsilon} = r_i\left(\cdot - \varepsilon\eta_i\right) = r_i - \varepsilon, \qquad r'_{j,\varepsilon} = r'_j\left(\cdot - \varepsilon\eta'_j\right) = r'_j - \varepsilon.$$

For the respective modified sets of lines we write

$$\Gamma_\varepsilon = \left\{r_0, \ldots, r_{i-1}, r_{i,\varepsilon}, r_{i+1}, \ldots, r_n\right\}, \; \Gamma'_\varepsilon = \left\{r'_0, \ldots, r'_{j-1}, r'_{j,\varepsilon}, r'_{j+1}, \ldots, r'_m\right\}.$$

Then we denote by P^ε either the Hermite interpolation based on Γ_ε and Γ' or the one based on Γ and Γ'_ε. Since we are treating reversible systems here, these two types of modification are essentially the same and we will not distinguish between them formally.

To formulate a suitable notion of convergence, we assume again that Ω is a convex and compact subset of $\mathbb{R}^2$ with nonempty interior and equip $f \in C^k(\Omega)$, $k \in \mathbb{N}_0$, with its standard norm. For that purpose we define the the semi–norms of the l-th derivatives as

$$|D^l f|_\Omega := \max_{u\in\Omega} \max_{\|\eta_1\|=\cdots=\|\eta_l\|=1} \left| \frac{\partial^l f}{\partial\eta_1 \cdots \partial\eta_l}(u) \right|, \qquad l = 0, \ldots, k,$$

where the derivatives are extended to the boundary by continuity, and set

$$\|f\|_{k,\Omega} = \sum_{l=0}^{k} |D^l f|_\Omega .$$

It is well known that, together with the norm $\|\cdot\|_{k,\Omega}$, the vector space $C^k(\Omega)$ becomes a Banach space.

Lemma 10. *Let Ω be any convex and compact set which contains the points $u_{hk}, u^\varepsilon_{hk}$, $(h,k) \in I$, in its interior. Then for either choice of P^ε there exists a continuous function $\mu : [0,a] \to \mathbb{R}$, $a > 0$, with $\mu(0) = 0$, such that for any $f \in C^{M+1}(\Omega)$*

$$\left\|L\left(f; P^\varepsilon\right) - L\left(f; P\right)\right\|_{M+1,\Omega} \leq \mu(\varepsilon) \left\|f\right\|_{M+1,\Omega}, \tag{57}$$

i.e., $L\left(\cdot; P^\varepsilon\right) \to L\left(\cdot; P\right)$ in the strong operator topology.

Proof. We will only consider the first case, i.e., the case that P^ε stems from Γ_ε and Γ', the second one is identical. It will become clear that the right boundary point a of the interval $[0,a]$ can always be chosen properly (i.e., sufficiently small), but will depend on the line systems Γ and Γ'.

Let us denote by u^ε_{hk} the points of the modified interpolation problem. It is easy to see that

$$u^\varepsilon_{hk} = \begin{cases} u_{hk} + \varepsilon \frac{\rho'_k}{\langle \eta_i, \rho'_k \rangle} & \text{if } h = i, \\ u_{hk} & \text{if } h \neq i, \end{cases} \qquad (h,k) \in I. \tag{58}$$

Finally, write ϕ^ε_{hk} for the respective basis functions. Since $\phi^\varepsilon_{hk} = \phi_{hk}$ if $h \leq i$ and

$$\phi^\varepsilon_{hk} = \frac{r_{i,\varepsilon}}{r_i} \phi_{hk}$$

otherwise, we observe that

$$\Pi_{h+k-1} \ni \phi^{\varepsilon}_{hk} - \phi_{hk} = \begin{cases} 0 & \text{if } h \le i, \\ -\varepsilon \frac{\phi_{hk}}{r_i} & \text{if } h > i, \end{cases} \qquad (h,k) \in I. \tag{59}$$

Also, for $l = 0, \dots, M$, let again P^{ε}_l and P_l denote the subproblems related to I_l; in particular, due to (26) and (29),

$$L\left(f, P^{\varepsilon}_l\right) = \sum_{(h,k)\in I_l} \lambda^{h+k}_{hk}\left[P^{\varepsilon}_l\right] f\, \phi^{\varepsilon}_{hk}, \qquad l = 0, \dots, M. \tag{60}$$

Note that

$$\lambda^{h+k}_{hk}\left[P_l\right] f = \lambda^{h+k}_{hk}\left[P\right] f \quad \forall (h,k), h+k \le l$$

and analogously for $\lambda^{h+k}_{hk}\left[P^{\varepsilon}_l\right] f$, and $\lambda^{h+k}_{hk}\left[P^{\varepsilon}\right] f$.

We also remark that the divided differences $\lambda^{h+k}_{hk}\left[P\right]$, related to the *original* interpolation problem, are *continuous linear functionals* on $C^{M+1}(\Omega)$, (see equations (30) and (31)), hence they are bounded, i.e.,

$$\left|\lambda^{h+k}_{hk}\left[P\right] f\right| \le C_{hk} \left\|f\right\|_{M+1,\Omega}, \qquad C_{hk} < \infty. \tag{61}$$

For convenience, we will set $C_l = \max_{(h,k)\in I_l} C_{hk}$, $l = 0, \dots, M$.

We will prove by induction on $l = h+k$ that the functionals $\lambda^{h+k}_{hk}\left[P^{\varepsilon}\right]$ converge strongly to $\lambda^{h+k}_{hk}\left[P\right]$, $(h,k) \in I_l$, which means that there exist continuous functions $\lambda_{hk} : [0,a] \to \mathbb{R}$ with $\lambda_{hk}(0) = 0$ such that

$$\left|\lambda^{h+k}_{hk}\left[P^{\varepsilon}\right] f - \lambda^{h+k}_{hk}\left[P\right] f\right| \le \lambda_{hk}(\varepsilon) \left\|f\right\|_{M+1,\Omega}, \qquad (h,k) \in I_l. \tag{62}$$

In the same spirit as above we set $\lambda_l(x) = \max_{(h,k)\in I_l} \lambda_{hk}(x)$.

We first remark that the validity of (62) for some $l \in \mathbb{N}_0$ implies the strong convergence $L\left(\cdot; P^{\varepsilon}_l\right) \to L\left(\cdot; P_l\right)$, i.e., there exist $\mu_l \in C[0,a]$ such that $\mu_l(0) = 0$ and

$$\left\|L\left(f; P^{\varepsilon}_l\right) - L\left(f; P_l\right)\right\|_{M+1,\Omega} \le \mu_l(\varepsilon) \left\|f\right\|_{M+1,\Omega}. \tag{63}$$

To prove this remark, we begin with the estimate

$$\begin{aligned} &\left\| L(f,P^{\varepsilon}_l) - L(f,P_l) \right\|_{M+1,\Omega} \\ &\quad = \left\| \sum_{(h,k)\in I_l} \lambda^{h+k}_{hk}\left[P^{\varepsilon}\right] f\, \phi^{\varepsilon}_{hk} - \sum_{(h,k)\in I_l} \lambda^{h+k}_{hk}\left[P\right] f\, \phi_{hk} \right\|_{M+1,\Omega} \\ &\quad \le \left\| \sum_{(h,k)\in I_l} \lambda^{h+k}_{hk}\left[P^{\varepsilon}\right] f\, \left(\phi^{\varepsilon}_{hk} - \phi_{hk}\right) \right\|_{M+1,\Omega} \\ &\qquad + \left\| \sum_{(h,k)\in I_l} \left(\lambda^{h+k}_{hk}\left[P^{\varepsilon}\right] f - \lambda^{h+k}_{hk}\left[P\right] f\right) \phi_{hk} \right\|_{M+1,\Omega} \end{aligned}$$

and observe from (62) and (63) that

$$\begin{aligned} \left|\lambda^{h+k}_{hk}\left[P^{\varepsilon}\right] f\right| &\le \left|\lambda^{h+k}_{hk}\left[P\right] f\right| + \left|\lambda^{h+k}_{hk}\left[P^{\varepsilon}\right] f - \lambda^{h+k}_{hk}\left[P\right] f\right| \\ &\le \left(C_l + \lambda_l(\varepsilon)\right) \left\|f\right\|_{M+1,\Omega}. \end{aligned} \tag{64}$$

Hence, by (59), the first term is bounded by

$$\varepsilon\left(C_l+\lambda_l(\varepsilon)\right)\|f\|_{M+1,\Omega}\sum_{\substack{(h,i)\in I_l\\ h>i}}\left\|\frac{\phi_{hk}}{r_i}\right\|_{M+1,\Omega},$$

while the second term is easily seen to be bounded by

$$\lambda_l(\varepsilon)\sum_{(h,k)\in I_l}\|\phi_{hk}\|_{M+1,\Omega}.$$

Combining these two inequalities readily gives a choice of μ_l with the required properties of being continuous and satisfying $\mu_l(0)=0$.

Starting with the induction for (62), the case $l=0$ is trivial ($\phi_{00}=\phi_{00}^{\varepsilon}=1$ and $\lambda_{00}^{0}\left[P^{\varepsilon}\right]f=f\left(u_{00}^{\varepsilon}\right)=f\left(u_{00}\right)$), hence we assume that the formula (62) and therefore also (63) are valid for some $l\geq 0$ and pick $(h,k)\in H_{l+1}$. From the recurrence relation (28) we obtain that

$$\lambda_{hk}^{h+k}\left[P^{\varepsilon}\right]f=\frac{1}{L_{hk}^{\varepsilon}\phi_{hk}^{\varepsilon}}L_{hk}^{\varepsilon}\left(f-L(f,P_l^{\varepsilon})\right). \tag{65}$$

as well as

$$\lambda_{hk}^{h+k}\left[P\right]f=\frac{1}{L_{hk}\phi_{hk}}L_{hk}\left(f-L(f,P_l)\right). \tag{66}$$

Here we have denoted by L_{hk}^{ε}, the (h,k) linear form which defines the corresponding interpolation datum of P^{ε}. We first consider the simpler case $h\neq i$; here, $u_{hk}^{\varepsilon}=u_{hk}$ and $L_{hk}^{\varepsilon}=L_{hk}$ and therefore

$$\lambda_{hk}^{h+k}\left[P^{\varepsilon}\right]f=\frac{1}{L_{hk}\phi_{hk}^{\varepsilon}}L_{hk}\left(f-L(f,P_l^{\varepsilon})\right).$$

Splitting this into

$$\begin{aligned}\left|\lambda_{hk}^{h+k}\left[P^{\varepsilon}\right]f-\lambda_{hk}^{h+k}\left[P\right]f\right| &\leq\left|\frac{1}{L_{hk}\phi_{hk}^{\varepsilon}}-\frac{1}{L_{hk}\phi_{hk}}\right|\,\left|L_{hk}\left(f-L\left(f,P_l^{\varepsilon}\right)\right)\right|\\ &\quad+\left|\frac{1}{L_{hk}\phi_{hk}}\right|\,\left|L_{hk}\left(L\left(f,P_l^{\varepsilon}\right)-L\left(f,P_l\right)\right)\right|\end{aligned}$$

and using the induction hypothesis on the second term as well as the simple fact that

$$\begin{aligned}\left\|f-L\left(f,P_l^{\varepsilon}\right)\right\|_{M+1,\Omega} &\leq\left\|f-L\left(f,P_l\right)\right\|_{M+1,\Omega}+\left\|L\left(f,P_l\right)-L\left(f,P_l^{\varepsilon}\right)\right\|_{M+1,\Omega}\\ &\leq\Big(1+\|L(\cdot,P_l)\|_{M+1,\Omega}+\mu_l(\varepsilon)\Big)\,\|f\|_{M+1,\Omega},\end{aligned} \tag{67}$$

we obtain that (63) holds true for

$$\lambda_{hk}(\varepsilon):=\frac{1}{|L_{hk}\phi_{hk}|}\begin{cases}\varepsilon\frac{\|\phi_{hk}/r_i\|_{M+1,\Omega}}{\left|L_{hk}\phi_{hk}^{\varepsilon}\right|}\left(1+\mu_l(\varepsilon)+\|L(\cdot,P_l)\|\right)+\mu_l(\varepsilon),\\ \mu_l(\varepsilon),\end{cases}$$

depending on whether $h>i$ or $h\leq i$. This quantity is again continuous on some neighborhood of 0 and satisfies $\lambda_{hk}(0)=0$ since $L_{hk}\phi_{hk}\neq 0$ and therefore, for sufficiently small ε, we also have that

$$\left|L_{hk}\phi_{hk}^{\varepsilon}\right|\geq\frac{1}{2}\left|L_{hk}\phi_{hk}\right|>0. \tag{68}$$

So, let us consider now the case $h = i$. In this case, due to (58) we find, like in Lemma 6, that

$$r_i\left(u_{ik}^{\varepsilon}\right) = r_i\left(u_{ik}\right) + \varepsilon \frac{\langle \eta_i, \rho_k' \rangle}{\langle \eta_i, \rho_k' \rangle} = \varepsilon,$$

which yields

$$\begin{aligned} L_{ik}^{\varepsilon}\phi_{ik}^{\varepsilon} &= t_k!\left\langle \rho_i, \eta_k' \right\rangle^{t_k} \varepsilon^s \prod_{r_l \neq r_i} r_l\left(u_{ik}^{\varepsilon}\right) \prod_{r_l' \neq r_k'} r_l'\left(u_{ik}^{\varepsilon}\right) \\ &=: t_k!\left\langle \rho_i, \eta_k' \right\rangle^{t_k} \varepsilon^s \tilde{\psi}_{ik}\left(u_{ik}^{\varepsilon}\right). \end{aligned} \tag{69}$$

We also recall that

$$L_{ik}\phi_{ik} = t_k!s!\left\langle \eta_i, \rho_k' \right\rangle^s \left\langle \rho_i, \eta_k' \right\rangle^{t_k} \tilde{\psi}_{ik}\left(u_{ik}\right) =: \psi_{ik}\left(u_{ik}\right), \tag{70}$$

and therefore define

$$\psi_{ik} := t_k!s!\left\langle \eta_i, \rho_k' \right\rangle^s \left\langle \rho_i, \eta_k' \right\rangle^{t_k} \tilde{\psi}_{ik}, \tag{71}$$

where now $\psi_{ik}\left(u_{ik}\right) \neq 0$.

Set $g := f - L\left(f, P_l^{\varepsilon}\right)$. Since

$$u_{\nu k}^{\varepsilon} = u_{ik} \quad \text{ and } \quad L_{\nu k}^{\varepsilon} = L_{\nu k}, \qquad \nu = 0, \ldots, i-1,$$

and, consequently,

$$\frac{\partial^{\nu + t_k}}{\partial {\rho_k'}^{\nu} \partial \rho_i^{t_k}} L\left(f, P_l^{\varepsilon}\right)\left(u_{ik}\right) = \frac{\partial^{\nu + t_k} f}{\partial {\rho_k'}^{\nu} \partial \rho_i^{t_k}}\left(u_{ik}\right), \qquad \nu = 0, \ldots, s-1,$$

we obtain that

$$\frac{\partial^{\nu + t_k} g}{\partial {\rho_k'}^{\nu} \partial \rho_i^{t_k}}\left(u_{ik}\right) = 0, \qquad \nu = 0, \ldots, s-1.$$

By the Taylor expansion for g at $u = u_{ik}$ there exists

$$\xi = (1-\alpha) u_{ik} + \alpha u_{ik}^{\varepsilon} = u_{ik} + \frac{\alpha \varepsilon}{\langle \eta_i, \rho_k' \rangle} \rho_k', \qquad \alpha \in [0,1],$$

such that

$$\begin{aligned} \frac{\partial^{t_k} g}{\partial \rho_i^{t_k}}\left(u_{ik}^{\varepsilon}\right) &= \sum_{\nu=0}^{s-1} \frac{1}{\nu!} \left(\frac{\varepsilon}{\langle \eta_i, \rho_k' \rangle} \right)^{\nu} \frac{\partial^{\nu + t_k} g}{\partial {\rho_k'}^{\nu} \partial \rho_i^{t_k}}\left(u_{ik}\right) \\ &\quad + \frac{1}{s!} \left(\frac{\varepsilon}{\langle \eta_i, \rho_k' \rangle} \right)^{s} \frac{\partial^{s + t_k} g}{\partial {\rho_k'}^{s} \partial \rho_i^{t_k}}(\xi), \end{aligned}$$

and we obtain that

$$\frac{\partial^{t_k} g}{\partial \rho_i^{t_k}}\left(u_{ik}^{\varepsilon}\right) = \frac{\varepsilon^s}{s!} \frac{1}{\langle \eta_i, \rho_k' \rangle^s} \frac{\partial^{s + t_k} g}{\partial {\rho_k'}^{s} \partial \rho_i^{t_k}}(\xi). \tag{72}$$

Substituting (69) and (72) into (65) and taking into account (71), we find that

$$\lambda_{ik}^{i+k}\left[P^{\varepsilon}\right] f = \frac{1}{\psi_{ik}\left(u_{ik}^{\varepsilon}\right)} L_{ik\xi}\left(f - L\left(f, P_l^{\varepsilon}\right)\right), \tag{73}$$

where we have denoted by $L_{ik\xi} f$ the value of the linear functional L_{ik} defined in (4) with u_{ik} replaced by ξ.

¿From (70) and (66) we also obtain

$$\lambda_{ik}^{i+k}[P]f = \frac{1}{\psi_{ik}(u_{ik})} L_{ik}(f - L(f, P_l)). \tag{74}$$

The rest of the proof is now straightforward again; indeed,

$$\begin{aligned}
&\left|\lambda_{ik}^{i+k}[P^\varepsilon]f - \lambda_{ik}^{i+k}[P]f\right| \\
&\qquad \le \left|\frac{1}{\psi_{ik}(u_{ik}^\varepsilon)} - \frac{1}{\psi_{ik}(u_{ik})}\right| \left|L_{ik\xi}(f - L(f, P_l^\varepsilon))\right| \\
&\qquad\quad + \frac{1}{|\psi_{ik}(u_{ik})|}\left|L_{ik\xi}(f - L(f, P_l^\varepsilon)) - L_{ik}(f - L(f, P_l^\varepsilon))\right| \\
&\qquad\quad + \frac{1}{|\psi_{ik}(u_{ik})|}\left|L_{ik}(L(f, P_l) - L(f, P_l^\varepsilon))(u_{ik})\right|.
\end{aligned}$$

The first and the third term can be estimated like above, while for the second term we put it in integral form and use (67) for the estimate

$$\begin{aligned}
&\frac{1}{|\psi_{ik}(u_{ik})|}\frac{\varepsilon}{\langle\eta_i, \rho_k'\rangle}\int_0^1 \left|\frac{\partial}{\partial\rho_k'}\frac{\partial^{s+t_k}}{\partial{\rho_k'}^s\partial\rho_i^{t_k}}(f - L(f, P_l^\varepsilon))\left(u_{ik} + \frac{\alpha\varepsilon}{\langle\eta_i, \rho_k'\rangle}\rho_k'\right)d\alpha\right| \\
&\qquad \le \frac{1}{|\psi_{ik}(u_{ik})|}\frac{\varepsilon}{\langle\eta_i, \rho_k'\rangle}\|f - L(f, P_l^\varepsilon)\|_{M+1,\Omega} \\
&\qquad \le \frac{1}{|\psi_{ik}(u_{ik})|}\frac{\varepsilon}{\langle\eta_i, \rho_k'\rangle}(1 + \|L(\cdot, P_l)\| + \mu_l(\varepsilon))\|f\|_{M+1,\Omega}.
\end{aligned}$$

Combining these estimates we get that (62) holds for

$$\begin{aligned}
\lambda_{ik}(\varepsilon) = &\frac{1}{|\psi_{ik}(u_{ik})|}\frac{\varepsilon}{\langle\eta_i, \rho_k'\rangle}\left(1 + \frac{\|\psi_{ik}\|_{M+1,\Omega}}{\psi_{ik}(u_{ik}^\varepsilon)}\right)(1 + \mu_l(\varepsilon) + \|L(\cdot, P_l)\|) \\
&+ \frac{\mu_l(\varepsilon)}{|\psi_{ik}(u_{ik})|}.
\end{aligned}$$

Since this function is again continuous with respect to ε in some neighborhood of 0 and satisfies $\lambda_{ik}(0) = 0$, we have advanced the induction hypothesis and hence, have completed the proof. $\square$

Now we can immediately deduce the main result on remainder formulae which says that Theorem 9 extends to Hermite interpolation.

THEOREM 11. *Let $\Omega \subset \mathbb{R}^2$ be a convex and compact set which contains the interpolation points u_{ij} $(i,j) \in I$ in its interior and let $f \in C^{M+1}(\Omega)$. Then, for any $u \in \Omega$,*

$$f(u) - L(f, P)(u) = \sum_{r=1}^{M+1}\sum_{\mu\in\hat\Lambda_r(I')}\frac{\phi_{\mu_r}^*(u)}{\theta_\mu}\int_{[X_\mu^{r-1}, u]}\frac{\partial^r f}{\partial\rho_{r-1}(\mu)\cdots\partial\rho_0(\mu)}. \tag{75}$$

Proof. The proof is now a simple induction on $\alpha = \max\{s_i + t_j \mid (i,j) \in I\}$. Indeed, $\alpha = 0$ is the statement of Theorem 9. In the case $\alpha > 0$ there is a finite number of pairs (i,j) such that the maximum is assumed. If there is only one pair then the respective multiplicity (s_i, t_i) satisfies at least one of the inequalities $s_i > 0$ or

$t_i > 0$. Assume that $s_i > 0$. Then we consider the reduced interpolation problem P^ε, choosing ε sufficiently small. Now, by the induction hypothesis, (75) can be applied and yields

$$f(u) - L\left(f, P^\varepsilon\right)(u) = \sum_{r=1}^{M+1} \sum_{\mu \in \hat{\Lambda}_r(I')} \frac{\phi_{\mu_r}^{\varepsilon *}(u)}{\theta_\mu} \int\limits_{\left[X_\mu^{r-1}, u\right]} \frac{\partial^r f}{\partial \rho_{r-1}(\mu) \cdots \partial \rho_0(\mu)}.$$

Hence,

$$\begin{aligned}
&\Big\| f - L(f,P) - \sum_{r=1}^{M+1} \sum_{\mu \in \hat{\Lambda}_r(I')} \frac{\phi_{\mu_r}^{*}}{\theta_\mu} \int\limits_{\left[X_\mu^{r-1}, \cdot\right]} \frac{\partial^r f}{\partial \rho_{r-1}(\mu) \cdots \partial \rho_0(\mu)} \Big\|_{M+1,\Omega} \\
&\quad \le \Big\| \sum_{r=1}^{M+1} \sum_{\mu \in \hat{\Lambda}_r(I')} \frac{\phi_{\mu_r}^{\varepsilon *} - \phi_{\mu_r}^{*}}{\theta_\mu} \int\limits_{\left[X_\mu^{r-1}, \cdot\right]} \frac{\partial^r f}{\partial \rho_{r-1}(\mu) \cdots \partial \rho_0(\mu)} \Big\|_{M+1,\Omega} \\
&\qquad + \left\| L\left(f, P^\varepsilon\right) - L\left(f, P\right) \right\|_{M+1,\Omega} \\
&\quad \le \varepsilon \Big\| \sum_{r=1}^{M+1} \sum_{\substack{\mu \in \hat{\Lambda}_r(I') \\ i_r > i}} \frac{\phi_{\mu_r}^{*}}{r_i\, \theta_\mu} \int\limits_{\left[X_\mu^{r-1}, \cdot\right]} \frac{\partial^r f}{\partial \rho_{r-1}(\mu) \cdots \partial \rho_0(\mu)} \Big\|_{M+1,\Omega} \\
&\qquad + \left\| L\left(f, P^\varepsilon\right) - L\left(f, P\right) \right\|_{M+1,\Omega} \\
&\quad \le \left(\frac{\varepsilon}{(r+1)!} \sum_{r=1}^{M+1} \sum_{\substack{\mu \in \hat{\Lambda}_r(I') \\ i_r > i}} \left\| \frac{\phi_{\mu_r}^{*}}{r_i} \right\|_{M+1,\Omega} \frac{1}{\theta_\mu} + \mu_r(\varepsilon) \right) \|f\|_{M+1,\Omega}\,,
\end{aligned}$$

by Lemma 10 and (43), which gives (75) as $\varepsilon \to 0$ and advances the induction hypothesis. If there are several pairs (i,j) such that $s_i + t_j = \alpha$, then we incorporate an additional induction on this number too: the case of one such pair has just been treated and more than one such pair is resolved by exactly the same limit argument as above. $\square$

Remarks.

i) If $m(i) = n - i$, $0 \le i \le n$, that is, if the interpolation space is Π_n, then $M = n$ and (48) and (75), in Theorems 9 and 11, reduce to

$$f(u) - L(f,P)(u) = \sum_{\mu \in \hat{\Lambda}_{M+1}(I')} \frac{\phi_{\mu_r}^{*}(u)}{\theta_\mu} \int\limits_{\left[X_\mu^{r-1}, u\right]} \frac{\partial^r f}{\partial \rho_{r-1}(\mu) \cdots \partial \rho_0(\mu)}.$$

ii) A particular case of this problem, namely that of interpolation at cardinal points $\{(i,j)\}$ with $0 \le i,j \le n$, $i+j \le n$, was studied by Sauer and Xu in [9], where a particular form of the remainder (48) can be seen.

iii) A very simple particular case of Theorem 11 arises by considering the totally coincident lines $r_i(u) := x - x_0$, $r'_j(u) := y - y_0$ for all $0 \le i,j \le n$ and $m(i) = n - i$. This gives rise to the Taylor interpolation problem of degree n at $u_0 = (x_0, y_0)$ and (75) is just the bivariate Taylor formula. In fact, simple calculations produce, in the right-hand side of (75),

$$\begin{aligned} f(u) - L(f,P)(u) \\ &= \sum_{i+j=n+1} \binom{n+1}{i} (x-x_0)^i (y-y_0)^j \int_{[\underbrace{u_0,\ldots,u_0}_{n},u]} \frac{\partial^{n+1} f}{\partial x^i \partial y^j} \\ &= \sum_{i+j=n+1} \frac{(x-x_0)^i (y-y_0)^j}{i!j!} \frac{\partial^{n+1} f}{\partial x^i \partial y^j} (\xi_{ij}, \eta_{ij}), \end{aligned}$$

where (ξ_{ij}, η_{ij}) is between $u = (x, y)$ and $u_0 = (x_0, y_0)$.

6.4. Error estimates. It is now very easy to derive error estimates. For that purpose we only have to notice that for $r = 0, \ldots, M+1$ the cardinality of $\hat{\Lambda}_r(I')$ is not greater than 2^r and to recall from [5] the formula

$$\int_{[x_0,\ldots,x_n]} 1 = \frac{1}{n!},$$

which has been used in the last proof.

We introduce two geometric quantities, depending on the lines and the domain Ω, namely a "radius" of Ω relative to Γ^* and Γ'^*, defined as

$$R_\Omega = \max \left\{ \|r_i\|_{0,\Omega}, \left\|r'_j\right\|_{0,\Omega} \mid 0 \le i, j \le M \right\}$$

and a "minimal angle of intersection" as

$$\theta = \min \left\{ \left| \left\langle \eta_i, \rho'_j \right\rangle \right| \mid (i,j) \in I \right\}.$$

Note that θ depends only on Γ and Γ'. Finally, we set

$$r(I) = \min \left\{ 0 \le r \le M+1 \mid I'_r \ne \emptyset \right\}.$$

Then we can easily obtain an error estimate from (75).

THEOREM 12. *Let $\Omega \subset \mathbb{R}^2$ be a convex compact set which contains the interpolation points u_{ij}, $(i,j) \in I$, and let $f \in C^{M+1}(\Omega)$. Then*

$$\|f - L(f,P)\|_\Omega \le \sum_{r=r(I)}^{M+1} \left(\frac{2R_\Omega}{\theta} \right)^r \frac{\|f\|_{r,\Omega}}{(r+1)!}. \tag{76}$$

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