THE WHITTAKER FUNCTION $M_{\kappa,\,\mu}$ QUA FUNCTION OF κ

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Abstract. The dependence of the Whittaker function $M_{\kappa,\,\mu}(z)$ on the parameter κ is considered. A convergent expansion in ascending powers and an asymptotic expansion in descending powers of κ are discussed. Some properties of the coefficients of the convergent expansion are shown.

Key words. Whittaker function, confluent hypergeometric function, Buchholz polynomials

AMS subject classifications. 33C15, 33E30

1. Introduction. The Whittaker function, closely related to the confluent hypergeometric function, has taken an ever increasing significance due to its frequent use in applications of mathematics to physical and technical problems. Most of its known properties are collected in monographies [3], [6], and in general treatises on special functions [2], [4, vol. 2]. Generally, $M_{\kappa,\,\mu}(z)$ is considered as a function of z, the parameters κ and μ taking fixed values. There are, however, interesting problems in nuclear, atomic, and molecular physics that requiere the use of $M_{\kappa,\,\mu}(z)$ with variable κ. Whenever electrically charged quantum particles, like nuclei, electrons, or ions, are involved, the interaction of the constituents of the physical system is taken into account by means of a potential that becomes purely Coulombian at distances larger than the range of the non-electric (nuclear, exchange, etc) forces. The solution of the Schrödinger equation in the non-Coulombian region, usually obtained by numerical integration, is obliged to match the Coulombian solution. This one can be immediately expressed in terms of the Whittaker function $M_{c/z,\,\mu}(z)$, the parameters c and μ being respectively related to the electric charges and the angular momentum of the system, and the variable z corresponding to the product of the distance and the square root of the energy. Both c and μ are kept fixed, whereas the energy, and therefore z, is varied to achieve the matching of the Coulombian and non-Coulombian wave functions at a given distance. This procedure determines the energies at which bound states or resonances occur.

In this paper we study the dependence of $M_{\kappa,\,\mu}(z)$ on the parameter κ. A convergent expansion in ascending powers of κ, useful for small $|\kappa|$, is given in §2. The coefficients of the expansion are analyzed in both cases of small and large values of $|z|$. For large $|\kappa|$, an asymptotic expansion is considered in §3.

The starting point is a convergent expansion of the Whittaker function in series of Bessel functions given by Buchholz [3, Sect. 7, Eq. (16)]. It reads

$$M_{\kappa,\,\mu}(z) = \Gamma(2\mu+1)\, 2^{2\mu} z^{\mu+\frac{1}{2}} \sum_{n=0}^{\infty} p_n^{(2\mu)}(z)\, \frac{J_{2\mu+n}(2\sqrt{z\kappa})}{(2\sqrt{z\kappa})^{2\mu+n}}, \tag{1.1}$$

where the $p_n^{(2\mu)}(z)$ represent polynomials in z^2, that we denominate Buchholz polynomials. These are defined by

$$p_n^{(\nu)}(z) = \frac{(iz)^n}{2\pi i} \int^{(0+)} \exp\left(\frac{iz}{2}\left(\cot v - \frac{1}{v}\right)\right) \left(\frac{\sin v}{v}\right)^{\nu-1} \frac{dv}{v^{n+1}}, \tag{1.2}$$

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and generated by the function

$$\exp\left(\frac{-z}{2}\left(\coth t-\frac{1}{t}\right)\right)\left(\frac{\sinh t}{t}\right)^{\nu-1}=\sum_{n=0}^{\infty}p_n^{(\nu)}(z)\left(-\frac{t}{z}\right)^n. \tag{1.3}$$

They can also be written in the form [1]

$$p_n^{(\nu)}(z)=\frac{(iz)^n}{n!}\sum_{s=0}^{[\frac{n}{2}]}\binom{n}{2s}f_s^{(\nu)}\,g_{n-2s}(z), \tag{1.4}$$

as a sum of products of polynomials in ν and in z, separately, easily obtainable by means of the recurrence relations

$$f_0^{(\nu)}=1,\qquad f_s^{(\nu)}=\frac{1-\nu}{2}\sum_{r=0}^{s-1}\binom{2s-1}{2r}\frac{4^{s-r}|B_{2(s-r)}|}{s-r}f_r^{(\nu)}, \tag{1.5}$$

$$g_0(z)=1,\qquad g_m(z)=-\frac{iz}{4}\sum_{k=0}^{[\frac{m-1}{2}]}\binom{m-1}{2k}\frac{4^{k+1}|B_{2(k+1)}|}{k+1}g_{m-2k-1}(z), \tag{1.6}$$

where the B_{2n} denote the Bernoulli numbers [2, Table 23.2]. A property of the Buchholz polynomials to be used later is contained in the following.

LEMMA 1.1. *The Buchholz polynomials are bounded in the form*

$$|p_n^{(\nu)}(z)|\le I^{(\nu)}(z,a)\frac{|z|^n}{a^n},\qquad \forall a\in(0,\pi), \tag{1.7}$$

where

$$I^{(\nu)}(z,a)=\frac{1}{2\pi a}\int_{|v|=a}\left|e^{\frac{iz}{2}\left(\cot v-\frac{1}{v}\right)}\right|\left|\left(\frac{\sin v}{v}\right)^{\nu-1}\right||dv|<\infty,\qquad \forall z,\nu\in C. \tag{1.8}$$

Proof. It follows trivially from the definition (1.2) of Buchholz polynomials, by choosing the contour $|v|=a$. □

In what follows $\mu\in C\setminus-\frac{1}{2}N$, the set $-\frac{1}{2}N=\left\{-\frac{1}{2},-1,-\frac{3}{2},-2,\ldots,\right\}$ being excluded since $M_{\kappa,\mu}(z)$ is irregular at those values of μ. As it is well known, the inclusion of a factor $1/\Gamma(1+2\mu)$, to give a new function [3, Sect. 2, Eq. (7)]

$$\mathcal{M}_{\kappa,\mu}(z)\equiv\frac{M_{\kappa,\mu}(z)}{\Gamma(2\mu+1)}, \tag{1.9}$$

eliminates that irregular behaviour and allows to extend μ to all C.

2. Expansion in ascending powers of κ. The Bessel functions in the right hand side of (1.1) can be written in terms of hypergeometric functions [2, Eq. 9.1.69] to give

$$M_{\kappa,\mu}(z)=z^{\mu+\frac{1}{2}}\sum_{n=0}^{\infty}\frac{p_n^{(2\mu)}(z)}{2^n(2\mu+1)_n}\;{}_0F_1(2\mu+1+n;-z\kappa), \tag{2.1}$$

the symbol $(\ )_n$ being the usual Pochhammer one.

PROPOSITION 2.1. *The Whittaker function $M_{\kappa,\,\mu}(z)$ has the following expansion in ascending powers of κ*

$$M_{\kappa,\,\mu}(z) = 2^{2\mu}\,\Gamma(\mu+1)\,z^{\frac{1}{2}} \sum_{m=0}^{\infty} F_m^{(\mu)}(z)\,\frac{(-\kappa)^m}{m!}, \tag{2.2}$$

where we have denoted

$$F_m^{(\mu)}(z) = \frac{(z/4)^{\mu}}{\Gamma(\mu+1)}\,z^m \sum_{n=0}^{\infty} \frac{p_n^{(2\mu)}(z)}{2^n(2\mu+1)_{n+m}}, \qquad m = 0, 1, 2, \ldots. \tag{2.3}$$

Proof. Substitution of ${}_0F_1$ in the right hand side of (2.1) by its definition gives a double series

$$M_{\kappa,\,\mu}(z) = z^{\mu+\frac{1}{2}} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{p_n^{(2\mu)}(z)}{2^n(2\mu+1)_n}\,\frac{(-z\kappa)^m}{(2\mu+1+n)_m\,m!}. \tag{2.4}$$

To invert the order of summations, we need to check that the iterated series of the moduli converges. In the case of $\Re\mu \geq 0$,we have

$$\sum_{m=0}^{\infty} \frac{|z\kappa|^m}{|(2\mu+1+n)_m|\,m!} \leq {}_0F_1(2\Re\mu+1; |z\kappa|), \qquad \forall z \in C, \quad \forall n \geq 0, \tag{2.5}$$

whereas for $\Re\mu < 0$ we can define

$$\varsigma(\mu) = \min\{|\mathrm{Frac}(2\Re\mu)|, 1 - |\mathrm{Frac}(2\Re\mu)|\} \tag{2.6}$$

and state

$$\sum_{m=0}^{\infty} \frac{|z\kappa|^m}{|(2\mu+1+n)_m|\,m!} \leq \frac{1}{(\varsigma(\mu))^2}\,e^{|z\kappa|}, \qquad \forall z \in C, \quad \forall n \geq 0. \tag{2.7}$$

On the other hand, due to Lemma 1.1 with, for example, $a = 2$,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\left|p_n^{(2\mu)}(z)\right|}{2^n\,|(2\mu+1)_n|} &\leq 1 + I^{(2\mu)}(z,2) \sum_{n=1}^{\infty} \frac{|z|^n}{2^{2n}|(2\mu+1)_n|} \\ &\leq 1 + \frac{I^{(2\mu)}(z,2)}{2\varsigma(\mu)} \sum_{n=1}^{\infty} \frac{|z|^n}{2^n(n-1)!} \\ &= 1 + \frac{|z|I^{(2\mu)}(z,2)}{4\varsigma(\mu)} e^{|z|/2} < \infty \qquad \forall z \in C. \end{aligned} \tag{2.8}$$

Therefore, the order of summations in the right hand side of (2.4) can be inverted. This leads to (2.2). □

The functions $F_m^{(\mu)}(z)$ possess several interesting properties collected in the following Lemmas.

LEMMA 2.2. *The functions $z^{-\mu}F_m^{(\mu)}(z)$ are integer functions of z.*

Proof. It is contained in the proof of Proposition 2.1. □

LEMMA 2.3. *The functions $F_m^{(\mu)}(z)$ obey the circuital relations*

$$F_m^{(\mu)}(e^{iq\pi}z) = (-1)^{qm} e^{iq\mu\pi} F_m^{(\mu)}(z), \qquad q \text{ integer}. \tag{2.9}$$

Proof. The definition (2.3) and the fact that the Buchholz polynomials contain only even powers of z make (2.9) evident. ◻

LEMMA 2.4. *The first coefficient, $F_0^{(\mu)}(z)$, of the expansion* (2.2) *is the modified Bessel function $I_\mu(z/2)$.*

Proof. It follows immediately from the known relation [3, App. I.A, Eq. (1a)], [6, Eq. (1.8.11)]

$$M_{0,\,\mu}(z) = 2^{2\mu}\,\Gamma(\mu+1)\,z^{\frac{1}{2}}I_\mu(z/2). \qquad ◻$$

This Lemma, together with the definition (2.3), provides with a representation of the modified Bessel function in terms of Buchholz polynomials,

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\sum_{n=0}^{\infty}\frac{p_n^{(2\nu)}(2z)}{2^n(2\nu+1)_n}, \tag{2.10}$$

or, alternatively, a sum rule for those polynomials,

$$\sum_{n=0}^{\infty}\frac{p_n^{(\nu)}(z)}{2^n(\nu+1)_n} = {}_0F_1\left(\frac{\nu}{2}+1;\frac{z^2}{16}\right). \tag{2.11}$$

LEMMA 2.5. *Let $\mathcal{D}_{(z,\,\mu)}$ denote the modified Bessel differential operator,*

$$\mathcal{D}_{(z,\,\mu)} \equiv z^2\frac{d^2}{dz^2} + z\frac{d}{dz} - (z^2+\mu^2).$$

The coefficients $F_m^{(\mu)}$ obey the differential equations

$$\mathcal{D}_{(z,\,\mu)}F_m^{(\mu)}(2z) = 2mzF_{m-1}^{(\mu)}(2z), \qquad m = 0,1,2,\ldots. \tag{2.12}$$

Proof. Substitution of the expansion (2.2) in the Whittaker differential equation gives a power series in κ that must vanish for any κ. Cancellation of the coefficient of each power leads to (2.12). ◻

The method of depression of the order applied to the differential equation (2.12) allows one to obtain $F_m^{(\mu)}$, by integration, in terms of $F_0^{(\mu)}$ and $F_{m-1}^{(\mu)}$.

PROPOSITION 2.6. *The coefficients $F_m^{(\mu)}(z)$ admit the integral representation*

$$F_m^{(\mu)}(z) = \frac{2^m\Gamma\left(\mu+\frac{1}{2}\right)}{\pi^{1/2}(z/4)^\mu}\,\frac{1}{2\pi i}\int_{\mathrm{C}_1}(\operatorname{arc\,coth} v)^m\,\frac{e^{zv/2}}{(v^2-1)^{\mu+\frac{1}{2}}}\,dv, \tag{2.13}$$

the integration contour C_1 *being represented in Fig.* 2.1.

Proof. From the definition (2.3),

$$F_m^{(\mu)}(z) = (z/4)^\mu\,\frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)}\,z^m\sum_{n=0}^{\infty}\frac{p_n^{(2\mu)}(z)}{2^n\Gamma(2\mu+1+n+m)}. \tag{2.14}$$

Now we use an integral representation for the reciprocal of the Gamma function [4, vol. 1, Sect. 1.6, Eq. (2)],

$$\frac{1}{\Gamma(2\mu+1+n+m)} = \frac{1}{2\pi i}\int_{\mathrm{C}_2}e^t t^{-(2\mu+1+n+m)}dt, \qquad |\arg t| \le \pi, \tag{2.15}$$

FIG. 2.1. *Contour for the integration in the right hand side of* (2.13)

the integration contour being shown in Fig. 2.2, and the duplication formula for the Gamma function [2, Eq. 6.1.18] to obtain

$$F_m^{(\mu)}(z) = \frac{z^{\mu+m}}{\pi^{\frac{1}{2}}}\,\Gamma(\mu+\tfrac{1}{2})\sum_{n=0}^{\infty}\frac{p_n^{(2\mu)}(z)}{2^n}\,\frac{1}{2\pi i}\int_{\mathrm{C}_2} e^t t^{-(2\mu+1+n+m)}dt. \tag{2.16}$$

The order of summation and integration can be inverted. This can be proved by choosing a contour C_2 such that $|t| \geq |z|$; then, remembering Lemma 1.1 with, for example, $a = 1$, one realizes that

$$\begin{aligned}\sum_{n=0}^{\infty}\int_{\mathrm{C}_2}\frac{|p_n^{(2\mu)}(z)|}{|2t|^n} &\left|e^t\right|\left|t^{-(2\mu+1+m)}\right||dt| \\ &\leq I^{(2\mu)}(z,1)\sum_{n=0}^{\infty}\frac{1}{2^n}\int_{\mathrm{C}_2}\left|e^t\right|\left|t^{-(2\mu+1+m)}\right||dt| < \infty.\end{aligned} \tag{2.17}$$

By inverting the order of summation and integration in (2.16) and using

$$\sum_{n=0}^{\infty}\frac{p_n^{(2\mu)}(z)}{(2t)^n} = \exp\left(\frac{z}{2}\coth\left(\frac{z}{2t}\right) - t\right)\left(\frac{\sinh\left(\frac{z}{2t}\right)}{\frac{z}{2t}}\right)^{2\mu-1}, \tag{2.18}$$

that follows immediately from (1.3), it becomes

$$F_m^{(\mu)}(z) = \frac{z^{\mu+m}}{\pi^{\frac{1}{2}}}\Gamma(\mu+\tfrac{1}{2})\frac{1}{2\pi i}\int_{\mathrm{C}_2} e^{\frac{z}{2}\coth\left(\frac{z}{2t}\right)}\left(\frac{\sinh\left(\frac{z}{2t}\right)}{\frac{z}{2t}}\right)^{2\mu-1} t^{-(2\mu+1+m)}dt. \tag{2.19}$$

The change of variable

$$v = \coth\left(\frac{z}{2t}\right) \tag{2.20}$$

transforms (2.19) in (2.13). $\square$

FIG. 2.2. *Integration contour used in* (2.15)

THEOREM 2.7. *For large values of $|z|$ and in the sector $|\arg z| < \pi/2$, the functions $F_m^{(\mu)}(z)$ can be written in the form*

$$F_m^{(\mu)}(z) = \frac{e^{z/2}}{\sqrt{\pi z}} \sum_{k=0}^{m} \binom{m}{k} I_k^{(\mu)}(z)\,(\ln z)^{m-k}\,, \tag{2.21}$$

where the functions $I_k^{(\mu)}(z)$ admit asymptotic expansions

$$I_k^{(\mu)}(z) \sim \Gamma(\mu+\tfrac{1}{2}) \sum_{n=0}^{\infty} (-1)^n \frac{a_{n,\,k}(\mu)}{n!z^n} \tag{2.22}$$

whose coefficients $a_{n,\,k}(\mu)$ can be obtained in the form

$$a_{n,\,k}(\mu) = \sum_{j=0}^{k} \binom{k}{j} \left(\frac{d^j\,(\mu+\frac{1}{2})_n}{d(-\mu)^j}\right) \frac{d^{k-j}}{d\mu^{k-j}} \left(\frac{1}{\Gamma(\mu-n+\frac{1}{2})}\right). \tag{2.23}$$

Proof. First, we will show that (2.21) is a formal expression of $F_m^{(\mu)}(z)$; then, we will prove the asymptotic nature of the expansions (2.22) for large $|z|$.

The change of variable

$$v = 1 + w/z, \tag{2.24}$$

transforms the integration contour C_1 in the right hand side of (2.13) in the contour C_3 shown in Fig. 2.3. On the other hand, for $v \in C_1$,

$$\begin{aligned} (\operatorname{arc\,coth} v)^m &= \left(\frac{1}{2}\ln\left(\frac{v+1}{v-1}\right)\right)^m \\ &= \frac{1}{2^m}\left(\ln z + \ln\left(\frac{2}{w}\left(1+\frac{w}{2z}\right)\right)\right)^m \\ &= \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}\left(\ln\left(\frac{2}{w}\left(1+\frac{w}{2z}\right)\right)\right)^k (\ln z)^{m-k}. \end{aligned} \tag{2.25}$$

This allows us to write $F_m^{(\mu)}(z)$ in the form (2.21), with

$$I_k^{(\mu)}(z) = 2^{2\mu}\Gamma(\mu+1/2) \frac{1}{2\pi i}\int_{C_3} e^{w/2}\left(\frac{1}{w^2}\right)^{\mu+1/2}(B(z,w))^{-(\mu+1/2)}\left(\ln\left(B(z,w)\right)\right)^k dw, \tag{2.26}$$

where we have used the abbreviation

$$B(z,w) = \frac{2}{w}\left(1+\frac{w}{2z}\right). \tag{2.27}$$

Then, using the property

$$(B(z,w))^{-(\mu+1/2)}\left(\ln\left(B(z,w)\right)\right)^k = (-1)^k\frac{d^k}{d\mu^k}\left(B(z,w)\right)^{-(\mu+1/2)}, \tag{2.28}$$

it becomes

$$I_k^{(\mu)}(z) = (-1)^k 2^{2\mu}\Gamma(\mu+1/2) \frac{1}{2\pi i}\int_{C_3} e^{w/2}w^{-(2\mu+1)}\frac{d^k}{d\mu^k}\left(\left(\frac{w}{2}\right)^{\mu+1/2}\left(1+\frac{w}{2z}\right)^{-(\mu+1/2)}\right)dw. \tag{2.29}$$

By using the formal expansion

$$\left(1+\frac{w}{2z}\right)^{-(\mu+1/2)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\mu+\tfrac{1}{2})_n\left(\frac{w}{2z}\right)^n, \tag{2.30}$$

strictly valid only in the region $|w/2z| < 1$, and inverting the order of summation and integration, one obtains the descending powers expansion (2.22) with coefficients given by

$$a_{n,\,k}(\mu) = \frac{(-1)^k 2^{2\mu-n}}{2\pi i}\int_{C_3} e^{w/2}w^{n-2\mu-1}\frac{d^k}{d\mu^k}\left(\left(\frac{w}{2}\right)^{\mu+1/2}(\mu+\tfrac{1}{2})_n\right)dw. \tag{2.31}$$

By expanding the derivative in the integrand and using the identity

$$\frac{1}{2\pi i}\int_{C_3} e^{w/2}w^{n-\mu-1/2}\left(\ln(w/2)\right)^j dw = (-1)^j\frac{d^j}{d\mu^j}\left(\frac{2^{n-\mu+1/2}}{\Gamma(\mu-n+\frac{1}{2})}\right), \tag{2.32}$$

one can write (2.31) in the form (2.23).

Now we want to prove that the expansions (2.22) are asymptotic. Let us consider the remainder

$$R_{N,\,k} = \frac{I_k^{(\mu)}(z)}{\Gamma(\mu+\frac{1}{2})} - \sum_{n=0}^{N}(-1)^n\frac{a_{n,\,k}(\mu)}{n!\,z^n}, \tag{2.33}$$

that, in view of (2.29) and (2.31), can be written in the form

$$R_{N,\,k} = \frac{(-1)^k 2^{2\mu}}{2\pi i}\int_{C_3} e^{w/2}w^{-2\mu-1}\frac{d^k}{d\mu^k}\left(\left(\frac{w}{2}\right)^{\mu+1/2}f_N^{(\mu)}(z,w)\right)dw, \tag{2.34}$$

where we have abbreviated

$$f_N^{(\mu)}(z,w) = \left(1+\frac{w}{2z}\right)^{-(\mu+1/2)} - \sum_{n=0}^{N}\frac{(-1)^n}{n!}(\mu+\tfrac{1}{2})_n\left(\frac{w}{2z}\right)^n. \tag{2.35}$$

FIG. 2.3. *Contour for the integral in* (2.26)

We need to prove that

$$\lim_{|z|\to\infty} \left|z^{N+1} R_{N,\,k}\right| < \infty, \qquad \forall\; N, k. \tag{2.36}$$

In order to make the integration contour in (2.34) independent of $|z|$, we deform it to get the contour C_4 shown in Fig. 2.4. Now we want to interchange the limit operation in (2.36) and the integration in (2.34). We can use the general convergence theorem if we show that the modulus of the integrand is bounded, in some neighbourhood of infinity, by a function not depending on $|z|$ and integrable over C_4. Such neighbourhood could be $\{z \in C,\ |z| > 1\}$ and the function can be chosen to be, for instance,

$$G_k(w) = e^{\Re w/2} \left|\frac{1}{(2w)^{\mu+1/2}}\right| \sum_{l=0}^{k} \left(\begin{array}{c} k \\ l \end{array}\right) \left|\ln\frac{w}{2}\right|^{k-l} g_l(w), \tag{2.37}$$

where

$$g_l(w) = |w|^{N+1}\, \max\{h_{l,\,1}, h_{l,\,2}(w)\}, \tag{2.38}$$

with the abbreviations

$$h_{l,\,1} = \sum_{n=N+1}^{\infty} \frac{1}{2^n n!} \left|\frac{d^l}{d\mu^l}(\mu+\tfrac{1}{2})_n\right|, \tag{2.39}$$

$$h_{l,\,2}(w) = e^{2|\Im\mu|\pi} |2w|^{|\Re\mu|+1} (\ln|2w| + 2\pi)^l + \sum_{n=0}^{N} \frac{|w|^n}{2^n n!} \left|\frac{d^l}{d\mu^l}(\mu+\tfrac{1}{2})_n\right|, \tag{2.40}$$

provided the contour C_4 be deformed in such a way that all its points w satisfy $|w| > 1$ and $|w + 2z| \geq 1$. One is led in this way to

$$\lim_{|z|\to\infty} z^{N+1} R_{N,\,k} = \frac{(-1)^k 2^{2\mu}}{2\pi i} \tag{2.41}$$

$$\int_{\mathrm{C}_4} e^{w/2} (2w)^{-\mu-1/2} \sum_{l=0}^{k} \left(\begin{array}{c} k \\ l \end{array}\right) \left(\ln\frac{w}{2}\right)^{k-l} \lim_{|z|\to\infty} z^{N+1} \left(\frac{d^l}{d\mu^l} f_N^{(\mu)}(z,w)\right) dw.$$

FIG. 2.4. *Integral contour used in* (2.41)

Now, in order to calculate the limit, we write for $|z| > |w|$

$$f_N^{(\mu)}(z,w) = \sum_{n=N+1}^{\infty} \frac{(-1)^n}{n!}(\mu+\tfrac{1}{2})_n \left(\frac{w}{2z}\right)^n . \tag{2.42}$$

It is immediate to see that

$$\left| \frac{1}{n!} \frac{d^l (\mu+\frac{1}{2})_n}{d\mu^l} \left(\frac{w}{2z}\right)^n \right| \begin{cases} = 0, & \text{for } n < l, \\ \leq \dfrac{1}{2^n (n-l)!}(|\mu_0| + r + l + \frac{1}{2})_{n-l}, & \forall\ n \geq l, \end{cases} \tag{2.43}$$

where μ_0 is the center of a ball of radius r, $V_r(\mu_0) \subset C \setminus -\frac{1}{2}N$, containig μ. Denoting by b_n the corresponding expressions in the right hand side of (2.43), we have, obviously,

$$\sum_{n=N+1}^{\infty} b_n < \infty, \tag{2.44}$$

Therefore, we can use once more the general convergence theorem in the right hand side of (2.41) to introduce the $\lim_{|z|\to\infty}$ and $\frac{d^l}{d\mu^l}$ symbols inside the sum in the expression of $f_N^{(\mu)}$ given by (2.42). Then,

$$\lim_{|z|\to\infty} z^{N+1} \frac{d^l}{d\mu^l} f_N^{(\mu)}(z,w) = \frac{(-1)^{N+1}}{(N+1)!} \left(\frac{w}{2}\right)^{N+1} \frac{d^l}{d\mu^l}(\mu+\tfrac{1}{2})_{N+1}, \tag{2.45}$$

that substituted in (2.41) proves (2.36). ◻

Remark 1. The expansion given by the precedent theorem for $F_0^{(\mu)}(z)$ is nothing but the known Hankel expansion [2, Eq. 9.7.1] of the modified Bessel function $I_\mu(z/2)$, as it should be.

Remark 2. If one substitutes the expressions (2.21), with the expansions (2.22), for the $F_m^{(\mu)}(z)$ in the differential relations (2.12), one obtains a recurrence relation for the coefficients $a_{n,\,k}(\mu)$, namely

$$\begin{aligned} a_{n,\,k}(\mu) &= \left(\mu^2 - (n-\tfrac{1}{2})^2\right) a_{n-1,\,k}(\mu) \\ &\quad + k(2n-1) a_{n-1,\,k-1}(\mu) - k(k-1) a_{n-1,\,k-2}(\mu), \qquad n = 1,2,3,\ldots . \end{aligned} \tag{2.46}$$

This recurrence, however, does not allow to determine the coefficients $a_{0,\,k}(\mu)$. These should be obtained directly from (2.23).

Remark 3. The thesis of the theorem can be checked by substituting the expression (2.21) for $F_m^{(\mu)}(z)$ in (2.2), grouping equal powers of $\ln z$ and reordering the sums. One obtains in this way the familiar asymptotic expansion of the Whittaker function.

Remark 4. Use of Lemma 2.3 allows one to evaluate $M_{\kappa,\,\mu}(z)$ in sectors different from $|\arg z| < \pi/2$ with the aid, for large $|z|$, of (2.21).

3. Expansion in descending powers of κ. For large values of $|\kappa|$, an expansion in descending powers can be obtained from (1.1). With the notation

$$t \equiv 2\sqrt{z\kappa}, \tag{3.1}$$

we can write, obviously,

$$M_{\kappa,\,\mu}(z) = \Gamma(2\mu+1) z^{\frac{1}{2}} \kappa^{-\mu} \left(\sum_{n=0}^{N} \frac{p_n^{(2\mu)}(z)}{2t^n} \left(H^{(1)}_{2\mu+n}(t) + H^{(2)}_{2\mu+n}(t) \right) + R_N \right), \tag{3.2}$$

where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ represent the Hankel functions and

$$R_N = \sum_{n=N+1}^{\infty} \frac{p_n^{(2\mu)}(z)}{t^n} J_{2\mu+n}(t). \tag{3.3}$$

Now we can use the asymptotic expansions [7, §7.2, Eqs. (3) and (4)]

$$H_\nu^{(1)}(t) = \sqrt{\frac{2}{\pi t}} e^{i(t-\nu\frac{\pi}{2}-\frac{\pi}{4})} \left(\sum_{m=0}^{q} \frac{(-1)^m}{(2it)^m} \frac{(\nu-m+\frac{1}{2})_{2m}}{m!} + O(t^{-q-1}) \right), \tag{3.4}$$

$$H_\nu^{(2)}(t) = \sqrt{\frac{2}{\pi t}} e^{-i(t-\nu\frac{\pi}{2}-\frac{\pi}{4})} \left(\sum_{m=0}^{q} \frac{1}{(2it)^m} \frac{(\nu-m+\frac{1}{2})_{2m}}{m!} + O(t^{-q-1}) \right), \tag{3.5}$$

simultaneously valid for all ν when z is confined to the sector $|\arg t| < \pi$. Then,

$$\begin{aligned} M_{\kappa,\,\mu}(z) = {} & \frac{\Gamma(2\mu+1)}{2\sqrt{\pi}} z^{\frac{1}{4}} \kappa^{-\mu-\frac{1}{4}} \left(e^{i(t-\mu\pi-\frac{\pi}{4})} \left(\sum_{n=0}^{N} \frac{T_n^{(1)}(z)}{(it)^n} + O(t^{-N-1}) \right) \right. \\ & \left. + e^{-i(t-\mu\pi-\frac{\pi}{4})} \left(\sum_{n=0}^{N} \frac{T_n^{(2)}(z)}{(it)^n} + O(t^{-N-1}) \right) + \sqrt{2\pi t} R_N \right), \end{aligned} \tag{3.6}$$

with the notation

$$T_n^{(1)}(z) = \sum_{m=0}^{n} p_m^{(2\mu)}(z)(-1)^{n-m} \frac{(2\mu-n+2m+\frac{1}{2})_{2(n-m)}}{2^{n-m}(n-m)!}, \tag{3.7}$$

$$T_n^{(2)}(z) = \sum_{m=0}^{n} p_m^{(2\mu)}(z)(-1)^{m} \frac{(2\mu-n+2m+\frac{1}{2})_{2(n-m)}}{2^{n-m}(n-m)!}. \tag{3.8}$$

To prove the asymptotic character of the expansion (3.6), we need the following.

LEMMA 3.1. *The Bessel functions satisfy the inequality*

$$\left| \frac{J_{2\mu+n}(t)}{e^{|\Im t|}} \right| < 3 e^{5|\Im\mu|\pi} \tag{3.9}$$

for sufficiently large values of n, $n \geq 1 - 2\Re\mu$.

Proof. Using the analytic continuation property [2, Eq. 9.1.35] of the Bessel functions and their integral representation [2, Eq. 9.1.22], and, for pure imaginary t, the relation [2, Eq. 9.6.3] between Bessel and modified Bessel functions and the integral representation [2, Eq. 9.6.20], one can easily check that

$$|J_{2\mu+n}(t)| < 3e^{|\Im t|+5|\Im\mu|\pi}, \qquad \text{for } n \geq 1 - 2\Re\mu. \tag{3.10}$$

This makes (3.9) evident. □

Finally we can enunciate the following.

PROPOSITION 3.2. *For fixed* z *and large values of* $|\kappa|$, *and therefore large values of* $|t|$, t *being given in* (3.1), *the Whittaker function admits the asymptotic expansion*

$$\begin{aligned} M_{\kappa,\,\mu}(z) = \; & \frac{\Gamma(2\mu+1)}{2\sqrt{\pi}} z^{\frac{1}{4}} \kappa^{-\mu-\frac{1}{4}} \left(e^{i(t-\mu\pi-\frac{\pi}{4})} \left(\sum_{n=0}^{N} \frac{T_n^{(1)}(z)}{(it)^n} + O(t^{-N-\frac{1}{2}}) \right) \right. \\ & \left. + e^{-i(t-\mu\pi-\frac{\pi}{4})} \left(\sum_{n=0}^{N} \frac{T_n^{(2)}(z)}{(it)^n} + O(t^{-N-\frac{1}{2}}) \right) \right), \end{aligned} \tag{3.11}$$

the polynomials $T_n^{(1)}(z)$ *and* $T_n^{(2)}(z)$ *being respectively given in* (3.7) *and* (3.8).

Proof. We need only to prove that the remainder R_N in the expansion (3.2), appearing also in the right hand side of (3.6), is

$$R_N = e^{|\Im t|} O(t^{-N-1}), \tag{3.12}$$

that is,

$$\lim_{|t|\to\infty} \left| t^{N+1} \frac{R_N}{e^{|\Im t|}} \right| < \infty. \tag{3.13}$$

By using (3.3), the left hand side of (3.13) reads

$$\lim_{|t|\to\infty} \left| \sum_{n=N+1}^{\infty} \frac{p_n^{(2\mu)}(z)}{t^{n-N-1}} \frac{J_{2\mu+n}(t)}{e^{|\Im t|}} \right|. \tag{3.14}$$

According to Lemmas 1.1, with $a = 2$, and 3.1, we have, for $|t| \geq |z|$ and $n \geq n_0 \geq \max\{N+1, 1-2\Re\mu\}$,

$$\left| \frac{p_n^{(2\mu)}(z)}{t^{n-N-1}} \frac{J_{2\mu+n}(t)}{e^{|\Im t|}} \right| < 3I^{(2\mu)}(z,2)|z|^{N+1} e^{5|\Im\mu|\pi} \frac{1}{2^n} \equiv c_n \tag{3.15}$$

and, of course,

$$\sum_{n=n_0}^{\infty} c_n < \infty. \tag{3.16}$$

Then, using the general convergence theorem,

$$\begin{aligned} \lim_{|t|\to\infty} \left| t^{N+1} \frac{R_N}{e^{|\Im t|}} \right| \; & \leq \; \sum_{n=N+1}^{\infty} \left| p_n^{(2\mu)}(z) \right| \lim_{|t|\to\infty} \left| t^{N+1-n} \frac{J_{2\mu+n}(t)}{e^{|\Im t|}} \right| \\ & < \; 3 \left| p_{N+1}^{(2\mu)}(z) \right| e^{5|\Im\mu|\pi} < \infty. \end{aligned} \tag{3.17}$$

This completes the proof of the proposition. □

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