Aitken–Neville sets, principal lattices and divided differences

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Abstract

In this paper we study multivariate polynomial interpolation on Aitken–Neville sets by relating them to generalized principal lattices. We express their associated divided differences in terms of spline integrals.

Key words: Polynomial interpolation, principal lattice, Aitken–Neville set, divided difference

Dedicated to Günter Mühlbach on occasion of his 65th birthday.

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[1] Partially supported by the Spanish Research Grant MTM2006-03388, by Gobierno de Aragón and Fondo Social Europeo.

1 Introduction

In 1932, A. C. Aitken [1] introduced a method which allowed for the simple computation of the value of an interpolation polynomial at a given point, with the main goal of "filling in" values in tables of functions. Aitken's approach was modified a little bit later by E. H. Neville [11] and nowadays the *Aitken–Neville scheme* can be found in most textbooks on Numerical Analysis. An extension to a more general situation has been provided by G. Mühlbach in [10]. From a more geometric point of view the Aitken–Neville scheme can be considered as repeated or iterated (which explains the title of [11]) *linear interpolation*, or *linear extrapolation*, to be precise, since in many cases the linear interpolant is evaluated "outside" the convex hull of the interpolation points.

The geometric idea of repeated linear interpolation can be generalized in a straightforward fashion to several variables by using a *multiindexed* set of interpolation points x_α, but it turns out that there is a restrictive geometric condition that has to be satisfied by these points in order to make the geometric Aitken–Neville scheme work. We repeat this condition, given in [15], in Theorem 1. Intuitively it says that whenever a multiindex α can be expressed as a *convex combination* of some other multiindices, then the associated point x_α must be representable as an *affine combination* of the respective points. This condition is always satisfied in the univariate case, but a very strong one in more than one variable.

In this paper, we relate Aitken–Neville configurations to another relevant concept from multivariate polynomial, namely to *generalized principal lattices* which are special point configurations satisfying the *geometric characterization* due to Chung and Yao [7]. In [9] concrete examples of sets satisfying the Chung and Yao condition are provided (more detailed information can be found in the book [12]). Such sets are also generalized principal lattices and Aitken-Neville sets.

It will turn out that any generalized principle lattice is an Aitken–Neville configuration and thus permits evaluation by iterated linear interpolation. The converse will be shown in this paper for dimension $d = 2$ and degree $n \geq 3$, but apparently in three or more variables there are difficulties beyond mere technicalities of the proof, see Example 15. Moreover, we consider the divided differences associated to Aitken–Neville configurations and derive a spline representation for them. Here, as in [14], the *divided difference* means the leading homogeneous form of the interpolation polynomials or, equivalently, the coefficient vector of this homogeneous form.

2 Notation and preliminaries

We consider polynomial interpolation in d variables by polynomials of a fixed maximal total degree n; the exponents appearing in such polynomials are *multiindices* and we denote by

$$\Gamma_k := \left\{ \alpha \in \mathbb{N}_0^d : k = |\alpha| = \alpha_1 + \cdots + \alpha_d \right\}$$

the set of all multiindices of *length* k. Here $\mathbb{N}_0$ denotes the set of all nonnegative numbers. Associated with Γ_k is the $1 \times \Gamma_k$ matrix

$$\boldsymbol{x}^k = (x^\alpha : \alpha \in \Gamma_k), \qquad k \in \mathbb{N}_0,$$

i.e., the row vector of all monomials of total degree k which spans Π_k^0, the vector space of all *homogeneous* polynomials of degree k, i.e.,

$$\Pi_k^0 = \left\{ \boldsymbol{x}^k \boldsymbol{c} : \boldsymbol{c} \in \mathbb{R}^{\Gamma_k \times 1} \right\}$$

and we can conveniently write any polynomial p as

$$p(x) = \sum_{k=0}^{n} \boldsymbol{x}^k \boldsymbol{p}_k, \qquad \boldsymbol{p}_k \in \mathbb{R}^{\Gamma_k \times 1}, k = 0, \ldots, n,$$

for some $n \in \mathbb{N}_0$. The *total degree* of p is then given as

$$\deg p = \max \left\{ k : \boldsymbol{p}_k \neq 0 \right\}.$$

It is well–known, cf. [8], that any finite set of points $X \subset \mathbb{R}^d$ that allows for unique interpolation from Π_n has to fulfill certain constraints. The cardinality $\# X$ of this set has to match the dimension of Π_n, which is $\binom{n+d}{d}$, but there is also the geometric condition on the points that requires them not to lie on an algebraic hypersurface of degree n, a property which is easy to phrase but hard to verify or to provide in general.

A particular class of interpolation points that not only allows for unique polynomial interpolation from Π_n but also provides a geometric way to evaluate the interpolant at any given point $x \in \mathbb{R}^d$ was considered in [15] by means of an extension of the classical *Aitken–Neville scheme* to several variables, using iterated linear interpolation as introduced in [1] and refined shortly afterwards in [11]. To explain this idea, we need the notion of *barycentric coordinates.* Given any vector $V = (v_0, \ldots, v_d)$ of $d+1$ points $v_j \in \mathbb{R}^d$, we say that they are *in general position* if the Lagrange interpolation problem on V by polynomials in Π_1 has a unique solution. The linear interpolant of a function f can be expressed by the Lagrange formula

$$p(x) = \sum_{j=0}^{d} f(v_j)\lambda_j(x|V),$$

where λ_j denotes the j-th *barycentric coordinate* with respect to V. We recall that the barycentric coordinates, determined by

$$\sum_{j=0}^{d} \lambda_j\,(x|V) = 1, \quad x = \sum_{j=0}^{d} \lambda_j\,(x|V)\,v_j$$

are the fundamental solutions of the affine interpolation problem at the vertices of the simplex $[V]$, hence they exist if and only if the associated interpolation problem is unisolvent in Π_1. By Cramer's rule, the barycentric coordinates can be explicitly written in terms of determinants as

$$\lambda_j\,(x|V) = \frac{\tau_j\,(x|V)}{\tau(V)},$$

where

$$\tau(V) := \det \begin{pmatrix} 1 & \dots & 1 \\ v_0 & \dots & v_d \end{pmatrix} \neq 0$$

and

$$\tau_j\,(x|V) := \det \begin{pmatrix} 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ v_0 & \dots & v_{j-1} & x & v_{j+1} & \dots & v_n \end{pmatrix}.$$

We can describe, for *any* $V = (v_0, \dots, v_k) \in \mathbb{R}^{k\times d}$, the *convex hull* $[V]$ and the *affine hull* $\langle V\rangle$ as

$$\Big\{\sum_{j=0}^{k} \lambda_j v_j : \lambda_j \geq 0, \sum_{j=0}^{k} \lambda_j = 1\Big\} =: [V] \subset \langle V\rangle := \Big\{\sum_{j=0}^{k} \lambda_j v_j : \sum_{j=0}^{k} \lambda_j = 1\Big\}.$$

To mimic the Aitken–Neville approach of iterated linear interpolation, [15] started with a set of points

$$X = \left\{x_\alpha : \alpha \in \Gamma_n^0\right\}, \tag{1}$$

where the set

$$\Gamma_n^0 = \left\{ \alpha = (\alpha_0, \dots, \alpha_d) \in \mathbb{N}_0^{d+1} \ : \ n = |\alpha| := \alpha_0 + \cdots + \alpha_d \right\}$$

of *homogenized* multiindices has the cardinality $\binom{n+d}{d}$ which coincides with $\dim \Pi_n$. We also assume that $\# X = \binom{n+d}{d}$, that is, different multiindices correspond to different points. Let us denote by $\epsilon_j \in \mathbb{N}_0^{d+1}$, $j \in \{0{:}d\} := \{0, \dots, d\}$, the multiindex whose j-th component is 1 and all other components are zero and define

$$V_\alpha^k := (x_{\alpha + k\epsilon_j} : j \in \{0{:}d\}), \quad \alpha \in \Gamma_{n-k}^0, \quad k = 1, 2, \dots, n.$$

We shall require that the points V_α^k are in general position, $\alpha \in \Gamma_{n-k}^0$, $k = 1, 2, \dots, n$, or, in other words, the interpolation problem at V_α^k is unisolvent in Π_1. Given the samples $f(x_\alpha)$ of a function f at X, the geometric generalization of the Aitken–Neville algorithm recursively computes, for given $x \in \mathbb{R}^d$ the values $f_\alpha^k(x)$, $\alpha \in \Gamma_{n-k}^0$, as

$$f_\alpha^0(x) = f(x_\alpha), \tag{2}$$

$$f_\alpha^k(x) = \sum_{j=0}^d \lambda_j \left(x | V_\alpha^k \right) f_{\alpha+\epsilon_j}^{k-1}(x), \qquad k = 1, \dots, n. \tag{3}$$

The outcome of this procedure is a polynomial $x \mapsto f_0^n(x)$ in Π_n which, in the univariate case, interpolates f at the prescribed sites of X. However, the univariate Aitken–Neville scheme gives more: any intermediate interpolant solves a well–structured subproblem of interpolation. Considering the subsets

$$X_\alpha^k := \left\{ x_{\alpha+\beta} : \beta \in \Gamma_k^0 \right\}, \qquad \alpha \in \Gamma_{n-k}^0,$$

then the multivariate extension of this "sub–interpolation" would be

$$f_\alpha^k \left(X_\alpha^k \right) = f \left(X_\alpha^k \right), \qquad \alpha \in \Gamma_{n-k}^0, \quad k = 0, \dots, n, \tag{4}$$

including the interpolation property of f_0^n. However, it cannot be expected any more that for arbitrary point configurations in $\mathbb{R}^d$, $d \geq 2$, the iteration (2), (3) would lead to interpolation properties as given in (4). There have to be additional constraints on the geometry of the point set X which were characterized as follows in [15].

Theorem 1 *Let $X = \{x_\alpha : \alpha \in \Gamma_n^0\}$ be a set such that all subsets V_β^k are unisolvent for Π_1 for all $k \in \{1, \dots, n\}$ and all $\beta \in \Gamma_{n-k}^0$. For all $f : X \to \mathbb{R}$*

the iteration (2), (3) gives solutions to the interpolation problems as in (4) if and only if for all $J \subset \{0{:}d\}$*, all* $0 \le k \le n$ *and all* $\beta \in \Gamma^0_{n-k}$*, we have that*

$$\gamma \in [\beta + k\epsilon_j : j \in J] \quad \Rightarrow \quad x_\gamma \in \left\langle x_{\beta+k\epsilon_j} : j \in J \right\rangle . \tag{5}$$

Observe that (5) means that the *convex* dependency structure of the multiindices must be reflected by the *affine* dependency structure of the associated points, that is, whenever a multiindex can be written as a convex combination of certain other multiindices, then the respective point is an *affine* combination of the points associated with these other multiindices. Since these are precisely the points that allow for a multivariate extension of the geometric Aitken–Neville scheme, the following definition makes sense.

Definition 2 A set X of $\binom{n+d}{d}$ points in $\mathbb{R}^d$ is called an *Aitken-Neville set* of order n if it can be so indexed as $X = \{x_\alpha : \alpha \in \Gamma^0_n\}$ that for all $k \in \{1, \dots, n\}$ and all $\beta \in \Gamma^0_{n-k}$, we have

$$V^k_\beta \text{ is unisolvent for } \Pi_1 \tag{6}$$

and

$$x_\gamma \in \langle x_{\beta+k\epsilon_j} : j \in J\rangle \text{ for all } \gamma \in [\beta + k\epsilon_j : j \in J], \qquad \emptyset \ne J \subset \{0{:}d\}. \tag{7}$$

Condition (6) means that all subsimplices V^k_β whose vertices are $x_{\beta+k\epsilon_j}$, $j \in \{0{:}d\}$, are nondegenerate and that the barycentric coordinates $\lambda_j(x|V^k_\alpha)$, $j \in \{0{:}d\}$, $\alpha \in \Gamma^0_{n-k}$, $k = 1, \dots, n$, in the Aitken-Neville recursion (3) are well-defined, which is an obvious minimal requirement. Condition (7) implies, according to Theorem 1, that the Aitken-Neville recursion (3) generates polynomials f^k_α interpolating at all points of the set X^k_α.

Let us remark that Aitken-Neville sets are sets equipped with an index structure. This means that there might be several ways of indexing the same set of points so that the set becomes an Aitken-Neville set. In Example 11, a set is labelled to become an Aitken-Neville set. In fact, it can be easily shown that the set in Example 11 can be labelled in essentially different ways.

Chung and Yao [7] introduced a geometric characterization (GC) of the unisolvent sets whose Lagrange polynomials are a product of first degree polynomials.

Definition 3 A set X of $\binom{n+d}{d}$ of points in $\mathbb{R}^d$ is a GC_n set if for each $x \in X$, there exists a set of n hyperplanes $\mathcal{H}_{x,X}$ such that their union contains $X \setminus \{x\}$ and not x.

Let us show that Aitken-Neville sets of order n are GC$_n$ sets. First we need to construct some hyperplanes which will be associated to each point $x \in X$.

Proposition 4 *Let X be an Aitken-Neville set of order n. The sets*

$$H_i^r := \langle x_{i\epsilon_r+(n-i)\epsilon_j} : j \in \{0{:}d\} \setminus \{r\}\rangle, \quad i = 0, \ldots, n-1, r \in \{0{:}d\}, \tag{8}$$

are hyperplanes such that

$$\gamma_r = i \Longrightarrow x_\gamma \in H_i^r. \tag{9}$$

PROOF. The set H_i^r is a hyperplane, $\dim H_i^r = d-1$, because by (6) the points

$$x_{i\epsilon_r+(n-i)\epsilon_j}, \quad j \in \{0{:}d\},$$

are in general position.

Observe that, if $\gamma \in \Gamma_n^0$ satisfies $\gamma_r = i$, then we can write it as

$$\gamma = \sum_{j\in\{0:d\}\setminus\{r\}} \frac{\gamma_j}{n-i}(i\epsilon_r + (n-i)\epsilon_j) \in [i\epsilon_r + (n-i)\epsilon_j : j \in \{0{:}d\} \setminus \{r\}].$$

Then we have

$$\gamma \in [i\epsilon_r + (n-i)\epsilon_j : j \in \{0{:}d\} \setminus \{r\}]$$

and by (7), we see that H_i^r contains the point x_γ as claimed. $\square$

Lemma 5 *If X is an Aitken-Neville set and $\gamma_r > i$ then $x_\gamma \notin H_i^r$.*

PROOF. Take $k := \gamma_r - i$ and $\beta := \gamma - k\epsilon_r \in \Gamma_{n-k}^0$ so that $\beta_r = i$. Then by (6) we have that $\dim\langle x_{\beta+k\epsilon_j} : j \in \{0{:}d\} \setminus \{r\}\rangle = d-1$ and since

$$x_{\beta+k\epsilon_j} \in H_{\beta_r}^0 = H_i^0, \quad j \in \{0{:}d\} \setminus \{r\},$$

we have that

$$\langle x_{\beta+k\epsilon_j} : j \in \{0{:}d\} \setminus \{r\}\rangle = H_i^r. \tag{10}$$

Recalling that $x_{\beta+k\epsilon_j}$, $j \in \{0{:}d\}$, form a nondegenerate simplex, we deduce that

$$x_\gamma = x_{\beta+k\epsilon_r} \notin \langle x_{\beta+k\epsilon_j} : j \in \{0{:}d\} \setminus \{r\}\rangle = H_i^r.$$

□

Theorem 6 *Any Aitken-Neville set of order n is a* GC_n *set.*

PROOF. If $\alpha \neq \gamma$, $\alpha, \gamma \in \Gamma_n^0$, then there exists some $j \in \{0{:}d\}$ such that $\alpha_j < \gamma_j$. Let H_i^r be the hyperplanes defined by (8). For each x_γ, $\gamma \in \Gamma_n^0$, the union of the n hyperplanes

$$H_i^r, \qquad i < \gamma_r, \quad r \in \{0{:}d\}, \tag{11}$$

contains all the points x_α, $\alpha \in \Gamma_n^0 \setminus \{\gamma\}$ by Proposition 4, but not x_γ according to Lemma 5. □

3 Generalized Principal Lattices

A simple example of Aitken-Neville sets of order n are *principal lattices of order n* of a nondegenerate d-simplex V, that is to say

$$X = \left\{ x_\alpha : \lambda(x_\alpha | V) = \frac{1}{n}\alpha, \alpha \in \Gamma_n^0 \right\}.$$

A generalization of planar principal lattices was introduced in [4,6]. Generalized principal lattices were analyzed in the multivariate case in [5].

Definition 7 *A* generalized principal lattice *of order n (GPL_n) is a set X that can be so indexed as $X = \{x_\alpha : \alpha \in \Gamma_n^0\}$ that, for $d+1$ families of hyperplanes*

$$H_i^r, \qquad i = 0, \ldots, n, \quad r \in \{0{:}d\},$$

containing altogether $(d+1)(n+1)$ distinct hyperplanes,

$$\{x_\alpha\} = \bigcap_{r=0}^{d} H_{\alpha_r}^r = \bigcap_{r \in \{0:d\} \setminus \{l\}} H_{\alpha_r}^r, \quad \forall \alpha \in \Gamma_n^0, \quad \forall l \in \{0{:}d\} \tag{12}$$

and

$$\bigcap_{r=0}^{d} H^r_{\alpha_r} \cap X \neq \emptyset \Longrightarrow \alpha \in \Gamma^0_n. \tag{13}$$

We also remark that, as in the case of Aitken-Neville sets, a set can be indexed in different ways to become a generalized principal lattice.

In this section we shall show that generalized principal lattices are Aitken-Neville sets. We shall also analyze under which conditions the converse holds. First we need to show a property of GPL_n sets.

Lemma 8 *If X is a GPL_n set, then for all $k \in \{1, \ldots, n\}$, $\beta \in \Gamma^0_{n-k}$ and $J \subseteq \{0{:}d\}$, we have*

$$\dim\langle x_{\beta+k\epsilon_j} : j \in J\rangle = \#J - 1, \quad \langle x_{\beta+k\epsilon_j} : j \in J\rangle = \bigcap_{r \notin J} H^r_{\beta_r}. \tag{14}$$

PROOF. We begin by showing that

$$\bigcap_{r=0}^{d} H^r_{\beta_r} = \emptyset. \tag{15}$$

Indeed, take any $j \in \{0{:}d\}$ and note that the the definition of a generalized principal lattice implies

$$\bigcap_{r \in \{0:d\}\setminus\{j\}} H^r_{\beta_r} = \bigcap_{r \in \{0:d\}\setminus\{j\}} H^r_{\beta_r} \cap H^j_{\beta_j+k} = \left\{x_{\beta+k\epsilon_j}\right\}. \tag{16}$$

If $\bigcap_{r=0}^{d} H^r_{\beta_r} \neq \emptyset$, then $\bigcap_{r=0}^{d} H^r_{\beta_r} \cap X = \left\{x_{\beta+k\epsilon_j}\right\} \neq \emptyset$ and by (13) we get $\beta \in \Gamma^0_n$, which contradicts the fact that $|\beta| = n - k < n$ and verifies (15).

Now we prove (14) by induction on the cardinality of J, where the case $\#J = 1$, say $J = \{j\}$, is exactly (16). Let $J' = J \cup \{j'\}$, $j' \notin J$, and assume that (14) holds for J, then we have, by (15),

$$\left\langle x_{\beta+k\epsilon_j} : j \in J\right\rangle \cap \left\{x_{\beta+k\epsilon_{j'}}\right\} = \bigcap_{r \notin J} H^r_{\beta_r} \cap \bigcap_{r \in \{0:d\}\setminus\{j'\}} H^r_{\beta_r} = \bigcap_{r=0}^{d} H^r_{\beta_r} = \emptyset,$$

that is, $x_{\beta+k\epsilon_{j'}} \notin \left\langle x_{\beta+k\epsilon_j} : j \in J\right\rangle$. Therefore,

$$\dim\left\langle x_{\beta+k\epsilon_j} : j \in J'\right\rangle = \dim\left\langle x_{\beta+k\epsilon_j} : j \in J\right\rangle + 1 = \#J = \#J' - 1.$$

Since, by Definition 7,

$$x_{\beta+k\epsilon_j} \in \bigcap_{r\notin J'} H^r_{\beta_r}, \qquad j \in J',$$

we deduce that $\left\langle x_{\beta+k\epsilon_j} : j \in J' \right\rangle \subseteq \bigcap_{r\notin J'} H^r_{\beta_r}$, and since

$$\#J' - 1 = \dim \left\langle x_{\beta+k\epsilon_j} : j \in J' \right\rangle \le \dim \bigcap_{r\notin J'} H^r_{\beta_r} \le d - (d + 1 - \#J') = \#J' - 1,$$

the identity $\left\langle x_{\beta+k\epsilon_j} : j \in J' \right\rangle = \bigcap_{r\notin J'} H^r_{\beta_r}$ follows and we have completed the proof by induction. □

Theorem 9 *Any generalized principal lattice of order n is an Aitken-Neville set of order n.*

PROOF. By Lemma 8, if $k \in \{1, \ldots, n\}$ and $\beta \in \Gamma^0_{n-k}$, then

$$\langle x_{\beta+k\epsilon_j} : j \in \{0{:}d\} \rangle = \mathbb{R}^d$$

and (6) follows. For any nonempty $J \subset \{0{:}d\}$, take $\gamma \in \Gamma^0_n$ such that $\gamma \in [\beta + k\epsilon_j : j \in J]$. By Definition 7, $\{x_\gamma\} = \bigcap_{r=0}^{d} H^r_{\gamma_r}$ and thus

$$x_\gamma \in \bigcap_{r\notin J} H^r_{\gamma_r} = \bigcap_{r\notin J} H^r_{\beta_r} = \langle x_{\beta+k\epsilon_j} : j \in J \rangle,$$

by Lemma 8. □

Now we analyze under which conditions an Aitken-Neville set of order n is a generalized principal lattice.

Theorem 10 *Let X be an Aitken-Neville set of order n and let H^r_i be the hyperplanes defined in (8). If*

$$x_\gamma \in H^r_i \Longrightarrow \gamma_r = i, \quad i = 0, \ldots, n-1, \quad \gamma \in \Gamma^0_n,\ \gamma_r \le i, \tag{17}$$

then X is a generalized principal lattice generated by the hyperplanes H^r_i.

PROOF. We choose H^r_i, $i < n$, according to (8), which is a hyperplane by Proposition 4. In addition, let H^r_n be *an arbitrary* hyperplane containing $x_{n\epsilon_r}$ and no other node of X.

To prove the theorem, we first show that for all $1 \le k \le n$, $|\beta| = n-k$, $J \subset \{0{:}d\}$, we have

$$\dim \bigcap_{r \notin J} H^r_{\beta_r} = \#J - 1, \quad \langle x_{\beta+k\epsilon_j} : j \in J \rangle = \bigcap_{r \notin J} H^r_{\beta_r}. \tag{18}$$

We shall prove (18) by (decreasing) induction on $\#J$, where the case $\#J = d+1$ is an immediate consequence of the assumption that X is an Aitken–Neville set. Taking into account that $\beta_r \in \{0, \ldots, n-1\}$ for any $r \in \{0{:}d\}$, we can apply Proposition 4 and obtain

$$x_{\beta+k\epsilon_j} \in \bigcap_{r \in \{0:d\} \setminus \{j\}} H^r_{\beta_r} \subseteq \bigcap_{r \notin J} H^r_{\beta_r}, \quad \forall j \in J,$$

which implies that

$$\langle x_{\beta+k\epsilon_j} : j \in J \rangle \subseteq \bigcap_{r \notin J} H^r_{\beta_r}.$$

Therefore, in order to show (18), it is sufficient to deduce that $\dim \bigcap_{r \notin J} H^r_{\beta_r} = \#J - 1$. If $\#J = d$, we conclude from the fact that (8) defines a hyperplane that $\dim H^r_{\beta_r} = d-1$ and then (18) holds. Assume that $J = J' \cup \{j\}$, $j \notin J'$, and that (18) holds for all sets J whose cardinality is greater than $\#J'$. By (10) and the induction hypothesis,

$$\langle x_{\beta+k\epsilon_l} : l \in \{0{:}d\} \setminus \{j\} \rangle = H^j_{\beta_j}, \quad \langle x_{\beta+k\epsilon_l} : l \in J \rangle = \bigcap_{r \notin J} H^r_{\beta_r}.$$

Then we have

$$\bigcap_{r \notin J'} H^r_{\beta_r} = H^j_{\beta_j} \cap \bigcap_{r \notin J} H^r_{\beta_r} = \langle x_{\beta+k\epsilon_l} : l \in \{0{:}d\} \setminus \{j\} \rangle \cap \langle x_{\beta+k\epsilon_l} : l \in J \rangle.$$

By (6), the points $x_{\beta+k\epsilon_l}$, $l = \{0{:}d\}$ are in general position, and therefore

$$\langle x_{\beta+k\epsilon_l} : l \in \{0{:}d\} \setminus \{j\} \rangle + \langle x_{\beta+k\epsilon_l} : l \in J \rangle = \mathbb{R}^d.$$

Hence

$$\dim \bigcap_{r \notin J'} H^r_{\beta_r} = \dim(H^j_{\beta_j} \cap \bigcap_{r \notin J} H^r_{\beta_r}) = (d-1) + (\#J - 1) - d = \#J' - 1$$

and formula (18) follows for J'. So we have proved formula (18) by induction on the cardinality of J.

Now, let us show that for each $l \in \{0{:}d\}$

$$\bigcap_{r \in \{0:d\} \setminus \{l\}} H^r_{\gamma_r}, \quad \gamma \in \Gamma^0_n, \tag{19}$$

consists exactly of one point.

If $\gamma_l > 0$, then we can apply (18), taking $J = \{l\}$, $k = 1$, $\beta := \gamma - \epsilon_l$. Then we have that $\bigcap_{r \in \{0:d\} \setminus \{l\}} H^r_{\gamma_r} = \bigcap_{r \in \{0:d\} \setminus \{l\}} H^r_{\beta_r}$ and

$$\{x_{\beta + k\epsilon_l}\} = \bigcap_{r \in \{0:d\} \setminus \{l\}} H^r_{\beta_r}.$$

So, $\bigcap_{r \in \{0:d\} \setminus \{l\}} H^r_{\beta_r}$ consists of exactly one point, which is $x_{\beta + \epsilon_l} = x_\gamma$.

If $\gamma_l = 0$ and $\gamma = n\epsilon_j$ for some $j \neq l$, then we recall that $\bigcap_{r \in \{0:d\} \setminus \{l,j\}} H^r_0$ is an affine submanifold of dimension 1 because the hyperplanes H^r_0, $r \in \{0{:}d\}$, are the facets of a simplex whose vertices are V^n_0. By Proposition 4, $\bigcap_{r \in \{0:d\} \setminus \{l,j\}} H^r_0$ contains the points $x_{n\epsilon_l}$ and $x_{n\epsilon_j}$. The choice of the hyperplanes H^j_n, $j \in \{0{:}d\}$, allows us to ensure that (19) consists of exactly one point.

In the remaining cases, $\gamma_l = 0$ and there exists some j with $\gamma_j \in \{1, \ldots, n-1\}$, we take $k = 1$, $\beta = \gamma - \epsilon_j$, $J = \{j, l\}$, and have by (18) that

$$\langle x_\gamma, x_{\gamma + \epsilon_l - \epsilon_j} \rangle = \bigcap_{r \in \{0:d\} \setminus \{j,l\}} H^r_{\gamma_r}.$$

By Proposition 4, $x_\gamma \in H^j_{\gamma_j}$ and, by the hypothesis (17), $x_{\gamma + \epsilon_l - \epsilon_j} \notin H^j_{\gamma_j}$ which implies that

$$\bigcap_{r \in \{0:d\} \setminus \{l\}} H^r_{\gamma_r} = \{x_\gamma\}.$$

The hypothesis (17), the choice of H^r_n and Proposition 4 then yield

$$H^r_i \cap X = \{x_\gamma : \gamma_r = i\}.$$

In particular, we have

$$x_\gamma \in \bigcap_{r=0}^{d} H^r_{\gamma_r} \neq \emptyset. \tag{20}$$

Finally, (17) also implies that (13) holds. □

The following example shows that not every Aitken-Neville set is a generalized principal lattice.

Example 11 We take hyperplanes $H_0, \ldots, H_{d+1}$ in $\mathbb{R}^d$, $d > 1$, in general position and define $y_{i,j}$ by

$$\{y_{i,j}\} := \bigcap_{k \neq i,j} H_k, \quad i \neq j \in \{0{:}d+1\}.$$

Then the set $X := \{y_{i,j} : i \neq j\}$ is a GC_2 set usually called a *natural lattice* of order 2 in $\mathbb{R}^d$.

For any $l \in \{0{:}d+1\}$,

$$H_i, \quad i \in \{0{:}d+1\} \setminus \{l\}$$

are the facets of a nondegenerate simplex because $H_0, \ldots, H_{d+1}$ are in general position. Therefore the points

$$y_{i,l}, \quad i \in \{0{:}d+1\} \setminus \{l\} \tag{21}$$

are the vertices of a nondegenerate simplex.

In an Aitken-Neville set of order 2, the multiindices $2\epsilon_i \in \Gamma_2^0$, $i = 0, \ldots, d$, correspond to the vertices of a simplex. Let us choose

$$x_{2\epsilon_i} := y_{i,d+1}, \qquad i \in \{0{:}d\},$$

as the vertices corresponding to multiindices $2\epsilon_i \in \Gamma_2^0$. The point y_{ij} in the one-dimensional affine manifold $\langle y_{i,d+1}, y_{j,d+1} \rangle$ must correspond to the multiindex $\epsilon_i + \epsilon_j$, $i \neq j \in \{0{:}d\}$. Therefore, we associate to each $y_{i,j}$, $i < j$, the multiindex

$$\alpha(i,j) = \begin{cases} 2\epsilon_i, & \text{if } j = d+1, \\ \epsilon_i + \epsilon_j, & \text{if } j < d+1, \end{cases}$$

in Γ_2^0 and relabel the points in X

$$x_{\alpha(i,j)} := y_{ij}, \quad i < j.$$

Using this notation, we have

$$x_\alpha = \begin{cases} \bigcap_{\alpha_r = 0} H_r & \text{if } \alpha = 2\epsilon_i, \\ \bigcap_{\alpha_r = 0} H_r \cap H_{d+1} & \text{otherwise.} \end{cases}$$

We now show that X is an Aitken-Neville set, where the validity of (6) is easily verified by direct computations.

In order to check (7), first take $k=2$, $\beta=0$, and assume that $\gamma \in [2\epsilon_j : j \in J]$, hence $\gamma_r = 0$ for $r \notin J$. Since $x_{2\epsilon_j} \in \bigcap_{r\in\{0:d\}\setminus J} H_r$, for all $j \in J$ and

$$\dim\langle x_{2\epsilon_j} : j \in J\rangle = \#J - 1 = \dim \bigcap_{r\in\{0:d\}\setminus J} H_r$$

we deduce that

$$\langle x_{2\epsilon_j} : j \in J\rangle = \bigcap_{r\in\{0:d\}\setminus J} H_r$$

and

$$x_\gamma \in \bigcap_{\{r:\gamma_r=0\}} H_r \subseteq \bigcap_{r\in\{0:d\}\setminus J} H_r = \langle x_{2\epsilon_j} : j \in J\rangle.$$

Now assume that $k=1$, $\beta=\epsilon_l$, and consider $\gamma \in [\epsilon_l + \epsilon_j : j \in J]$. Then we have two cases $l \notin J$ and $l \in J$.

The case $l \in J$ is similar to the case $\beta = 0$ discussed above. We have that $\gamma_r = 0$, for any $r \notin J$. Taking into account that $x_{\epsilon_l+\epsilon_j} \in \bigcap_{r\in\{0:d\}\setminus J} H_r$ and that

$$\dim \bigcap_{r\in\{0:d\}\setminus J} H_r = \#J - 1 = \dim\langle x_{\epsilon_l+\epsilon_j} : j \in J\rangle,$$

we have

$$x_\gamma \in \bigcap_{r\in\{0:d\},\gamma_r=0} H_r \subseteq \bigcap_{r\in\{0:d\}\setminus J} H_r = \langle x_{\epsilon_l+\epsilon_j} : j \in J\rangle.$$

If $l \notin J$, then $\gamma_r = 0$, for any $r \notin J \cup \{l\}$ and $x_\gamma \in H_{d+1}$. Taking into account that $x_{\epsilon_l+\epsilon_j} \in \bigcap_{r\in\{0:d\}\setminus(J\cup\{l\})} H_r \cap H_{d+1}$ and that

$$\dim \bigcap_{r\in\{0:d\}\setminus(J\cup\{l\})} H_r \cap H_{d+1} = \#J - 1 = \dim\langle x_{\epsilon_l+\epsilon_j} : j \in J\rangle,$$

we obtain

$$x_\gamma \in \bigcap_{\{r:\gamma_r=0\}} H_r \cap H_{d+1} \subseteq \bigcap_{r\in\{0:d\}\setminus(J\cup\{l\})} H_r = \langle x_{\epsilon_l+\epsilon_j} : j \in J\rangle.$$

So, we have shown that X is an Aitken-Neville set. Let us now show that X cannot be a generalized principal lattice.

First we observe that if X is a generalized principal lattice, then the hyperplanes H_0^r, $r \in \{0{:}d\}$, contain exactly $(d+1)d/2$ nodes. From the definition of the set X as a natural lattice of degree 2, we see that the only hyperplanes containing $(d+1)d/2$ nodes are $H_0, \dots, H_{d+1}$. Reordering the hyperplanes, if necessary, we can assume that

$$H_0^r = H_r, \quad r \in \{0{:}d\}.$$

The hyperplane H_1^r must contain $d(d-1)/2$ nodes in $X \setminus H_0^r$ and must be different from the hyperplanes H_0^j, $j \in \{0{:}d\}$. Then we find that

$$H_1^r = H_{d+1}, \quad r \in \{0{:}d\}.$$

This contradicts the definition of generalized principal lattices, since all hyperplanes H_1^r, $r \in \{0{:}d\}$ must be distinct.

As a consequence of Theorem 10, the property (17) has to be violated by X which, however, can be seen directly here because $x_{\epsilon_i+\epsilon_j} \in H_1^k$ for any distinct i, j and any k.

Although this example shows that there exists an Aitken–Neville set of order 2 that violates (17), condition (17) nevertheless holds in the plane for all degrees greater than 2.

Theorem 12 *Let X be an Aitken-Neville set of order $n > d = 2$ and let H_i^r be the hyperplanes defined in (8). Then (17) holds.*

We will split the proof of this theorem into several parts. They will all be kept multivariate and so it will become clear from the proof where the restrictions $n > d$ and $d = 2$ become relevant and why it is unlikely that Theorem 12 holds for three and more dimensions. The following Lemmas show that condition (17) can be weakened by proving that $\gamma_r \neq i$ implies that $x_\gamma \notin H_i^r$.

While the situation $\gamma_r > i$ has already been considered in Lemma 5, the case $\gamma_r < i$ is more intricate and will be resolved in the following results.

Lemma 13 *If X is an Aitken-Neville set, $\gamma_r < i$ and $\gamma_j > i - \gamma_r$ for some $j \in \{0{:}d\} \setminus \{r\}$ then $x_\gamma \notin H_i^r$.*

PROOF. Since

$$\beta := \gamma + (i - \gamma_r)(\epsilon_r - \epsilon_j) = \left(1 - \frac{i - \gamma_r}{\gamma_j}\right)\gamma + \frac{i - \gamma_r}{\gamma_j}\left(\gamma + \gamma_j(\epsilon_r - \epsilon_j)\right)$$

is a strictly convex combination of multiindices, it follows that there exists $\lambda \in \mathbb{R}$ so that

$$x_\beta = \lambda x_\gamma + (1 - \lambda)\, x_{\gamma + \gamma_j(\epsilon_r - \epsilon_j)}$$

where $\lambda \notin \{0,1\}$ since the points form an Aitken–Neville set and therefore have to be distinct. Since $\beta_r = i$, hence $x_\beta \in H_i^r$, the assumption $x_\gamma \in H_i^r$ would yield that

$$x_{\gamma + \gamma_j(\epsilon_r - \epsilon_j)} = \frac{1}{1 - \lambda} x_\beta - \frac{\lambda}{1 - \lambda} x_\gamma \in H_i^r.$$

But since $\left(\gamma + \gamma_j(\epsilon_r - \epsilon_j)\right)_r = \gamma_r + \gamma_j > i$ we then obtain a contradiction to Lemma 5. □

Lemma 14 *If X is an Aitken-Neville set, $n > d > 1$, $i \le n - d + 1$, $\gamma_r < i$ and $\gamma_j \le i - \gamma_r$ for all $j \in \{0{:}d\} \setminus \{r\}$, then $x_\gamma \notin H_i^r$.*

PROOF. Suppose that $x_\gamma \in H_i^r$ and choose any $\beta \in \Gamma_n^0$ with $\beta_r = i - 1 \le n - d$ as well as $\beta_j \ge 1$ for $j \in \{0{:}d\} \setminus \{r\}$, which is possible because $\sum_{j \in \{0:d\} \setminus \{r\}} \beta_j = n - \beta_r \ge d$. It is easily verified that β can be expressed as the convex combination

$$\beta = \frac{1}{i - \gamma_r}\gamma + \sum_{j \in \{0:d\} \setminus \{r\}} \frac{\beta_j - \gamma_j / (i - \gamma_r)}{n - i}\left(i\epsilon_r + (n - i)\epsilon_j\right) \tag{22}$$

from which it follows that either $\gamma = \beta$ if $\gamma_r = i - 1$ or that the convex combination (22) is a strict one. In both cases this implies that $x_\beta \in H_i^r$.

If there exists $j \in \{0{:}d\} \setminus \{r\}$ such that $\beta_j > 1$, which is definitely the case when $i < n - d + 1$, then $i - \beta_r = 1 < \beta_j$ for some $j \ne r$ immediately gives a contradiction to Lemma 13. If, on the other hand, $\beta_j = 1$, $j \in \{0{:}d\} \setminus \{r\}$, hence $i = n - d + 1$, the points $x_{\beta + \epsilon_r - \epsilon_j}$, $j \in \{0{:}d\} \setminus \{r\}$, belong to H_i^r as well and since β is a convex combination of $\beta + \epsilon_r - \epsilon_j$ and $\beta + k(\epsilon_j - \epsilon_r)$, $k = 1, \dots, n - d$, we can again apply the above affine combination argument to find that

$$x_{\beta + k(\epsilon_j - \epsilon_r)} \in H_i^r, \qquad j \in \{0{:}d\} \setminus \{r\}, \quad k = 1, \dots, n - d.$$

This set of points is non–empty for $n > d$. Specifically, this implies for $k = n-d = i-1 = \beta_r$ that

$$x_{\beta+(n-d)(\epsilon_j-\epsilon_r)} \in H_0^r \cap H_i^r, \qquad j \in \{0{:}d\} \setminus \{r\}, \tag{23}$$

and since X is an Aitken–Neville set, the points from (23) span a hyperplane contained in $H_0^r \cap H_i^r$ which implies $H_0^r = H_i^r$, thus $x_\beta \in H_0^r$. But this contradicts Lemma 13 since $\beta_r = i-1 > 0$. $\square$

These lemmas complete the proof of Theorem 12. Choose any γ such that $\gamma_r \neq i$. The case that $i < \gamma_r$ is covered by Lemma 5, the cases where $\gamma_r < i \leq n-d+1$ are treated successively in Lemmas 13 and 14. For $d = 2$, this includes all values of i from 0 to $n-1$ and proves Theorem 12. Note that the assumption $n > d$ in the statement of Theorem 12 was needed only in Lemma 14 and even there only for the case $i = n-d+1$. However, the condition is definitely necessary since Example 11 shows that even in the simplest case $n = d = 2$ there are Aitken-Neville configurations for which (17) does not hold.

The other question left open by Theorem 12 is what happens in the case $n > d \geq 3$ where the values $i = n-d+2, \ldots, n-1$ are not covered by the above lemmas. We conjecture that in these cases Theorem 12 is not valid. The reason for this conjecture lies in the nature of the above proofs: whenever a point x_γ was located on some hyperplane H_i^r with $i \neq \gamma_r$, the simplex generated by the vertices $i\epsilon_r + (n-i)\epsilon_j$, $j \in \{0{:}d\} \setminus \{r\}$, and γ also contained at least one other multiindex β which we used to construct a contradiction. Now this "coupling" between the hyperplane H_i^r and the additional point need not be present for the hyperplanes H_i^r, $i = n-d+2, \ldots, n-1$ as the following example shows.

Example 15 Consider the hyperplane $H_i^r = \langle i\epsilon_r + (n-i)\epsilon_j : j \in \{0{:}d\} \setminus \{r\} \rangle$ with $i > n-d+1$ and the multiindex $\gamma = (n-d+1)\epsilon_l + \sum_{j\in\{0:d\}\setminus\{r,l\}} \epsilon_j$ for some $l \neq r$. Let β be any convex combination of the form

$$\beta = \lambda_r \gamma + \sum_{j\in\{0:d\}\setminus\{r\}} \lambda_j (i\epsilon_r + (n-i)\epsilon_j), \quad \lambda_0, \ldots, \lambda_d \geq 0, \quad \sum_{j=0}^{d} \lambda_j = 1.$$

Then we can write

$$\beta = ((n-d+1)\lambda_r + (n-i)\lambda_l)\epsilon_l + i(1-\lambda_r)\epsilon_r + \sum_{j\in\{0:d\}\setminus\{r,l\}} (\lambda_r + (n-i)\lambda_j)\epsilon_j.$$

But now β cannot be a multiinteger in Γ_n^0 which is the index of a point $x_\beta \neq x_\gamma$ that also belongs to H_i^r. This follows since $x_\beta \in H_i^r$ would require that $0 < \beta_r < i$ and $0 < \lambda_r < 1$, but taking into account that

$$\beta_j = \lambda_r + (n-i)\lambda_j \geq 1, \qquad j \in \{0{:}d\} \setminus \{r,l\},$$

we obtain the contradiction

$$\begin{aligned} d-1 &\leq \sum_{j\in\{0:d\}\setminus\{r,l\}} \beta_j = \sum_{j\in\{0:d\}\setminus\{r,l\}} (\lambda_r + (n-i)\lambda_j) \\ &= (n-i)\,(1-\lambda_l) + (i-n+d-1)\lambda_r < d-1. \end{aligned}$$

Finally, we interpret Theorem 10 in the context of *lattice transformations*. Recall that a lattice transformation, as introduced in [7], consists of a point mapping Φ and a hyperplane mapping Ψ which preserves the incidence relation between points and hyperplanes of the lattice:

$$x \in H \Longleftrightarrow \Phi(x) \in \Psi(H). \tag{24}$$

While originally Chung and Yao required neither of the maps to be injective, it can be seen quite easily that (24) already implies that Φ and Ψ are even bijections. While any two generalized principal lattices of the same order can be connected by a lattice transformation, one can also show that a lattice transformation preserves the structure of generalized principal lattices, i.e., maps GPL_n to GPL_n, provided that for $r \in \{0{:}d\}$ one has

$$\dim \bigcap_{j\in\{0:d\}\setminus\{r\}} \Psi(H^j_{\alpha_j}) = 0, \quad \alpha \in \Gamma^0_n,\ \alpha_j \leq n-1,\ j \in \{0{:}d\} \setminus \{r\}. \tag{25}$$

On the other hand, this property (25) was exactly what was verified in the proof of Theorem 10.

4 The divided difference

In this section we want to derive a spline representation for the divided difference associated with any Aitken–Neville set. In general, the notion of a multivariate divided difference is far from being agreed upon, just see [2,3] as well as [8] and the references therein. Here we follow the point of view taken in [14] as well as in [13] that defines the divided difference as the leading term of the interpolation polynomial, interpreted either as a multiindexed vector or as a multilinear form. To be more specific, we assume that X is an Aitken–

Neville set of order n and write the intermediate interpolation polynomials as

$$f_\alpha^k(x) = \sum_{j=0}^k \boldsymbol{x}^j \boldsymbol{f}_j(\alpha), \qquad \alpha \in \Gamma_{n-k}^0, \quad k = 0, \ldots, n,$$

where $\boldsymbol{f}_j(\alpha) \in \mathbb{R}^{\Gamma_k}$ is the vector containing the coefficients of the leading form of the interpolant f_α^k. Since f_α^k depends on X_α^k and the values of f there, we can thus define the divided difference as in [14] by

$$\Delta \left(X_\alpha^k \right) f := \boldsymbol{f}_k(\alpha), \qquad \alpha \in \Gamma_{n-k}^0, \quad k = 0, \ldots, n. \tag{26}$$

A recurrence relation for the divided difference has been derived in [14] by dualizing the recurrence (3). To formulate it, we denote for any set V of $d+1$ points in $\mathbb{R}^d$ by $\tau_{jk}(V)$ the determinant obtained by deleting in $\begin{pmatrix} 1 & \ldots & 1 \\ v_0 & \ldots & v_d \end{pmatrix}$ the jth column and the kth row, $j = 0, \ldots, d$, $k = 1, \ldots, d$.

Theorem 16 *The divided difference $\Delta \left(X_\alpha^k \right) f$ satisfies the recurrence relation*

$$\Delta \left(X_\alpha^k \right)_\beta f = \sum_{j=0}^d \sum_{\ell=1}^d (-1)^{j+\ell} \frac{\tau_{j\ell} \left(V_\alpha^k \right)}{\tau \left(V_\alpha^k \right)} \Delta \left(X_{\alpha+\epsilon_j}^{k-1} \right)_{\beta-\epsilon_\ell} f, \quad \alpha \in \Gamma_{n-k}^0, \quad \beta \in \Gamma_k. \tag{27}$$

We will use this recurrence to derive a spline representation of the divided difference and show how it behaves when the points coalesce in a uniform way. To that end, we fix an arbitrary point $u \in \mathbb{R}^d$. We shall use the notation u^n to indicate that u is a point which appears n times in a sequence.

Theorem 17 *For any Aitken–Neville set X and any $u \in \mathbb{R}^d$ the associated divided difference takes the form*

$$\Delta (X)_\beta f = \sum_{\gamma \in \Gamma_n^0} \mu_{\beta,\gamma} \int_{[x_\gamma, u^n]} D_{x_\gamma - u}^n f, \qquad \beta \in \Gamma_n, \tag{28}$$

where the coefficients $\mu_{\beta,\gamma}$ satisfy the identity

$$\sum_{\gamma \in \Gamma_n^0} \mu_{\beta,\gamma} D_{x_\gamma - u}^n = \frac{1}{\beta!} D^\beta, \qquad \beta \in \Gamma_n, \tag{29}$$

and can be recursively computed as

$$\mu_{\beta,\eta}(\alpha) = \sum_{\ell=1}^{d}\sum_{j=0}^{d}(-1)^{j+\ell}\frac{\tau_{j\ell}\left(V_\alpha^{k+1}\right)}{\tau\left(V_\alpha^{k+1}\right)}\mu_{\beta-\epsilon_\ell,\eta-\epsilon_j}\left(\alpha+\epsilon_j\right). \tag{30}$$

A special property of the divided difference is the fact that it becomes a derivative in the case of multiple points. Indeed, this property motivates the extension of divided differences to the limit situation. In the multivariate case, however, we must be more careful with such limits, taking care that all points tend to the limit with "the same speed". To that end, we consider, for $h > 0$, the interpolation set

$$X(h) := u + h\,(X-u) = \left\{u + h\,(x_\gamma - u) : \gamma \in \Gamma_n^0\right\}$$

that clearly satisfies $X = X(1)$ and define the divided difference at u^n as

$$\Delta\left(u^n\right)f = \lim_{h\to 0}\Delta\left(X(h)\right)f.$$

This notion makes sense and is even independent of the "initial" Aitken–Neville set as the following result shows.

Theorem 18 *For any Aitken–Neville set X, any $u \in \mathbb{R}^d$ and any $f \in C^n\left(\mathbb{R}^d\right)$ we have that*

$$\Delta\left(u^n\right)f = \lim_{h\to 0}\Delta\left(X(h)\right)f = \left(\frac{1}{\alpha!}D^\alpha(u) : \alpha \in \Gamma_n\right). \tag{31}$$

The remainder of this section is dedicated to proving Theorems 17 and 18, which will be done by simultaneously establishing them for the Aitken–Neville subsets X_α^k, $\alpha \in \Gamma_{n-k}^0$, of increasing orders $k = 0, 1, \dots, n$. More precisely, we show that there exist coefficients $\mu_{\beta,\gamma}$, $\beta \in \Gamma_k$, $\gamma \in \Gamma_k^0$, such that

$$\Delta\left(X_\alpha^k\right)_\beta f = \sum_{\gamma\in\Gamma_k^0}\mu_{\beta,\gamma}(\alpha)\int\limits_{\left[x_{\alpha+\gamma},u^k\right]} D^k_{x_{\alpha+\gamma}-u}f, \qquad \beta \in \Gamma_k, \quad \alpha \in \Gamma_{n-k}^0, \tag{32}$$

and

$$\sum_{\gamma\in\Gamma_k^0}\mu_{\beta,\gamma}(\alpha)D^k_{x_{\alpha+\gamma}-u} = \frac{1}{\beta!}D^\beta, \qquad \beta \in \Gamma_k, \quad \alpha \in \Gamma_{n-k}^0. \tag{33}$$

Since the case $k = 0$ is trivial, let us begin with $k = 1$. Here we take into account that for $\ell = 1, \dots, d$

$$\sum_{j=0}^{d} (-1)^{j+\ell} \tau_{j\ell}\left(V_\alpha^k\right) = \det \begin{pmatrix} 1 & \dots & 1 \\ x_{\alpha+k\epsilon_0,1} & \dots & x_{\alpha+k\epsilon_d,1} \\ \vdots & \ddots & \vdots \\ x_{\alpha+k\epsilon_0,\ell-1} & \dots & x_{\alpha+k\epsilon_d,\ell-1} \\ 1 & \dots & 1 \\ x_{\alpha+k\epsilon_0,\ell+1} & \dots & x_{\alpha+k\epsilon_d,\ell+1} \\ \vdots & \ddots & \vdots \\ x_{\alpha+k\epsilon_0,d} & \dots & x_{\alpha+k\epsilon_d,d} \end{pmatrix} = 0,$$

and obtain from the recurrence relation (27) for $\alpha \in \Gamma_{n-1}^0$ and $\ell = 1, \dots, d$ that

$$\begin{aligned} \Delta\left(X_\alpha^1\right)_{\epsilon_\ell} f &= \sum_{j=0}^{d} (-1)^{j+\ell} \frac{\tau_{j\ell}\left(V_\alpha^1\right)}{\tau\left(V_\alpha^1\right)} \Delta\left(X_{\alpha+\epsilon_j}^0\right)_0 f = \sum_{j=0}^{d} (-1)^{j+\ell} \frac{\tau_{j\ell}\left(V_\alpha^1\right)}{\tau\left(V_\alpha^1\right)} f\left(x_{\alpha+\epsilon_j}\right) \\ &= \sum_{j=0}^{d} (-1)^{j+\ell} \frac{\tau_{j\ell}\left(V_\alpha^1\right)}{\tau\left(V_\alpha^1\right)} \left(f\left(x_{\alpha+\epsilon_j}\right) - f(u)\right) \\ &= \sum_{j=0}^{d} (-1)^{j+\ell} \frac{\tau_{j\ell}\left(V_\alpha^1\right)}{\tau\left(V_\alpha^1\right)} \int_{\left[x_{\alpha+\epsilon_j}, u\right]} D_{x_{\alpha+\epsilon_j}-u} f \\ &=: \sum_{j=0}^{d} \mu_{\epsilon_\ell,\epsilon_j}(\alpha) \int_{\left[x_{\alpha+\epsilon_j}, u\right]} D_{x_{\alpha+\epsilon_j}-u} f, \end{aligned}$$

yielding (32) in the case $k = 1$. To see that this representation is invariant under dilation, we first note that $\tau_{j\ell}\left(V_\alpha^k - u\right) = \tau_{j\ell}\left(V_\alpha^k\right)$ as well as $\tau\left(V_\alpha^k - u\right) = \tau\left(V_\alpha^k\right)$ and then conclude that for $h > 0$ we have that

$$\tau\left(V_\alpha^k(h)\right) = \tau\left(V_\alpha^k(h) - u\right) = \tau\left(h\left(V_\alpha^k - u\right)\right) = h^d \tau\left(V_\alpha^k\right).$$

In the same way, we also find that $\tau_{j\ell}\left(V_\alpha^k(h)\right) = h^{d-1} \tau_{j\ell}\left(V_\alpha^k\right)$. Substituting this into the integral representation we thus find that

$$\Delta\left(X_\alpha^1(h)\right)_{\epsilon_\ell} f = \sum_{j=0}^d (-1)^{j+\ell}\frac{\tau_{j\ell}\left(V_\alpha^1\right)}{h\tau\left(V_\alpha^1\right)} \int_{\left[(1-h)u+hx_{\alpha+\epsilon_j},u\right]} D_{h\left(x_{\alpha+\epsilon_j}-u\right)} f$$
$$= \sum_{j=0}^d \mu_{\epsilon_\ell,\epsilon_j}(\alpha) \int_{\left[(1-h)u+hx_{\alpha+\epsilon_j},u\right]} D_{x_{\alpha+\epsilon_j}-u} f.$$

Therefore, the limit $h \to 0$ exists for any $f \in C^1\left(\mathbb{R}^d\right)$ and satisfies

$$\lim_{h\to 0} \Delta\left(X_\alpha^1(h)\right)_{\epsilon_\ell} f = \sum_{j=0}^d \mu_{\epsilon_\ell,\epsilon_j}(\alpha) D_{x_{\alpha+\epsilon_j}-u} f(u).$$

On the other hand, the interpolation operator of order n is a projection on Π_n, hence the interpolant to any monomial of degree n reproduces this monomial. Consequently, there is a duality between monomials and the divided difference which we record for our purposes as follows.

Lemma 19 *Let X be an Aitken–Neville set of order n. Then we have that*

$$\Delta(X)_\beta (\cdot)^\alpha = \delta_{\alpha,\beta}, \qquad \alpha,\beta \in \Gamma_n. \tag{34}$$

Applying Lemma 19 in the case $n = 1$, we now obtain

$$\delta_{\ell,m} = \Delta\left(X_\alpha^1\right)_{\epsilon_\ell} (\cdot)^{\epsilon_m} = \lim_{h\to 0} \Delta\left(X_\alpha^1(h)\right)_{\epsilon_\ell} (\cdot)^{\epsilon_m} = \left(\sum_{j=0}^d \mu_{\epsilon_\ell,\epsilon_j} D_{x_{\alpha+\epsilon_j}-u} (\cdot)^{\epsilon_m}\right)(u)$$

independently of u, hence,

$$\sum_{j=0}^d \mu_{\epsilon_\ell,\epsilon_j} D_{x_{\alpha+\epsilon_j}-u} = D^{\epsilon_\ell}, \qquad \ell = 1,\ldots,d,$$

which is (29) in the case $k = 1$.

The inductive step from k to $k+1$ proceeds along the same lines, it is only slightly more complicated in a technical way. Again we employ (27) which enables us to use the induction hypothesis to obtain for $\alpha \in \Gamma_{n-k-1}^0$ and $\beta \in \Gamma_{k+1}$

$$\Delta\left(X_\alpha^{k+1}\right)_\beta f = \sum_{j=0}^d \sum_{\ell=1}^d (-1)^{j+\ell} \frac{\tau_{j\ell}\left(V_\alpha^{k+1}\right)}{\tau\left(V_\alpha^{k+1}\right)} \Delta\left(X_{\alpha+\epsilon_j}^k\right)_{\beta-\epsilon_\ell} f$$

$$
\begin{aligned}
&= \sum_{\ell=1}^{d} \sum_{j=0}^{d} (-1)^{j+\ell} \frac{\tau_{j\ell}\left(V_\alpha^{k+1}\right)}{\tau\left(V_\alpha^{k+1}\right)} \sum_{\gamma\in\Gamma_k^0} \mu_{\beta-\epsilon_\ell,\gamma}\left(\alpha+\epsilon_j\right) \int\limits_{\left[x_{\alpha+\gamma+\epsilon_j},u^k\right]} D^k_{x_{\alpha+\gamma+\epsilon_j}-u} f \\
&= \sum_{\ell=1}^{d} \sum_{j=0}^{d} (-1)^{j+\ell} \frac{\tau_{j\ell}\left(V_\alpha^{k+1}\right)}{\tau\left(V_\alpha^{k+1}\right)} \\
&\qquad \times \left(\sum_{\gamma\in\Gamma_k^0} \mu_{\beta-\epsilon_\ell,\gamma}\left(\alpha+\epsilon_j\right) \int\limits_{\left[x_{\alpha+\gamma+\epsilon_j},u^k\right]} D^k_{x_{\alpha+\gamma+\epsilon_j}-u} f - \frac{D^{\beta-\epsilon_\ell} f(u)}{(\beta-\epsilon_\ell)!} \right),
\end{aligned}
$$

with the standard convention that all terms with negative entries in any multiindex are zero. According to (29), we can write

$$
\frac{1}{(\beta-\epsilon_\ell)!} D^{\beta-\epsilon_\ell} = \sum_{\gamma\in\Gamma_k^0} \mu_{\beta-\epsilon_\ell,\gamma}\left(\alpha+\epsilon_j\right) D^k_{x_{\alpha+\gamma+\epsilon_j}-u}
$$

and

$$
\frac{1}{(\beta-\epsilon_\ell)!} D^{\beta-\epsilon_\ell} f(u) = \sum_{\gamma\in\Gamma_k^0} \mu_{\beta-\epsilon_\ell,\gamma}\left(\alpha+\epsilon_j\right) \int\limits_{\left[u^{k+1}\right]} D^k_{x_{\alpha+\gamma+\epsilon_j}-u}
$$

which we substitute into the above identity to obtain

$$
\begin{aligned}
\Delta\left(X_\alpha^{k+1}\right)_\beta f &= \sum_{\ell=1}^{d} \sum_{j=0}^{d} (-1)^{j+\ell} \frac{\tau_{j\ell}\left(V_\alpha^{k+1}\right)}{\tau\left(V_\alpha^{k+1}\right)} \\
&\qquad \times \sum_{\gamma\in\Gamma_k^0} \mu_{\beta-\epsilon_\ell,\gamma}\left(\alpha+\epsilon_j\right) \left(\int\limits_{\left[x_{\alpha+\gamma+\epsilon_j},u^k\right]} D^k_{x_{\alpha+\gamma+\epsilon_j}-u} f - \int\limits_{\left[u^{k+1}\right]} D^k_{x_{\alpha+\gamma+\epsilon_j}-u} f \right) \\
&= \sum_{\ell=1}^{d} \sum_{j=0}^{d} (-1)^{j+\ell} \frac{\tau_{j\ell}\left(V_\alpha^{k+1}\right)}{\tau\left(V_\alpha^{k+1}\right)} \sum_{\gamma\in\Gamma_k^0} \mu_{\beta-\epsilon_\ell,\gamma}\left(\alpha+\epsilon_j\right) \int\limits_{\left[x_{\alpha+\gamma+\epsilon_j},u^{k+1}\right]} D^{k+1}_{x_{\alpha+\gamma+\epsilon_j}-u} f \\
&= \sum_{\eta\in\Gamma_0^{k+1}} \sum_{\ell=1}^{d} \sum_{j=0}^{d} (-1)^{j+\ell} \frac{\tau_{j\ell}\left(V_\alpha^{k+1}\right)}{\tau\left(V_\alpha^{k+1}\right)} \mu_{\beta-\epsilon_\ell,\eta-\epsilon_j}\left(\alpha+\epsilon_j\right) \int\limits_{\left[x_{\alpha+\eta},u^{k+1}\right]} D^{k+1}_{x_{\alpha+\eta}-u} f \\
&=: \sum_{\eta\in\Gamma_0^{k+1}} \mu_{\beta,\eta}(\alpha) \int\limits_{\left[x_{\alpha+\eta},u^{k+1}\right]} D^{k+1}_{x_{\alpha+\eta}-u} f,
\end{aligned}
$$

from which we can read off the recurrence (30).

This proves our desired identity (32) for $k+1$.

Now we can prove by induction that the coefficients $\mu_{\beta,\gamma}(\alpha, h)$ in the divided differences of order k for the rescaled sets $X_\alpha^k(h)$ satisfy

$$\mu_{\beta,\gamma}(\alpha, h) = h^{-k}\mu_{\beta,\gamma}(\alpha), \qquad \alpha \in \Gamma_{n-k}^0, \beta \in \Gamma_k, \gamma \in \Gamma_k^0, \quad h > 0, \tag{35}$$

a fact that we already verified for $k = 1$. Substituting this identity into the recurrence (30) immediately shows by the same argument as used for $k = 1$ that (35) also holds for k replaced by $k + 1$.

Using (35), we get

$$\begin{aligned}\Delta\left(X_\alpha^{k+1}(h)\right)_\beta f &= \sum_{\gamma \in \Gamma_0^{k+1}} \mu_{\beta,\gamma}(\alpha, h) h^{k+1} \int_{\left[h x_{\alpha+\gamma} + (1-h)u, u^{k+1}\right]} D_{x_{\alpha+\eta}-u}^{k+1} f \\ &= \sum_{\gamma \in \Gamma_0^{k+1}} \mu_{\beta,\gamma}(\alpha) \int_{\left[h x_{\alpha+\gamma} + (1-h)u, u^{k+1}\right]} D_{x_{\alpha+\eta}-u}^{k+1} f\end{aligned}$$

so that also

$$\lim_{h \to 0} \Delta\left(X_\alpha^{k+1}(h)\right)_\beta f = \sum_{\gamma \in \Gamma_0^{k+1}} \mu_{\beta,\gamma}(\alpha)\, D_{x_{\alpha+\eta}-u}^{k+1} f(u). \tag{36}$$

Combined with Lemma 19 this also extends (33) to the case $k + 1$. This completes the proof of Theorem 17. Theorem 18, on the other hand, is just a combination of (32) and (33) with Lemma 19.

Acknowledgement: We appreciate very much the suggestions and discussions with Carl de Boor which helped us to improve the paper.

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