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Bivariate Hermite–Birkhoff polynomial interpolation with asymptotic conditions

J.M. Carnicer *, M. Gasca

Departamento de Matemática Aplicada, Universidad de Zaragoza, Edificio de Matemáticas, Planta 1a, 50009 Zaragoza, Spain

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Abstract

Some asymptotic conditions along prescribed directions are added to the usual interpolation data in bivariate problems. These asymptotic conditions are written in terms of interpolation and then the new problem is studied in the frame of the interpolation systems introduced by Gasca and Maeztu some years ago. A Newton type interpolation formula is obtained for the enlarged problem and then some particular cases are studied.

1. Introduction

A Newton-like approach to multivariate interpolation problems was suggested in [5]. The interpolation space associated with a given problem is described as the space spanned by a Newton basis. A natural question which arises is how to characterize these spaces. A desirable description should be intuitive and suggested by the data.

The interpolation spaces are subspaces of the space of polynomials not exceeding a given degree, say M. A common feature of those subspaces is that some of the monomial terms of the highest degrees are missing. We try to describe this property by ensuring that some limits of the polynomials are zero when the variables tend to infinity. This idea leads to the concept of asymptotic conditions.

On the one hand, Dyn and Ron [4] studied some interpolation problems, whose associated interpolation spaces can be interpreted in terms of vanishing asymptotic conditions (see [2]). On the other, asymptotic conditions have a precedent in the work of Bojanov et al. in [1,7]. Each polynomial, when restricted to an affine submanifold of $\mathbb{R}^s$ (a trace of a polynomial), can be interpreted as a polynomial in less than s variables. Improper submanifolds can be seen as intersections of proper

* Corresponding author.

E-mail addresses: carnicer@posta.unizar.es (J.M. Carnicer), gasca@posta.unizar.es (M. Gasca).

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affine submanifolds when they tend to be parallel and a limiting polynomial trace can be defined for improper manifolds.

Our aim is to define general asymptotic conditions for describing interpolation spaces in [5].

Let M be a given positive integer, p a bivariate polynomial of total degree not greater than M, u_0 a point of the affine space $\mathbb{R}^2$ and $v=(v_x,v_y)$ a vector of $\mathbb{R}^2$ different from 0. As usual, we denote by D_v the *directional derivative* operator defined by

$$D_v f := v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} \tag{1}$$

and then, for a parameter $\lambda = 0$, we have

$$\lambda^M p\left(u_0 + \frac{1}{\lambda} v\right) = \sum_{i=0}^{M} \frac{1}{i!} D_v^i p(u_0) \lambda^{M-i} = q(\lambda),$$

where $q(\lambda)$ is a univariate polynomial of degree not greater than M.

For $0 \leqslant k \leqslant M$ we define

$$D^k_{\infty,M,u_0,v} p := D^k q(0) = \frac{k!}{(M-k)!} D_v^{M-k} p(u_0). \tag{2}$$

Equivalently, one has

$$D^k_{\infty,M,u_0,v} p = k! \lim_{\mu\to\infty} \frac{p(u_0+\mu v) - \sum_{i=0}^{k-1} (1/i!) \mu^{M-i} D^i_{\infty,M,u_0,v} p}{\mu^{M-k}}, \quad k \geqslant 1 \tag{3}$$

with

$$D^0_{\infty,M,u_0,v} p = \lim_{\mu\to\infty} \frac{p(u_0+\mu v)}{\mu^M}. \tag{4}$$

In other words, $D^k_{\infty,M,u_0,v} p$ is the result of multiplying by $k!$ the coefficient of μ^{M-k} in the univariate polynomial $p(u_0+\mu v)$.

Obviously, if the degree of p is less than M then $D^k_{\infty,M,u_0,v} p = 0$ for $0 \leqslant k \leqslant M - \deg(p) - 1$ while, if the exact degree of p is M and we write

$$p(x,y) = \sum_{i=0}^{M} p_i(x,y)$$

with p_i homogeneous polynomial of degree i, then one has

$$D^0_{\infty,M,u_0,v} p = p_M(v_x, v_y).$$

In this paper, we shall be concerned with the problem of determining a bivariate polynomial p of prescribed total degree not greater than M, which satisfies some usual interpolation conditions and some additional asymptotic conditions. Here, by usual interpolation conditions we understand values of p and/or some of its directional derivatives at prescribed points. By asymptotic conditions we mean that the values of some operators of the type $D^k_{\infty,M,u_0,v}$ applied to p and/or some of its derivatives are also prescribed. The problem will be stated in a precise form in Section 2 and its solution will be constructed in a recursive way in Section 3. In Section 4, we study some particular problems with vanishing asymptotic conditions which give rise to some interesting interpolation spaces: spaces of polynomials of a given total degree which are of other given lower degrees when restricted to any straight line of some prescribed directions. Finally, some examples are provided in Section 5.

2. Statement of the problem

Let us consider a straight line in $\mathbb{R}^2$ of equation $ax + by + c = 0$, $(a, b) \neq (0, 0)$. In the sequel, we denote by r indistinctly the line and the polynomial $ax + by + c$. Since this polynomial is determined by the line only up to a constant factor, we may choose, for example, $(a, b) = (\cos\theta, \sin\theta)$, for some $\theta \in (-\pi/2, \pi/2]$ or any other suitable choice. Once the polynomial r associated to the line has been chosen, it will remain unchanged in the rest of the paper.

A general interpolation problem consists of finding a function of a certain (simple) linear space of dimension N satisfying a set of N linear conditions. Since in our case we shall consider bivariate interpolation problems, it will be convenient to introduce the following set of indices:

$$I := \{(i, j) \mid i = 0, 1, \ldots, n; j = 0, 1, \ldots, m(i)\}, \tag{5}$$

lexicographically ordered

$$(i, j) < (k, l) \quad \Leftrightarrow \quad i < k \text{ or } (i = k \text{ and } j < l). \tag{6}$$

The interpolation conditions are determined by a set of linear functionals L_{ij}, $(i, j) \in I$, on the interpolation space. Each of these functionals will be supported on a single point u_{ij}, unless an asymptotic condition is involved. We shall use a set of flags w_{ij} indicating which kind of condition is introduced. If $w_{ij} = 1$ then $L_{ij} f$ is the value of f or some derivative of f at u_{ij}, whereas if $w_{ij} = 0$ we associate to the index (i, j) an asymptotic condition.

For each i all the points $\{u_{ij} \mid j = 0, \ldots, m(i)\}$ will be collinear, belonging to the same line r_i, and we shall use a set of *transversal* (that is, neither parallel nor coincident with r_i) lines r_{ij} to determine the position of the points u_{ij}, $j = 0, \ldots, m(i)$, on the line r_i. In other words, we shall have $r_i \cap r_{ij} = u_{ij}$.

Therefore, $n + 1$ indicates the number of lines r_i $(i = 0, \ldots, n)$, containing all points. Usually, the lines r_i are numbered in order to have

$$m(0) \geqslant \cdots \geqslant m(n) \tag{7}$$

to keep the total degree of the interpolating functions as low as possible. Under this condition the interpolation space will be a subspace of the space Π_M of bivariate polynomials of degree not greater than

$$M := \max_{(i,j) \in I} \{i + j\}. \tag{8}$$

The integer M can be seen as an upper bound for the total degree of all polynomials and a starting (or reference) degree for describing sequences of polynomials with decreasing degree.

Some years ago, a Newton approach was introduced in [5,6] to deal with bivariate interpolation problems. The idea was recently extended in [2] to allow one asymptotic condition on some lines. In order to extend this situation again, to allow several asymptotic conditions at the same line, we define an *interpolation system* as a set of triples

$$S := \{(r_i, r_{ij}, w_{ij}) \mid (i, j) \in I)\} \tag{9}$$

with I given by (5), satisfying the following conditions for all (i, j):

Condition 1. *The lines r_i, r_{ij} intersect at exactly one point u_{ij}.*

Condition 2. *If $w_{ij} = 0$ for some $(i, j) \in I$, then $m(i) = M - i$ (with M given by (8)) and moreover, $\forall k > j$, one has $w_{ik} = 0$.*

Let us denote, for each i with some $w_{ij} = 0$, $m_i := \min\{j \mid w_{ij} = 0\}$. Then, due to Condition 2 we have, for each of these i's,

$$w_{ij} = \begin{cases} 1 \text{ for } j = 0, \ldots, m_i - 1, \\ 0 \text{ for } j = m_i, m_i + 1, \ldots, M - i. \end{cases} \tag{10}$$

Obviously, for the other i's,

$$w_{ij} = 1, \quad j = 0, 1, \ldots, m(i).$$

Remark that parallel or coincident lines are allowed in S taking into account the specified conditions above: the only lines which have to be transversal are r_i and r_{ij} for all $(i, j) \in I$.

If $w_{ij} = 1$ for all (i, j), the interpolation system will give rise to a Hermite–Birkhoff interpolation problem of the type considered in [5]. Else we will have a Hermite–Birkhoff interpolation problem with one or several asymptotic conditions along some lines. In [2] only one asymptotic condition was allowed on each line r_i and a third condition was assumed in order to use only asymptotic data of the form (4).

For a given interpolation system S we define a set of linear functionals on Π_M. These linear functionals L_{ij} will be called the *interpolation data*. First, we associate to each $(i, j) \in I$ two numbers s_i, t_{ij}. The number of lines r_h, $h < i$, which are coincident with r_i will be denoted by s_i. If $w_{ij} = 1$, then the number of lines $r_0, r_1, \ldots, r_{i-1}, r_{i0}, r_{i1}, \ldots, r_{i,j-1}$, which contain u_{ij} but are not coincident with r_i will be denoted by t_{ij}. The same notation will be used, in the case $w_{ij} = 0$, to denote the number of lines $r_0, r_1, \ldots, r_{i-1}$ which are parallel, not coincident, with r_i, plus the number of indices (i, k) with $k < j$ and $w_{ik} = 0$.

Now, the linear functionals L_{ij} are defined for $f \in \Pi_M$ and $(i, j) \in I$ in the form

$$L_{ij} f := \begin{cases} D^{s_i}_{\rho_{ij}} D^{t_{ij}}_{\rho_i} f(u_{ij}) & \text{if } w_{ij} = 1, \\ D^{s_i + t_{ij}}_{\infty, M, u_{ij}, \rho_i} D^{s_i}_{\rho_{ij}} f & \text{if } w_{ij} = 0, \end{cases} \tag{11}$$

where $\rho_i = (-b_i, a_i)$, $\rho_{ij} = (-b_{ij}, a_{ij})$ are directional vectors of the lines $r_i = a_i x + b_i y + c_i$, $r_{ij} = a_{ij} x + b_{ij} y + c_{ij}$, respectively.

As it can be easily deduced from the proof of our next theorem, the data of the type $D^{s_i + t_{ij}}_{\infty, M, u_{ij}, \rho_i} D^{s_i}_{\rho_{ij}} f$ can be replaced in (11) by $D^{t_{ij}}_{\infty, M - s_i, u_{ij}, \rho_i} D^{s_i}_{\rho_{ij}} f$.

We also associate to the system S a *Newton basis* $B(S)$, the set of polynomials ϕ_{ij}, $(i, j) \in I$, defined in the form

$$\phi_{ij} := \begin{cases} \displaystyle\prod_{h=0}^{i-1} r_h \prod_{k=0}^{j-1} r_{ik} & \text{if } w_{ij} = 1, \\ \displaystyle\prod_{h=0}^{i-1} r_h \prod_{k=0}^{m_i - 1} r_{ik} \prod_{k=j}^{M-i-1} r_{ik} & \text{if } w_{ij} = 0, \end{cases} \tag{12}$$

where empty products (when $i = 0$ or $j = 0$) equal 1. Observe that the degree of ϕ_{ij} is $i + j$ if $w_{ij} = 1$, and $M - j + m_i$ elsewhere.

We shall see later that these polynomials are linearly independent and why we may call $B(S)$ a Newton basis.

The interpolation space $V(S)$ of our problem is the space spanned by the polynomials ϕ_{ij} of $B(S)$. Consequently, it will be a subspace of Π_M.

We denote by $P(S)$ the interpolation problem defined by the linear functionals L_{ij}, $(i,j) \in I$ on the space $V(S)$.

Let us now analyse the existence of solutions of this interpolation problem.

Theorem 1. *Let S be an interpolation system satisfying conditions* 1 *and* 2. *Let $P(S)$ be the interpolation problem defined by the linear functionals L_{ij} of* (11) *and the space $V(S)$ spanned by the polynomials ϕ_{ij} of* (12). *For each set of real numbers z_{ij}, $(i,j) \in I$, there exists a unique polynomial $p \in V(S)$ such that*

$$L_{ij}\, p = z_{ij} \quad \textit{for all } (i,j) \in I.$$

Moreover, the matrix

$$(L_{ij}\phi_{hk})_{(i,j),(h,k)\in I} \tag{13}$$

is lower triangular, for the row indices (i,j) and column indices (h,k) ordered by (6).

Proof. Any polynomial $p \in V(S)$ can be written in the form

$$p = \sum_{(h,k)\in I} a_{hk}\phi_{hk} \tag{14}$$

with ϕ_{ij} given by (12). Then the existence and uniqueness of solution of $P(S)$ is equivalent to the nonsingularity of the matrix (13). In order to show that this matrix is nonsingular, we shall prove that it is lower triangular with nonzero diagonal elements:

$$L_{hk}\phi_{ij} = 0 \quad \text{for all } (h,k) < (i,j), \quad L_{ij}\phi_{ij} \neq 0 \quad \text{for all } (i,j) \in I. \tag{15}$$

We first consider $(h,k) \leqslant (i,j)$ with $w_{hk} = 1$. In this case the arguments of the proof of Theorem 1 of [5] still work to prove (15). That proof is based on the following results which are direct consequences of the definitions and of the Leibniz rule for differentiation:

(a) If $i,h \in \{0,1,\ldots,n\}$, then

$$\frac{\partial r_h}{\partial \rho_i} = A_{hi} = -\frac{\partial r_i}{\partial \rho_h} \tag{16}$$

with $A_{hi} \neq 0$ except if r_i and r_h are parallel or coincident.

(b) If $(i,j) \in I$ then

$$\frac{\partial r_i}{\partial \rho_{ij}} = -\frac{\partial r_{ij}}{\partial \rho_i} = B_{ij} \neq 0. \tag{17}$$

(c) If w is any polynomial, then for any nonnegative integer s and $(i,j) \in I$ one has

$$\frac{\partial^s r_i w}{\partial \rho_i^s} = r_i \frac{\partial^s w}{\partial \rho_i^s}, \tag{18}$$

$$\frac{\partial^s r_i^s w}{\partial \rho_{ij}^s} = s! B_{ij}^s w + r_i w_1, \tag{19}$$

$$\frac{\partial^s r_i^{s+1} w}{\partial \rho_{ij}^s} = r_i w_2, \tag{20}$$

where w_1, w_2 are also polynomials.

The main interest of these formulas is to prove that if r_i vanishes then the left-hand sides of (18) and (20) also vanish, while the left-hand side of (19) still depends on whether w vanishes or not.

(d) If $v_1, \ldots, v_k$ are affine polynomials and w is an arbitrary polynomial, then for any nonnegative integer s and any vector $\rho \in \mathbb{R}^2$ one has

$$\frac{\partial^s v_1 \cdots v_k w}{\partial \rho^s} = \sum_{t=0}^{k} t! \binom{s}{t} \frac{\partial^{s-t} w}{\partial \rho^{s-t}} \sum_{h_1,\ldots,h_t} \frac{\partial v_{h_1}}{\partial \rho} \cdots \frac{\partial v_{h_t}}{\partial \rho} \frac{v_1 \cdots v_k}{v_{h_1} \cdots v_{h_t}},$$

where the summation $\sum_{h_1,\ldots,h_t}$ ranges over all subsets of $\{1, 2, \ldots, k\}$ having t different elements $h_1 < \cdots < h_t$. When $t = 0$ this summation reduces to $v_1 \cdots v_k$.

Coming back to our present proof, if $w_{hk} = 0$ and $w_{ij} = 1$, then

$$L_{hk} \phi_{ij} = D^{s_h + t_{hk}}_{\infty, M, u_{hk}, \rho_h} D^{s_h}_{\rho_{hk}} \phi_{ij}. \tag{21}$$

Since Condition 2 implies that $h < i$, the polynomial ϕ_{ij} contains at least $s_h + 1$ times the factor r_h and consequently $D^{s_h}_{\rho_{hk}} \phi_{ij}$ contains this factor once at least. When we replace (x, y) in $D^{s_h}_{\rho_{hk}} \phi_{ij}(x, y)$ by $u_{hk} + \mu \rho_h$ to compute (21) we get the null polynomial and therefore $L_{hk} \phi_{ij} = 0$. The same argument can be used for the case $w_{ij} = 0$, $w_{hk} = 0$ with $h < i$.

Consequently, it only remains to prove the case $h = i$ with $k \leqslant j$ and $w_{hk} = 0$.

As it has been noted after Eq. (12), the degree of ϕ_{ij} is $M - j + m_i$, hence the degree of $D^{s_h}_{\rho_{hk}} \phi_{ij}$ is $M - j + m_i - s_h$.

For the affine polynomial $r_l(x, y)$, associated to the line r_l, the result of replacing the variable (x, y) by $u_{hk} + \mu \rho_h$ is a univariate polynomial of exact degree 1 in μ, if r_l has not the direction of r_h, but it is a constant different from zero when r_l is parallel, not coincident, with r_h. Obviously, it is zero when r_l and r_h are coincident lines. On the other hand, by definition of t_{hk} there are exactly $t_{hk} - (k - m_h)$ lines r_l, $l < h$, parallel, not coincident with r_h. Due to all these facts and taking into account that in the present case $h = i$, one has

$$\deg(D^{s_i}_{\rho_{ik}} \phi_{ij}(u_{ik} + \mu \rho_i)) = M - j + k - s_i - t_{ik}. \tag{22}$$

Now we have two possibilities. If $k < j$, then the coefficient of $\mu^{M - s_i - t_{ik}}$ is zero and consequently

$$L_{ik} \phi_{ij} = D^{s_i + t_{ik}}_{\infty, M, u_{ik}, \rho_i} D^{s_i}_{\rho_{ik}} \phi_{ij} = 0.$$

Otherwise, if $k = j$, (22) becomes

$$\deg(D^{s_i}_{\rho_{ij}} \phi_{ij}(u_{ij} + \mu \rho_i)) = M - s_i - t_{ij}. \tag{23}$$

By applying Leibniz rule to compute $D^{s_i}_{\rho_{ij}} \phi_{ij}$ we easily see that it is a sum with only one summand with no factor coincident with r_i, namely

$$K \prod_{l}' r_l \prod_{v=0}^{m_i - 1} r_{iv} \prod_{v=j}^{M-i-1} r_{iv}, \tag{24}$$

where $\prod_l' r_l$ is the product of the factors $r_0, r_1, \ldots, r_{i-1}$ excluding the s_i ones which are coincident with r_i. K is a constant factor which, according to (17) is not 0. Hence, only this summand will remain when we replace (x, y) by $u_{ij} + \mu\rho_i$ in $D^{s_i}_{\rho_{ij}}\phi_{ij}(x, y)$ to compute $D^{s_i}_{\rho_{ij}}\phi_{ij}(u_{ij} + \mu\rho_i)$.

The degree of (24) is $M - j + m_i - s_i$. On the other hand, in that product there are exactly $t_{ij} - (j - m_i)$ factors corresponding to lines r_l parallel (not coincident) with r_i and all the rest corresponding to lines with directions different from that of r_i. By computing $D^{s_i}_{\rho_{ij}}\phi_{ij}(u_{ij} + \mu\rho_i)$ we get a univariate polynomial of *exact* degree $M - j + m_i - s_i - (t_{ij} - (j - m_i)) = M - s_i - t_{ij}$ in μ.

Therefore, $L_{ij}\phi_{ij} = D^{s_i+t_{ij}}_{\infty,M,u_{ij},\rho_i} D^{s_i}_{\rho_{ij}}\phi_{ij} \neq 0$. $\square$

As a consequence of Theorem 1, we get that $B(S)$ is a basis of $V(S)$ and moreover that it is a Newton basis in the sense that (15) holds.

Sometimes we have a bivariate function f whose asymptotic behavior along some lines is known in the following sense:

$$\lim_{\mu\to\infty} \frac{f(u_0 + \mu v)}{\mu^M} = A_0,$$

$$\lim_{\mu\to\infty} \frac{f(u_0 + \mu v) - \sum_{i=0}^{k-1}(1/i!)\mu^{M-i}A_i}{\mu^{M-k}} = \frac{A_k}{k!}, \quad k \geqslant 1$$

for some finite A_i, $0 \leqslant i \leqslant k$.

If we want to approximate that function (for example a rational function) by an interpolating polynomial, we can denote $D^i_{\infty,M,u_0,v}f := A_i$ and use these data as asymptotic data. For such a function, if the problem of finding a polynomial p satisfying

$$L_{ij}\,p = L_{ij}f \quad \forall (i,j) \in I$$

has a unique solution we can say that p interpolates f (in the sense of the data L_{ij}).

Remark 1. Observe that, for any $(i, m(i))$ with $w_{i,m(i)} = 1$, the role of $r_{i,m(i)}$ is only to indicate the point $u_{i,m(i)}$ used in $L_{i,m(i)}$, but $r_{i,m(i)}$ does not appear as a factor in any of the polynomials ϕ_{hk}. On the other hand if $w_{ij} = 0$ the lines r_{ik}, $k \geqslant j$ are taken transversal to r_i only in order that Theorem 1 hold but neither the directions of these lines nor the positions of the points u_{ik} are essential and can be chosen arbitrarily.

3. Construction of the solution

As another consequence of Theorem 1, we derive that the solution p of the interpolation problem $P(S)$ can be constructed by recurrence. For p written in form (14), the fact that matrix (13) is lower triangular implies that

$$L_{ij}\,p = \sum_{(h,k)\leqslant(i,j)} a_{hk} L_{ij}\phi_{hk} \tag{25}$$

and, therefore, the coefficients a_{hk} can be computed using the following recurrence relations:

$$a_{00} := \frac{z_{00}}{L_{00}\phi_{00}},$$

$$a_{ij} := \frac{z_{ij} - \sum_{(h,k)<(i,j)} a_{hk} L_{ij}\phi_{hk}}{L_{ij}\phi_{ij}}. \tag{26}$$

The similarity between this recurrence relation and the one for defining univariate divided differences justifies the name Newton basis which we give to $B(S)$. Analogously, formula (14) can be called a *Newton interpolation formula* for the problem $P(S)$ and the coefficients a_{ij} of (14) defined by the recursion (26) can be seen as *divided differences* associated to this problem, playing a similar role to that of univariate divided differences.

We want to remark a particular case of special interest for us. If one has

$$w_{ij} = 0 \Rightarrow L_{ij}\phi_{hk} = 0 \quad \forall (h,k) < (i,j) \quad \text{with } w_{hk} = 1, \tag{27}$$

then the row indexed with (i,j) in matrix (13), namely $(L_{ij}\phi_{hk})_{(h,k)\in I}$, has zeros on all columns indexed by (h,k) with $w_{hk} = 1$ and, as always, also on all columns with $(h,k) > (i,j)$. So, we can write

$$a_{ij} = \frac{z_{ij} - \sum_{\{(h,k)<(i,j);(h,k)\in I, w_{hk}=0\}} a_{hk} L_{ij}\phi_{hk}}{L_{ij}\phi_{ij}}. \tag{28}$$

Note that the sum includes only terms with $(h,k) < (i,j)$ satisfying $w_{hk} = 0$.

If (27) holds for all (i,j) with $w_{ij}=0$, all the coefficients a_{ij} corresponding to asymptotic conditions can be computed first, independently of the other ones and the problem can be easily reduced to a standard Hermite–Birkhoff interpolation problem of the type considered in [5].

In the sequel, we shall say (as usually is said) that *two lines have the same direction* if they are either parallel or coincident.

Proposition 2. *Condition* (27) *holds for a given* (i,j) *with* $w_{ij} = 0$ *if for all* $(h,k) < (i,j)$ *with* $w_{hk} = 1$ *and* $h + k \geqslant M - t_{ij}$ *there exist at least* $h + k - M + t_{ij} + s_i + 1$ *straight lines in the set* $\{r_0, r_1, \ldots, r_{h-1}, r_{h0}, \ldots, r_{h,k-1}\}$ *having the same direction as* r_i.

Proof. For (i,j) and (h,k) with $w_{ij} = 0$ and $w_{hk} = 1$, one has

$$L_{ij}\phi_{hk} = D^{s_i+t_{ij}}_{\infty,M,u_{ij},\rho_i} D^{s_i}_{\rho_{ij}} r_0 \cdots r_{h-1} r_{h0} \cdots r_{h,k-1}. \tag{29}$$

Here M is defined by (8).

The degree of the polynomial

$$D^{s_i}_{\rho_{ij}} r_0 \cdots r_{h-1} r_{h0} \cdots, r_{h,k-1}$$

is not greater than $h+k-s_i$ and, by (3), $L_{ij}\phi_{hk}$ is the result of multiplying by $(s_i+t_{ij})!$ the coefficient of $\mu^{M-s_i-t_{ij}}$ in the univariate polynomial obtained replacing (x,y) in $D^{s_i}_{\rho_{ij}} r_0 \cdots r_{h-1} r_{h0} \cdots r_{h,k-1}$ by $u_{ij} + \mu\rho_i$.

Obviously, if $h + k - s_i < M - s_i - t_{ij}$, that coefficient is 0. Hence, for $(h,k) < (i,j)$ satisfying $h+k < M-t_{ij}$ we get $L_{ij}\phi_{hk}=0$ as in (27), and for this reason those indices (h,k) are not mentioned in the statement of Proposition 2.

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If $h+k \geqslant M - t_{ij}$ we have to analyse $D^{s_i}_{\rho_{ij}}\phi_{hk}$ more in detail. It is a sum with each summand the product of a real number by $h+k-s_i$ of the $h+k$ factors $r_0, r_1, \ldots, r_{h-1}, r_{h0}, r_{h1}, \ldots, r_{h,k-1}$. If at least $h+k-M+t_{ij}+s_i+1$ of these $h+k$ factors decrease their degree when we restrict the variable to a straight line of the direction of ρ_i, it is easy to deduce that each summand of the sum mentioned above is a univariate polynomial of degree not greater than

$$h+k-s_i-(h+k-M+t_{ij}+1) = M - s_i - t_{ij} - 1$$

and consequently the same happens for the whole sum.

Hence $L_{ij}\phi_{hk} = D^{s_i+t_{ij}}_{\infty,M,u_{ij},\rho_i}(D^{s_i}_{\rho_{ij}}\phi_{hk}) = 0$, that is (27). □

A particularly simple and important case happens when

$$n = M, \quad m(i) = M - i, \;\; i = 0, \ldots, M, \tag{30}$$

that is, equivalently, when $I = \{(i,j) \mid i+j \leqslant M\}$. In this case one has $\#I = \binom{M+2}{2}$, which is exactly the dimension of Π_M and therefore $V(S) = \Pi_M$. Conversely, it is straightforward to see that the unique choice of I which produces $V(S) = \Pi_M$ in our approach is (30).

4. Interpolation spaces and vanishing asymptotic conditions

Let S be an interpolation system (9) indexed by a set I given by (5). Let $I^* := \{(i,j) \in I \mid w_{ij} = 0\}$ and define $\hat{I} := I \setminus I^*$ and

$$\hat{V}(S) := \{p \in V(S) \mid L_{ij}p = 0, \; \forall (i,j) \in I^*\}. \tag{31}$$

From Theorem 1, we know that the interpolation problem $P(S)$ always has a unique solution. In particular there exists a unique $p \in V(S)$ such that

$$\begin{aligned} L_{ij}p &= z_{ij}, \quad \forall (i,j) \in \hat{I}, \\ L_{ij}p &= 0, \quad \forall (i,j) \in I^*. \end{aligned} \tag{32}$$

This is equivalent to saying that the problem $\hat{P}(S)$ of finding a function $p \in \hat{V}(S)$ such that $L_{ij}p = L_{ij}f$, for all $(i,j) \in \hat{I}$ has always a unique solution. Moreover, p can be found recursively by (26), which means that p is written as a linear combination of the basis of $V(S)$. This seems not natural because p belongs to the subspace $\hat{V}(S)$ of $V(S)$.

Trying to find a basis of $\hat{V}(S)$ one could be tempted to think of the set $\{\phi_{ij} \mid (i,j) \in \hat{I}\}$, but in general, there are no reasons for the space spanned by it

$$W(S) = \mathrm{span}\{\phi_{ij} \mid (i,j) \in \hat{I}\}$$

to coincide with $\hat{V}(S)$. They are just subspaces of the same dimension $\#\hat{I}$ of $V(S)$ which are coincident only under some extra condition.

Proposition 3. *Under the notation above, if condition* (27) *holds for all* $(i,j) \in I^*$, *then one has* $W(S) = \hat{V}(S)$ *and* $\{\phi_{ij} \mid (i,j) \in \hat{I}\}$ *is a basis of* $\hat{V}(S)$. *In this case the solution* p *of the problem* $\hat{P}(S)$ *can be written*

$$p = \sum_{(h,k)\in I^*} a_{hk}\phi_{hk}$$

and constructed by (26) *with the indices* $(i,j),(h,k)$ *always in* I^*.

Proof. As it has been seen in Section 3, under condition (27) all the coefficients a_{ij} with $(i,j) \in I^*$ can be computed first and since in our present problem the interpolation conditions are given by (32) those coefficients are all 0. Since the numbers z_{ij} in (32) are arbitrary, this means that in this case $W(S) = \hat{V}(S)$ and $\{\phi_{ij} \mid (i,j) \in \hat{I}\}$ is a basis of $\hat{V}(S)$. The rest of the proposition is straightforward. □

In a recent paper [2], we have studied the particular case of at most one asymptotic condition per direction.

5. Examples

Let us consider some simple examples. Let

$$I = \{(0,0),(0,1),(0,2),(1,0),(1,1),(2,0)\},$$

$r_0 = x - y,\ r_1 = x - y + 1,\ r_2 = x - y + 2,\ r_{00} = r_{10} = r_{20} = x,\ w_{00} = w_{10} = w_{20} = 1,\ w_{01} = w_{02} = w_{11} = 0.$ The lines r_{01}, r_{02}, r_{11} can be chosen arbitrarily.

Here $M = 2$, the space is Π_2, the Newton basis is

$$\{1, x, x(x-1), x-y, (x-y)x, (x-y)(x-y+1)\}$$

and the interpolation data

$$L_{00}p = p(0,0), \quad L_{01}p = D^0_{\infty,2,(1,1),(1,1)}p, \quad L_{02}p = D^1_{\infty,2,(2,2),(1,1)}p$$

$$L_{10}p = p(0,1), \quad L_{11}p = D^1_{\infty,2,(1,2),(1,1)}p, \quad L_{20}p = p(0,2).$$

Let us take $z_{00} = f(0,0),\ z_{01} = z_{02} = 0,\ z_{10} = f(0,1),\ z_{11} = 0,\ z_{20} = f(0,2)$. Then the solution of the problem can be found recursively by (26) in the following order: $a_{00} = f(0,0),\ a_{0,1} = 0 = a_{02},\ a_{10} = f(0,0) - f(1,0),\ a_{11} = 0,\ a_{20} = (f(0,2) - 2f(0,1) + f(0,0))/2$:

$$p(x,y) = f(0,0) + (f(0,0) - f(1,0))(x-y) + \frac{f(0,2) - 2f(0,1) + f(0,0)}{2}(x-y)(x-y+1).$$

We observe that (27) holds and therefore, since $z_{01} = z_{02} = z_{11} = 0$, we could have deduced in advance that $a_{01} = a_{02} = a_{11} = 0$ and obtained immediately a_{00}, a_{10} and a_{20}. The solution p belongs to the subspace of Π_2 which satisfies $L_{01} = L_{02} = L_{11} = 0$, that is, the space of quadratic polynomials which become constant (degree 0) when restricted to any line parallel to $x - y = 0$.

Analogously, we can consider for example the case when $r_0 = r_1 = x - y,\ r_2 = x - y + 2$ with all the rest of the lines the same as above. Now $L_{10}p = (\partial/\partial y)p(0,0)$ and

$$L_{11}p = D^1_{\infty,2,(1,1),(1,1)}\frac{\partial}{\partial y}p.$$

The rest of the data and the space are the same as above, and the solution is

$$p(x,y) = f(0,0) - \frac{\partial f}{\partial y}(0,0)(x-y) + \frac{f(0,2) - 2\frac{\partial f}{\partial y}(0,0) - f(0,0)}{4}(x-y)^2.$$

Let us come back to the first example and just replace $w_{01}=0$ by $w_{01}=1$. The Newton basis and space remain, but two of the interpolation data change: now one has $L_{01}p=p(1,1)$ and $L_{02}p=D^0_{\infty,2,(2,2),(1,1)}p$. Assume, as in the first example, that we take $z_{02}=0$. The coefficients of p can be constructed recursively again, but observe that in this case (27) does not hold because $L_{11}x\neq 0$. The solution is

$$\begin{aligned}p(x,y)=&\,f(0,0)+(f(1,1)-f(0,0))x+(f(0,0)-f(1,0))(x-y)\\&+(f(1,1)-f(0,0))(x-y)x\\&+\frac{f(0,2)-2f(0,1)+f(0,0)}{2}(x-y)(x-y+1).\end{aligned}$$

It belongs to the space of quadratic polynomials which are of degree one when restricted to any line parallel to $x-y=0$ and in particular, become constant when restricted to the line $x-y+1=0$.

6. Uncited References

[3]

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