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Asymptotic Expansions of Mellin Convolution Integrals*

José L. López†

Abstract. We present a new method for deriving asymptotic expansions of $\int_0^\infty f(t)h(xt)dt$ for small x. We only require that $f(t)$ and $h(t)$ have asymptotic expansions at $t=\infty$ and $t=0$, respectively. Remarkably, it is a very general technique that unifies a certain set of asymptotic methods. Watson's lemma and other classical methods, Mellin transform techniques, McClure and Wong's distributional approach, and the method of analytic continuation turn out to be simple corollaries of this method. In addition, the most amazing thing about it is that its mathematics are absolutely elemental and do not involve complicated analytical tools as the aforementioned methods do: it consists of simple "sums and subtractions." Many known and new asymptotic expansions of important integral transforms are trivially derived from the approach presented here.

Key words. asymptotic expansions of integrals, Mellin convolution integrals, Mellin transforms

AMS subject classifications. 41A60, 30B40, 46F10

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1. Introduction.

The subject of asymptotics can be divided into two main areas. The first area is concerned with solutions of differential equations. The most famous book where this topic is presented is that of Olver [20]. Olver introduces in his book a "universal" method to obtain asymptotic expansions of solutions of linear differential equations of the second order, including error bounds. The second area deals with functions that are expressible in the form of definite integrals or contour integrals. The most complete and modern book in this area is perhaps Wong's [24]. Other excellent books are, for example, the classical Blestein and Handelsman [2] or the more recent book of Paris and Kaminski [21], where a new perspective on asymptotics from Mellin–Barnes integrals is introduced. See also other reference books cited in [2], [21], and [24]. As distinct from the first area, in this second area we cannot speak about a "universal" method. On the contrary, many asymptotic methods have been designed to obtain asymptotic expansions of different kinds of integrals: Watson's lemma for Laplace transforms, Laplace's method, saddle point methods and steepest descents for contour integrals, summability methods for Fourier transforms, stationary phase methods, Mellin transform techniques, distributional methods, analytic continuation methods, etc. This variety of methods gives a certain ad hoc aspect to the asymptotic theory of integrals.

In the last few decades, some investigations have tried to unify the classical methods for integrals by looking for a common root. The first idea was suggested in 1963

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http://www.siam.org/journals/sirev/50-2/65352.html
†Departamento de Matemática e Informática, Universidad Pública de Navarra, 31006-Pamplona, Spain (jl.lopez@unavarra.es).

by Erdélyi and Wyman [4], [25]. In their work, they show that Darboux's method, Watson's lemma, steepest descents, and stationary phase can be viewed as particular cases of the method of Laplace. More recently, following the work of Wong [24], this author suggested that Watson's lemma and integration by parts should be considered as "fundamental classical methods": steepest descents, Laplace's method, and Perron's method, for example, are based on Watson's lemma, whereas stationary phase or summability methods, for example, are based on the integration by parts technique [13].

Despite the effort of those authors, it seems impossible to design a unique asymptotic method valid for any kind of integral containing an asymptotic parameter x, $\int_\Gamma f(x,t)dt$. Nevertheless, a first step toward this goal can be taken: we show here that a simple method is possible to bring a certain order and shed some light on the "unification" of asymptotic methods of integrals of the form

$$I(x) \equiv \int_0^\infty f(t)h(xt)dt. \tag{1}$$

The ideas developed in this paper may be generalized to complex x, but for the sake of clarity, we restrict ourselves to positive values of x. Without loss of generality we can think of x as a small parameter. (If x is large, perform the change of variable $t \to t/x$ and replace the roles of f and h in (1).) Many integral transforms can be put in the form (1): Laplace, Fourier, Stieltjes, Hankel, Poisson, Glasser, Lambert, and so on [26].

If we want to approximate (1) for small x, we may think that only the behavior of $h(t)$ near the origin is relevant. Then we require for $h(t)$ an expansion at $t = 0$,

$$h(t) = \sum_{k=0}^{n-1} b_k t^{k+\beta} + h_n(t), \tag{2}$$

substitute this expansion for $h(xt)$ in (1), and interchange summation and integration. We obtain, formally, an asymptotic expansion for small x:

$$I(x) = \sum_{k=0}^{n-1} \left[b_k \int_0^\infty f(t)t^{k+\beta}dt \right] x^{k+\beta} + \int_0^\infty f(t)h_n(xt)dt. \tag{3}$$

In fact, classical methods such as Watson's lemma, Laplace's method, and saddle point techniques are based on this idea. This is the message established in [4], [13], and [25]. After some manipulations of the integrals to put them in the form (1) and some hypotheses on f and h, those methods show that the above expansion is not only formal, but a valid asymptotic expansion.

On the other hand, we may write (1) in a different form:

$$I(x) \equiv x^{-1} \int_0^\infty f\left(\frac{t}{x}\right) h(t)dt. \tag{4}$$

Written in this form, it seems plausible that only the behavior of $f(t)$ at infinity is relevant to approximate $I(x)$ when $x \to 0$. Then we require for $f(t)$ an expansion at $t = \infty$:

$$f(t) = \sum_{k=0}^{n-1} \frac{a_k}{t^{k+\alpha}} + f_n(t), \qquad t \to \infty. \tag{5}$$

Substituting this expansion in (4) and interchanging summation and integration we obtain the formal expansion

$$I(x) = \sum_{k=0}^{n-1} \left[a_k \int_0^\infty t^{-k-\alpha} h(t) dt \right] x^{k+\alpha-1} + \int_0^\infty f_n(t) h(xt) dt. \tag{6}$$

If the negative moments of $h(t)$ exist, we can think of this formula as generating a new family of classical methods. Then, somehow, "the two natural and easy possibilities" to obtain asymptotic expansions of $I(x)$ for small x are as follows:

(i) Try an expansion of $h(t)$ at $t = 0$ if the positive moments of $f(t)$ exist (classical methods I).

(ii) Try an expansion of $f(t)$ at $t = \infty$ if the negative moments of $h(t)$ exist (classical methods II).

From (3) and (6) we see that classical methods require the existence of either all the positive moments of $f(t)$ or all the negative moments of $h(t)$. But what happens if $f(t)$ does not converge fast enough to 0 when $t \to \infty$ and $h(t)$ does not converge fast enough to 0 when $t \to 0$? Under these circumstances, the coefficients of the expansion (3) or (6) are not defined and the classical expansion makes no sense. Important examples of failure are Laplace transforms for small parameter [24, Chap. 6, sec. 5], Stieltjes transforms for large parameter [24, Chap. 6, sec. 2], the elliptic integrals [14], [15], the Epstein–Hubbel integral [8], the Appell function [10], and the Poisson transform for small parameter [12], among others.

McClure and Wong (M&W) solved this problem for certain families of functions $f(t)$ and $h(t)$ by using the theory of distributions and analytic continuation techniques [18], [19], [24, Chaps. 5, 6], or using also convolutions of distributions [23], [24, Chap. 6, sec. 7]. Different and more general proofs using only analytic continuation (AC) were proposed in [22], [17], and [16]. A different solution to this problem was proposed by Handlesman and Lew using the method of Mellin transforms (MTs) [5], [6], [7], [24, Chap. 3]. Their technique writes $I(x)$ in the Mellin transformed space and uses analytic continuation and the Cauchy residue theorem. M&W, AC, and MT are all more difficult techniques than classical methods I or II.

Having reached this point, a little bit of simplification would be very welcome. This is exactly the main idea to be presented in this work: the appearance of divergences in cases (i) or (ii) above is an artificial problem. An unnecessary problem is created when expanding h or f in (2) or (5) up to n terms, substituting this expansion in (1), and interchanging sum and integral. And, if we do not create the problem, we will not have to solve it by using M&W, AC, MT, or any other repairing tool. So, do not expand only h or only f up to n terms. The idea is as simple as this: *expand h and f simultaneously and substitute these expansions in* (1) *in such a way that you do not create any divergence.* The aim of this paper is to show that in fact this idea not only works, but it also generates an extraordinarily simple method which contains, as straightforward corollaries, classical methods I and II, M&W theory, the AC method, and MT techniques. It will be shown that this procedure works for a family of functions f and h at least as large as the families considered in the classical methods, M&W theory, MT techniques, or in the AC method. Moreover, all of the aforementioned methods can be seen as one unique method viewed from different angles, the easiest angle being the one presented here, the easiest perspective coming by the hand of simplicity.

In the next section we give some definitions and technical results and MT techniques, the AC method, and M&W theory are briefly resumed. Section 3 presents the main result of the paper: a unified and simple method to obtain asymptotic expansions of $I(x)$ for small x. In section 4, error bounds for the remainders are derived. Section 5 rederives some classical results, MT techniques, the AC method, and M&W theory as corollaries of the fundamental theorem of section 3. An example which shows the applicability of the method is given in section 6. Section 7 contains some conclusions and final remarks.

2. Preliminaries.

This section recalls very briefly the MT techniques, M&W theory, and the AC method. Some definitions and two important formulas are needed to formulate accurately the concepts mentioned in the introduction.

2.1. Definitions and Technical Results.

DEFINITION 1. *We denote by $\mathcal{F}$ the set of functions $f \in L^1_{\mathrm{Loc}}(0,\infty)$ such that*

(i) *f has an asymptotic expansion at infinity,*

$$f(t) = \sum_{k=0}^{n-1} \frac{a_k}{t^{\alpha_k}} + f_n(t), \qquad n = 1, 2, 3, \ldots, \tag{7}$$

where, for $k = 0, 1, 2, \ldots$, $\{a_k\}$ and $\{\alpha_k\}$ are sequences of complex and real numbers, respectively, with α_k strictly increasing and $f_n(t) = \mathcal{O}(t^{-\alpha_n})$ as $t \to \infty$;

(ii) *$f(t) = \mathcal{O}(t^{-a})$ as $t \to 0^+$ with $a \in \mathbb{R}$.*

DEFINITION 2. *We denote by $\mathcal{H}$ the set of functions $h \in L^1_{\mathrm{Loc}}(0,\infty)$ such that*

(i) *h has an asymptotic expansion at $t = 0^+$,*

$$h(t) = \sum_{k=0}^{n-1} b_k t^{\beta_k} + h_n(t), \qquad n = 1, 2, 3, \ldots, \tag{8}$$

where, for $k = 0, 1, 2, \ldots$, $\{b_k\}$ and $\{\beta_k\}$ are sequences of complex and real numbers, respectively, with β_k strictly increasing and $h_n(t) = \mathcal{O}(t^{\beta_n})$ as $t \to 0^+$;

(ii) *$h(t) = \mathcal{O}(t^{-b})$ when $t \to \infty$ with $b \in \mathbb{R}$.*

DEFINITION 3. *Let $g \in L^1_{\mathrm{Loc}}(0,\infty)$. We denote by $M[g; z]$ the Mellin transform of g, $\int_0^\infty t^{z-1} g(t)dt$ (when this integral exists), or its analytic continuation as a function of z.*

The following remark and the two following formulas are proved in [16] for the particular case $\alpha_k = k + \alpha$, $\beta_k = k + \beta$, α, $\beta \in \mathbb{R}$. The proof in the general case is a straightforward generalization and is omitted here.

Remark 1. In the foregoing discussion we require the parameters a, b, α_0, and β_0 to satisfy, without loss of generality, the following relations.

Condition I. $a - \beta_0 < 1 < b + \alpha_0$.

Condition II. $-\beta_0 < b$ and $a < \alpha_0$.

The Mellin transform $M[f; z]$ of every function $f \in \mathcal{F}$ exists and defines a meromorphic function of z in the half plane $\Re z > a$. More precisely, for any $n \in \mathbb{N}$,

$$M[f; z] = \begin{cases} \displaystyle\int_0^\infty t^{z-1} f(t)dt \ \text{ for } \ a < \Re z < \alpha_0, \\ \displaystyle\int_0^1 t^{z-1} f(t)dt - \sum_{k=0}^{n-1} \frac{a_k}{z - \alpha_k} + \int_1^\infty t^{z-1} f_n(t)dt \ \text{ for } \ a < \Re z < \alpha_n, \\ \displaystyle\int_0^\infty t^{z-1} f_n(t)dt \ \text{ for } \ \alpha_{n-1} < \Re z < \alpha_n. \end{cases} \tag{9}$$

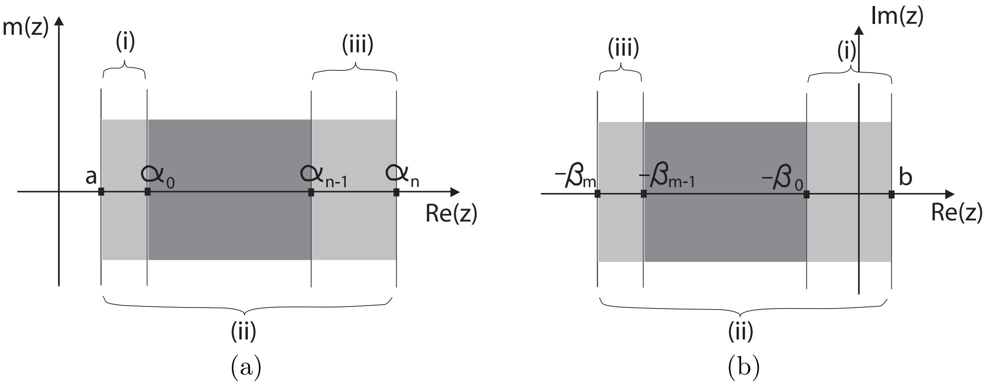

Fig. 1 *The different strips of analyticity of the different representations of the Mellin transforms* $M[f;z]$ *and* $M[h;z]$ *given in* (9) *and* (10)*, respectively. In* (a)*, the strips* (i), (ii), *and* (iii) *are the regions of analyticity of the respective lines in the right-hand side of* (9)*. In* (b)*, the strips* (i), (ii), *and* (iii) *are the regions of analyticity of the respective lines in the right-hand side of* (10)*.*

Observe that $M[f;z]$ has simple poles at the points $z = \alpha_k$, $k = 0, 1, 2, \ldots,$ with residues $-a_k$ (see Figure 1(a)).

The Mellin transform $M[h;z]$ of every function $h \in \mathcal{H}$ exists and defines a meromorphic function of z in the half plane $\Re z < b$. More precisely, for any $m \in \mathbb{N}$,

(10)

$$M[h;z] = \begin{cases} \displaystyle\int_0^\infty t^{z-1}h(t)dt \ \text{ for } \ -\beta_0 < \Re z < b, \\ \displaystyle\int_0^1 t^{z-1}h_m(t)dt + \sum_{k=0}^{m-1}\frac{b_k}{z+\beta_k} + \int_1^\infty t^{z-1}h(t)dt \ \text{ for } \ -\beta_m < \Re z < b, \\ \displaystyle\int_0^\infty t^{z-1}h_m(t)dt \ \text{ for } \ -\beta_m < \Re z < -\beta_{m-1}. \end{cases}$$

Observe that $M[h;z]$ has simple poles at the points $z = -\beta_k$, $k = 0, 1, 2, \ldots,$ with residues b_k (see Figure 1(b)).

2.2. MT Techniques. Roughly speaking, the MT technique proceeds as follows. Let $h \in \mathcal{H}$ and $f \in \mathcal{F}$ and let c be any real number satisfying $-\beta_0 < c < b$ and $1 - \alpha_0 < c < 1 - a$. If $M[f; 1 - c - i\cdot] \in L_1(-\infty, \infty)$ or $M[h; c + i\cdot] \in L_1(-\infty, \infty)$, then $I(x)$ may be written in the form [24, Chap. 3]

$$I(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-z} M[f;1-z]M[h;z]dz. \tag{11}$$

A displacement of the integration contour to the straight line $\Re z = d < c$ and the use of the Cauchy residue theorem gives

$$I(x) = \sum_{d<\Re z<c} \text{Res}\{x^{-z}M[f;1-z]M[h;z]; z = 1-\alpha_k, -\beta_k\} + \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} x^{-z}M[f;1-z]M[h;z]dz.$$

Then, from (9) and (10), when $\alpha_k - \beta_j \neq 1 \;\forall\; k, j \in \mathbb{N}\bigcup\{0\}$, and for appropriate n and $m \in \mathbb{N}$ [24, Chap. 3],

$$\begin{aligned}\int_0^\infty h(xt)f(t)dt = &\sum_{k=0}^{n-1} a_k M[h; 1-\alpha_k]x^{\alpha_k - 1} + \sum_{j=0}^{m-1} b_j M[f; \beta_j + 1]x^{\beta_j} \\ &+ \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} x^{-z} M[f; 1-z]M[h; z]dz.\end{aligned} \tag{12}$$

If $\alpha_k - \beta_j = 1$ for some $k, j \in \mathbb{N}\bigcup\{0\}$, the pole $z = 1 - \alpha_k$ of $M[f; 1-z]$ and the pole $z = -\beta_j$ of $M[h; z]$ coalesce and, then, the integrand $x^{-z}M[f; 1-z]M[h; z]$ in (11) has a double pole. In this case the first line in the right-hand side of (12) must be replaced by

$$\lim_{z\to 0}\left\{x^{\beta_j}\left[a_k x^{-z} M[h; 1+z-\alpha_k] + b_j M[f; z+\beta_j+1]\right]\right\}.$$

Formally, the sum (12) yields an asymptotic expansion of $I(x)$ for small x. The difficulty of this method lies in the technical results required to write $I(x)$ in the form (11) and on the proof of the asymptotic character of (12). Moreover, from the expansion given above it is not always clear how to obtain appropriate error bounds for the remainder $\int_{d-i\infty}^{d+i\infty} x^{-z}M[f; 1-z]M[h; z]dz$.

2.3. McClure and Wong's Distributional Theory. Roughly speaking, M&W theory proceeds as follows. Consider the tempered distributions $\mathbf{f}$, $\mathbf{t}_+^{-\mathbf{k}-\alpha}$, and $\mathbf{f_n}$ associated to the corresponding functions $f(t)$, $t^{-k-\alpha}$, and $f_n(t)$ in (7) for the particular case $\alpha_k = k + \alpha$, $k = 0, 1, 2, \ldots$. Consider first the case $0 < \alpha < 1$. Those distributions act over functions $h \in \mathcal{S}[0, \infty)$ (the Schwarz class of $\mathcal{C}^{(\infty)}[0, \infty)$ rapidly decreasing functions) in the following way [24, Chap. 6]:

$$\begin{aligned}\langle \mathbf{f}, h\rangle = \int_0^\infty f(t)h(t)dt, \qquad &\langle \mathbf{f_n}, h\rangle = (-1)^n \int_0^\infty f_{n,n}(t)h^{(n)}(t)dt, \\ \left\langle \mathbf{t}_+^{-\mathbf{k}-\alpha}, h\right\rangle = &\frac{1}{(\alpha)_k}\int_0^\infty t^{-\alpha}h^{(k)}(t)dt\end{aligned} \tag{13}$$

for $k = 0, 1, 2, \ldots$, where

$$f_{n,n}(t) \equiv \frac{(-1)^n}{(n-1)!}\int_t^\infty (u-t)^{n-1} f_n(u)du. \tag{14}$$

From [24, Chap. 6, Lem. 1], we have that these distributions are related by the equality

$$\mathbf{f} = \sum_{k=0}^{n-1} a_k \mathbf{t}_+^{-\mathbf{k}-\alpha} + \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} M[f; k+1]\delta^{(\mathbf{k})} + \mathbf{f_n}, \tag{15}$$

where $\delta^{(k)}$ is the kth derivative of the delta distribution at the origin: $\left\langle \delta^{(k)}, h\right\rangle = (-1)^k h^{(k)}(0)$. Applying (15) to specific kernels $h(xt) \in \mathcal{S}[0, \infty)$ and using (13), we can derive asymptotic expansions of certain integral transforms $I(x)$. For example, if $h(t) = e^{-t}$, we derive the asymptotic expansion of the Laplace transform near the origin for functions $f(t) \in \mathcal{F}$ [24, Chap. 6, Thm. 13]:

$$\begin{aligned}\int_0^\infty e^{-xt}f(t)dt = &\sum_{k=0}^{n-1} a_k \Gamma(1-k-\alpha)x^{k+\alpha-1} + \sum_{k=0}^{n-1}(-1)^k \frac{M[f; k+1]}{k!}x^{k-1} \\ &+ x^n \int_0^\infty e^{-xt} f_{n,n}(t)dt.\end{aligned}$$

The case $\alpha = 1$ is more complicated. In this case, the second line of (13) is replaced by

$$\langle \mathbf{t}_+^{-\mathbf{k}-\mathbf{1}}, h \rangle = -\frac{1}{k!}\int_0^\infty h^{(k+1)}(t)\log t\, dt, \qquad k = 0, 1, 2, \ldots. \tag{16}$$

Formula (15) must be also replaced by [24, Chap. 6, Lem. 2]

$$\mathbf{f} = \sum_{k=0}^{n-1} a_k \mathbf{t}_+^{-\mathbf{k}-\mathbf{1}} + \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} c_k \delta^{(\mathbf{k})} + \mathbf{f_n}, \tag{17}$$

with

$$c_k \equiv \lim_{z\to k}\left[M[f; z+1] + \frac{a_k}{z-k}\right] + a_k(\gamma + \psi(k+1)).$$

The asymptotic expansion of the Laplace transform for $\alpha = 1$ is also given in [24, Chap. 6, Thm. 13]. Asymptotic expansions of Stieltjes, Fourier, and Hilbert transforms are also derived in [24, Chap. 6] using this technique. Asymptotic expansions of the three standard elliptic integrals and the Appell function F_1 were derived in [10], [14], [15] using this technique.

The complexity of M&W's method lies in the derivation of (15) and (17) and their a posteriori implementation in specific kernels $h(t)$. Moreover, the calculation of a general error bound for the remainder is still a challenge.

Similar results may be found in [23] and [24, Chap. 6, sec. 7] using convolutions of distributions. Essentially, that method (called the regularization method) requires both $f, g \in \mathcal{F}$ and $f, g \in \mathcal{H}$. For brevity, we do not reproduce them here and instead refer the reader to [24, Chap. 6, sec. 7].

2.4. The AC Technique. As well as M&W's method, the AC method considered in [17] requires $f \in \mathcal{F}$ for the particular case $\alpha_k = k + \alpha$, $k = 0, 1, 2$. But for h it only requires $h \in \mathcal{C}^{(\infty)}[0,\infty)$ and not the more stringent condition $h \in \mathcal{S}[0,\infty)$ required in M&W's method. This method uses AC techniques instead of distributions. Nevertheless, it gives rise to a particular case of (12) with $\alpha_k = k + \alpha$, $k = 0, 1, 2, \ldots$, $M[h; 1-\alpha_k]$ replaced by $\frac{1}{(\alpha)_k}\int_0^\infty t^{-\alpha} h^{(k)}(t)dt$, b_j replaced by $h^{(j)}(0)/j!$, and an "M&W form" for the remainder [17, Theorems 1 and 2]:

$$\int_0^\infty h(xt)f(t)dt = \sum_{k=0}^{n-1} \frac{a_k}{(\alpha)_k} x^{k+\alpha-1} \int_0^\infty t^{-\alpha} h^{(k)}(t)dt + \sum_{k=0}^{n-1} \frac{M[f; k+1]}{k!} h^{(k)}(0) x^k$$
$$+ (-1)^n x^{n-1} \int_0^\infty f_{n,n}\left(\frac{t}{x}\right) h^{(n)}(t)dt \qquad \text{if } 0 < \alpha < 1$$

and

$$\int_0^\infty h(xt)f(t)dt = -\sum_{k=0}^{n-1} \frac{a_k}{k!} x^k \int_0^\infty h^{(k+1)}(t)\log t dt$$
$$+ \sum_{k=0}^{n-1}\left[a_k(\psi(k+1) + \gamma - \log x) + \lim_{z\to k+1}\left(M[f; z] + \frac{a_k}{z-k-1}\right)\right] \frac{h^{(k)}(0)}{k!} x^k$$
$$+ (-1)^n x^{n-1} \int_0^\infty f_{n,n}\left(\frac{t}{x}\right) h^{(n)}(t)dt \qquad \text{if } \alpha = 1.$$

This method is generalized in [22] and [16] to the case $h \in \mathcal{H}$. A formula similar to (12) is obtained in [22] and [16] for the particular case $\alpha_k = k+\alpha$ and $\beta_k = k+\beta$, $k = 0, 1, 2, \ldots$. Also, a different form for the remainder is obtained there.

3. The "Sum Up and Subtract" Method. We have seen in the previous section that the M&W method is a particular case of the AC method considered in [17], which is a particular case of the method introduced in [22] and revisited in [16]. The asymptotic formula given in [22] and [16] is just (12) for the particular case $\alpha_k = k+\alpha$ and $\beta_k = k+\beta$, $k = 0, 1, 2, \ldots$, and a different form for the remainder. Then, in principle, we may think that the AC method is a particular case of the MT method. But the MT method requires the additional hypothesis $M[f; 1-c-i\cdot] \in L_1(-\infty, \infty)$ or $M[h; c+i\cdot] \in L_1(-\infty, \infty)$, which is not required in the AC method. We present here a trivial proof of (12) without the restrictions $\alpha_k = k+\alpha$ or $\beta_j = j+\beta$ which does not require $M[f; 1-c-i\cdot] \in L_1(-\infty, \infty)$ or $M[h; c+i\cdot] \in L_1(-\infty, \infty)$. Moreover, it gives a simpler expression for the remainder from which a universal error bound is derived. Then we derive a method which results in a generalization of the M&W, AC, and MT methods.

We define $\alpha_{-1} \equiv a$ and $\beta_{-1} \equiv -b$ and observe that $\alpha_{-1} < \alpha_0$ and $\beta_{-1} < \alpha_0$ (see Remark 1). We make the following observation stated in the form of a lemma.

LEMMA 1. *For any $n \in \mathbb{N} \bigcup \{0\}$, $\exists\ m \in \mathbb{N} \bigcup \{0\}$ such that $\alpha_{n-1} - \beta_m < 1 < \alpha_n - \beta_{m-1}$.*

Proof. The lemma is true for $n = 0$ with $m = 0$: $\alpha_{-1} - \beta_0 < 1 < \alpha_0 - \beta_{-1}$ (see Remark 1). From this we have that $\alpha_n - \beta_{-1} > 1\ \forall\ n \ge 0$. Now, for a given $n \in \mathbb{N}$, take the unique $m \in \{0, 1, 2, \ldots\}$ such that $\alpha_n - \beta_{m-1} > 1$ and $\alpha_n - \beta_m \le 1$. Then $\alpha_{n-1} - \beta_m < 1$. ∎

The main result of this paper is stated in the following two theorems.

THEOREM 1. *Let $f \in \mathcal{F}$ and $h \in \mathcal{H}$. Then, for any n, $m \in \mathbb{N}$ such that $\alpha_{n-1} - \beta_m < 1 < \alpha_n - \beta_{m-1}$,*

$$\int_0^\infty h(xt) f(t) dt = \sum_{k=0}^{n-1} a_k M[h; 1-\alpha_k] x^{\alpha_k - 1} + \sum_{j=0}^{m-1} b_j M[f; \beta_j + 1] x^{\beta_j} + \int_0^\infty f_n(t) h_m(xt) dt. \tag{18}$$

If $\alpha_k - \beta_j = 1$ for some pair (k, j), then, in this formula, the sum of terms

$$a_k M[h; 1-\alpha_k] x^{\alpha_k - 1} + b_j M[f; \beta_j + 1] x^{\beta_j}$$

must be replaced by

$$\begin{aligned} &\lim_{z\to 0} \left\{ x^{\beta_j} \left[a_k x^{-z} M[h; 1+z-\alpha_k] + b_j M[f; z+\beta_j+1] \right] \right\} \\ &= x^{\beta_j} \left\{ \lim_{z\to 0} \left[a_k M[h; 1+z-\alpha_k] + b_j M[f; z+\beta_j+1] \right] - a_k b_j \log x \right\}. \end{aligned} \tag{19}$$

Proof. Define $f_0(t) = f(t)$ and $h_0(t) = h(t)$. From Lemma 1, for any $k \in \mathbb{N} \bigcup \{0\}$ it is always possible to find a $j \in \mathbb{N} \bigcup \{0\}$ such that $\alpha_{k-1} - \beta_j < 1 < \alpha_k - \beta_{j-1}$. For the given (k, j) we have that the following integral exists:

$$\int_0^\infty f_k(t) h_j(xt) dt. \tag{20}$$

We start at $k=j=0$ (the inequalities $\alpha_{-1}-\beta_0<1<\alpha_0-\beta_{-1}$ hold) and switch on the following algorithm which increases (k,j) step by step from $(0,0)$ up to (n,m):

(a) For a given (k,j) satisfying $\alpha_{k-1}-\beta_j<1<\alpha_k-\beta_{j-1}$ do the following. If $\alpha_k-\beta_j<1$, go to (b). If $\alpha_k-\beta_j>1$, go to (c). If $\alpha_k-\beta_j=1$, go to (d).

(b) Use $f_k(t)=a_k t^{-\alpha_k}+f_{k+1}(t)$ in (20) and the third line of (10):

$$\int_0^\infty f_k(t)h_j(xt)dt = a_k x^{\alpha_k-1}M[h;1-\alpha_k]+\int_0^\infty f_{k+1}(t)h_j(xt)dt.$$

Go to (a) with k replaced by $k+1$.

(c) Use $h_j(xt)=b_j(xt)^{\beta_j}+h_{j+1}(xt)$ in (20) and the third line of (9):

$$\int_0^\infty f_k(t)h_j(xt)dt = b_j x^{\beta_j}M[f;\beta_j+1]+\int_0^\infty h_{j+1}(xt)f_k(t)dt.$$

Go to (a) with j replaced by $j+1$.

(d) Use first $f_k(t)=a_k t^{-\alpha_k}+f_{k+1}(t)$ and then $h_j(xt)=b_j(xt)^{\beta_j}+h_{j+1}(xt)$ in (20):

$$\begin{aligned}\int_0^\infty & f_k(t)h_j(xt)dt \\ &= \int_0^\infty \left[a_k t^{-\alpha_k}h_j(xt)+b_j x^{\beta_j}t^{\beta_j}f_{k+1}(t)\right]dt+\int_0^\infty h_{j+1}(xt)f_{k+1}(t)dt.\end{aligned} \tag{21}$$

Define the function

$$F_{k,j}(z,t)\equiv t^z\left[a_k t^{-\alpha_k}h_j(xt)+b_j x^{\beta_j}t^{\beta_j}f_{k+1}(t)\right], \qquad z\in\mathbb{C}.$$

Then

$$\int_0^\infty f_k(t)h_j(xt)dt=\int_0^\infty F_{k,j}(0,t)dt+\int_0^\infty f_{k+1}(t)h_{j+1}(xt)dt.$$

On the one hand, $h_j(t)=b_j t^{\beta_j}+\mathcal{O}(t^{\beta_{j+1}})$ when $t\to 0^+$. On the other hand, $f_{k+1}(t)=-a_k t^{-\alpha_k}+\mathcal{O}(t^{-\alpha_{k-1}})$ when $t\to 0^+$. Hence, $F_{k,j}(z,\cdot)\in L^1[0,\infty)$ for $\mathrm{Max}\{\alpha_k-\beta_{j+1},\alpha_{k-1}-\beta_j\}-1<\Re z<\mathrm{Min}\{\alpha_k-\beta_{j-1},\alpha_{k+1}-\beta_j\}-1$. Choose two numbers z_0 and z_1 satisfying $\mathrm{Max}\{\alpha_k-\beta_{j+1},\alpha_{k-1}-\beta_j\}-1<z_0<0$ and $0<z_1<\mathrm{Min}\{\alpha_k-\beta_{j-1},\alpha_{k+1}-\beta_j\}-1$. Then we have that, for $z_0\le\Re z\le z_1$,

$$|F_{k,j}(t,z)|\le G_{k,j}(t)\equiv\begin{cases}|F_{k,j}(t,z_0)| & \text{for } t\in[0,1],\\ |F_{k,j}(t,z_1)| & \text{for } t\in[1,\infty),\end{cases}$$

and that $G_{k,j}(t)\in L^1[0,\infty)$. Using the dominated convergence theorem we have that

$$\begin{aligned}\int_0^\infty F_{k,j}(0,t)dt &= \lim_{z\to 0}\int_0^\infty F_{k,j}(z,t)dt\\ &= x^{\beta_j}\lim_{z\to 0}\int_0^\infty\left[a_k t^{z-\alpha_k}x^{-z}h_j(t)+b_j t^{z+\beta_j}f_{k+1}(t)\right]dt.\end{aligned}$$

From $\alpha_k = \beta_j + 1$ and from (9) and (10) we have that $M[h; z+1-\alpha_k]$ and $M[f; z+\beta_j+1]$ have a common strip of analyticity: $a-\beta_j-1 < \Re z < \beta_j+b$. From Conditions I and II we have $a-\beta_0-1 < 0 < \beta_0+b$, and then the point $z=0$ belongs to that strip of analyticity. Then

$$\int_0^\infty f_k(t)h_j(xt)dt = x^{\beta_j}\lim_{z\to 0}\left\{a_k x^{-z}M[h; z+1-\alpha_k] + b_j M[f; z+\beta_j+1]\right\} + \int_0^\infty h_{j+1}(xt)f_{k+1}(t)dt.$$

Using $x^{-z} = 1 - z\log x + \mathcal{O}(z^2)$ when $z\to 0$ and

$$M[h; z+1-\alpha_k] = \int_0^\infty t^{z-\alpha_k}h_j(t)dt = \frac{b_j}{z} + \mathcal{O}(1) \quad \text{when} \quad z\to 0,$$

we find that the above expression can also be written in the form

$$\int_0^\infty f_k(t)h_j(xt)dt = x^{\beta_j}\left\{\lim_{z\to 0}\left[a_k M[h; 1+z-\alpha_k] + b_j M[f; z+\beta_j+1]\right] - a_k b_j \log x\right\} + \int_0^\infty h_{j+1}(xt)f_{k+1}(t)dt.$$

Go to (a) with k replaced by $k+1$ and j replaced by $j+1$.

This algorithm generates (18)–(19). □

THEOREM 2. *Within the hypothesis of Theorem* 1, *the expansion* (18) *is an asymptotic expansion for small x:*

$$\int_0^\infty f_n(t)h_m(xt)dt = \mathcal{O}(x^{\beta_m} + x^{\alpha_n-1}) \qquad \textit{when} \;\; x\to 0 \;\; \textit{and} \;\; \alpha_n \neq \beta_m + 1 \tag{22}$$

and

$$\int_0^\infty f_n(t)h_m(xt)dt = \mathcal{O}(x^{\beta_m}\log x) \qquad \textit{when} \;\; x\to 0 \;\; \textit{and} \;\; \alpha_n = \beta_m + 1. \tag{23}$$

Proof. On the one hand, from (i) of Definition 1, there is a $c_n^1 > 0$ and a t_0^1 such that $|f_n(t)| \le c_n^1 t^{-\alpha_n}$ for $t \ge t_0^1$. From (i) of Definition 2, there is a $c_m^2 > 0$ and a t_0^2 such that $|h_m(xt)| \le c_m^2 (xt)^{\beta_m}$ for $xt \le t_0^2$. For small enough x we have that $t_0^1 < t_0^2/x$ and we can choose a $t_0 \in [t_0^1, t_0^2/x]$. Then

$$\int_0^\infty f_n(t)h_m(xt)dt = \int_0^{t_0} f_n(t)h_m(xt)dt + \int_{t_0}^\infty f_n(t)h_m(xt)dt,$$

$$\begin{aligned}\left|\int_0^\infty f_n(t)h_m(xt)dt\right| &\le c_m^2 x^{\beta_m}\int_0^{t_0}|f_n(t)|t^{\beta_m}dt + c_n^1 x^{\alpha_n-1}\int_{xt_0}^\infty |h_m(t)|t^{-\alpha_n}dt \\ &\le (x^{\beta_m} + x^{\alpha_n-1})\left[c_m^2\int_0^{t_0}|f_n(t)|t^{\beta_m}dt + c_n^1\int_0^\infty |h_m(t)|t^{-\alpha_n}dt\right].\end{aligned} \tag{24}$$

On the other hand, from (i) of Definition 1, there is a $c_n^1 > 0$ and a t_0^1 such that $|f_n(t/x)| \le c_n^1 x^{\alpha_n}t^{-\alpha_n}$ for $t/x \ge t_0^1$. From (i) of Definition 2, there is a $c_m^2 > 0$ and a t_0^2 such that $|h_m(t)| \le c_m^2 t^{\beta_m}$ for $t \le t_0^2$. For small enough x we have that $t_0^1 x < t_0^2$ and we can choose a $t_0 \in [t_0^1 x, t_0^2]$. Then,

$$\begin{aligned}\int_0^\infty f_n(t)h_m(xt)dt &= \frac{1}{x}\int_0^\infty f_n(t/x)h_m(t)dt \\ &= \frac{1}{x}\left\{\int_0^{t_0} f_n(t/x)h_m(t)dt + \int_{t_0}^\infty f_n(t/x)h_m(t)dt\right\},\end{aligned}$$

$$\begin{aligned}\left|\int_0^\infty f_n(t)h_m(xt)dt\right| &\le c_m^2 x^{\beta_m}\int_0^{t_0/x} |f_n(t)|t^{\beta_m}dt + c_n^1 x^{\alpha_n-1}\int_{t_0}^\infty |h_m(t)|t^{-\alpha_n}dt \\ &\le (x^{\beta_m}+x^{\alpha_n-1})\left[c_m^2\int_0^\infty |f_n(t)|t^{\beta_m}dt + c_n^1\int_{t_0}^\infty |h_m(t)|t^{-\alpha_n}dt\right].\end{aligned} \tag{25}$$

The integrals between brackets in the last line of (24) are finite for $\alpha_n < \beta_m + 1$ (and $\alpha_{n-1} - \beta_m < 1 < \alpha_n - \beta_{m-1}$). The integrals between brackets in the last line of (25) are finite for $\alpha_n > \beta_m + 1$ (and $\alpha_{n-1} - \beta_m < 1 < \alpha_n - \beta_{m-1}$). Then (22) follows from (24) and (25).

If $\alpha_n = \beta_m + 1$, the second integral in the last line of (24) and the first integral in the last line of (25) are divergent and those inequalities are true but useless. In this case, from the first line of (24) and $h_m(t) = h_{m+1}(t) + b_m t^{\beta_m}$,

$$\begin{aligned}\int_{xt_0}^\infty |h_m(t)|t^{-\alpha_n}dt &\le \int_1^\infty |h_m(t)|t^{-\alpha_n}dt + \int_{xt_0}^1 (|h_{m+1}(t)| + |b_m t^{\beta_m}|)t^{-\alpha_n}dt \\ &\le \int_1^\infty |h_m(t)|t^{-\alpha_n}dt + \int_0^1 |h_{m+1}(t)|t^{-\alpha_n}dt + |b_m \log(xt_0)|.\end{aligned}$$

The integrals in the last line above are finite and (23) follows. □

Remark 2. The above theorem applies also to integrals of the form $\int_c^\infty h(xt)f(t)dt$ with $0 < c < \infty$ and with the same hypotheses for f and h except for one concerning the asymptotic behavior of f at $t = 0$:

$$\int_c^\infty h(xt)f(t)dt = \int_0^\infty h(xt)f_c(t)dt$$

with $f_c(t) = f(t)\chi_{(c,\infty)}(t)$, $\chi_{(c,\infty)}(t)$ being the characteristic function of the interval $(0,\infty)$: $\chi_{(c,\infty)}(t) = 1$ if $t \in (c,\infty)$ and $\chi_{(c,\infty)}(t) = 0$ if $t \notin (c,\infty)$. In this case we have $f_c(t) = \mathcal{O}(t^{-a})$ as $t \to 0^+$ with $a < 0$ and $|a|$ as large as we wish. The theorem also applies to integrals of the form $\int_c^d h(xt)f(t)dt$ by writing

$$\int_c^d h(xt)f(t)dt = \int_c^\infty h(xt)f(t)dt - \int_d^\infty h(xt)f(t)$$

and using the above considerations.

The coefficients of the expansion are given in terms of Mellin transforms of f and h. Several representations of those Mellin transforms are given in (9) and (10), but a simpler representation which does not use the concept of analytic continuation is possible when f and h are differentiable.

LEMMA 2. *Suppose that* $\alpha_k = k + \alpha$ *with* $\alpha + b > 1$ *and that* $\beta_j = j + \beta$ *with* $a - \beta < 1$. *Define* $\tilde{h}(t) \equiv t^{-\beta}h(t)$ *and* $\tilde{f}(t) \equiv t^{-\alpha}f(t^{-1})$.

(i) *If* $\tilde{h} \in \mathcal{C}^{(\infty)}[0,\infty)$ *with* $\tilde{h}^{(j)}(t) = \mathcal{O}(t^{-\beta-b})$ *when* $t \to \infty$ $\forall$ $j = 0,1,2,\ldots$ *and* $\alpha - \beta \notin Z$, *then*

$$M[h; 1-k-\alpha] = \frac{(-1)^j}{(\beta-k-\alpha+1)_j} \int_0^\infty t^{\beta+j-\alpha-k}\tilde{h}^{(j)}(t)dt,$$

with $j = \lfloor k+\alpha-\beta \rfloor$ *(this means* $\alpha_k - \beta_j < 1 < \alpha_k - \beta_{j-1}$*).*

(ii) *If* $\tilde{f} \in \mathcal{C}^{(\infty)}[0,\infty)$ *with* $\tilde{f}^{(k)}(t) = \mathcal{O}(t^{a-\alpha})$ *when* $t \to \infty$ $\forall$ $k = 0,1,2,\ldots$ *and* $\alpha - \beta \notin Z$, *then*

$$M[f; 1+j+\beta] = \frac{(-1)^k}{(\alpha-\beta-j-1)_k} \int_0^\infty t^{k+\alpha-\beta-j-2}\tilde{f}^{(k)}(t)dt,$$

with $k = \lfloor \beta+j+2-\alpha \rfloor$ *(this means* $\alpha_{k-1} - \beta_j < 1 < \alpha_k - \beta_j$*).*

(iii) *If the hypotheses of both* (i) *and* (ii) *hold but with* $\alpha - \beta \in Z$ *and* $b + \beta > 1$, $\alpha - a > 1$, *then*

$$\lim_{z\to 0}\left\{x^{\beta_j}\left[a_k x^{-z} M[h; 1+z-\alpha_k] + b_j M[f; z+\beta_j+1]\right]\right\}$$
$$= -x^{\beta+j}\left\{\frac{a_k}{j!}\int_0^\infty \tilde{h}^{(j+1)}(t)\log\left(\frac{t}{x}\right)dt + \frac{b_j}{k!}\int_0^\infty \tilde{f}^{(k+1)}(t)\log t\, dt\right\},$$

with $k = \beta+j+1-\alpha$ *(this means* $\alpha_k - \beta_j = 1$*).*

Proof. To prove (i), write

$$M[h; 1-\alpha_k] = \int_0^\infty t^{-k-\alpha}h_j(t)dt = \int_0^\infty t^{\beta-k-\alpha}\left[\tilde{h}(t) - \sum_{l=0}^{j-1} b_l t^l\right]dt$$

and integrate by parts j times.

To prove (ii), write

$$M[f; 1+\beta_j] = \int_0^\infty t^{j+\beta}f_k(t)dt = \int_0^\infty t^{\alpha-\beta-j-2}\left[\tilde{f}(t) - \sum_{l=0}^{k-1} a_l t^l\right]dt$$

and integrate by parts k times.

To prove (iii), write

$$\lim_{z\to 0}\left\{x^{\beta_j}\left[a_k x^{-z} M[h; 1+z-\alpha_k] + b_j M[f; z+\beta_j+1]\right]\right\}$$
$$= \int_0^\infty \left[a_k t^{-\alpha_k}h_j(xt) + b_j x^{\beta_j} t^{\beta_j} f_{k+1}(t)\right]dt$$
$$= x^\beta\left\{a_k\int_1^\infty t^{\beta-\alpha-k}\left[\tilde{h}(xt) - \sum_{l=0}^{j-1} b_l x^l t^l\right]dt + b_j x^j \int_1^\infty t^{\alpha-\beta-j-2}\left[\tilde{f}(t) - \sum_{l=0}^{k-1} a_l t^l\right]dt\right\}$$
$$+ x^\beta\left\{a_k\int_0^1 t^{\beta-\alpha-k}\left[\tilde{h}(xt) - \sum_{l=0}^{j} b_l x^l t^l\right]dt + b_j x^j \int_0^1 t^{\alpha-\beta-j-2}\left[\tilde{f}(t) - \sum_{l=0}^{k} a_l t^l\right]dt\right\}$$

and integrate by parts $j+1$ times in the first and third integrals and $k+1$ times in the second and fourth integrals. ∎

4. Error Bounds. Theorem 2 does not offer a precise bound for the remainder. We show in this section that a precise bound for the remainder may be obtained if the bound $|f_n(t)| \le c_n^1 t^{-\alpha_n}$ holds $\forall\ t \in (0,\infty)$ and not only for $t \in [t_0^1,\infty)$ and the bound $|h_m(t)| \le c_m^2 t^{\beta_m}$ holds $\forall\ t \in (0,\infty)$ and not only for $t \in (0,t_0^2]$ (see the proof of Theorem 2).

In order to obtain a precise bound for the remainder $\int_0^\infty f_n(t)h_m(xt)dt$, let the remainder $f_n(t)$ in the expansion (7) satisfy the bound $|f_n(t)| \le F_n t^{-\alpha_n}\ \forall\ t \in (0,\infty)$ and let the remainder $h_m(t)$ in (8) satisfy the bound $|h_m(t)| \le H_m t^{\beta_m}\ \forall\ t \in (0,\infty)$ for some positive constants F_n and H_m.

Consider first the case $\beta_m + 1 \ne \alpha_n$. Write

$$\int_0^\infty f_n(t)h_m(xt)dt = \int_0^1 f_n(t)h_m(xt)dt + \int_1^\infty f_n(t)h_m(xt)dt. \tag{26}$$

If $\beta_m > \alpha_n - 1$, perform the change of variable $t \to t/x$ and use $h_m(t) = h_{m-1}(t) - b_{m-1}t^{\beta_{m-1}}$ in the second integral in the right-hand side above. If $\beta_m < \alpha_n - 1$, use $f_n(t) = f_{n-1}(t) - a_{n-1}t^{-\alpha_{n-1}}$ in the first integral in the right-hand side above. Using the inequalities $\alpha_{n-1} - \beta_m < 1 < \alpha_n - \beta_{m-1}$, $|f_n(t)| \le F_n t^{-\alpha_n}$, and $|h_m(t)| \le H_m t^{\beta_m}$ in (26) and straightforward operations we obtain

$$\left|\int_0^\infty f_n(t)h_m(xt)dt\right| \le \begin{cases} C_{n,m}^1 x^{\alpha_n - 1} & \text{if } \beta_m > \alpha_n - 1, \\ C_{n,m}^2 x^{\beta_m} & \text{if } \beta_m < \alpha_n - 1, \end{cases} \tag{27}$$

with

$$C_{n,m}^1 \equiv F_n\left[\frac{H_m}{1+\beta_m-\alpha_n} + \frac{|b_{m-1}| + H_{m-1}}{\alpha_n - \beta_{m-1} - 1}\right] \tag{28}$$

and

$$C_{n,m}^2 \equiv H_m\left[\frac{F_n}{\alpha_n-\beta_m-1} + \frac{|a_{n-1}| + F_{n-1}}{\beta_m + 1 - \alpha_{n-1}}\right]. \tag{29}$$

Consider now the case $\beta_m + 1 = \alpha_n$. Write

$$\int_0^\infty f_n(t)h_m(xt)dt = \int_0^1 f_n(t)h_m(xt)dt + \int_1^{1/x} f_n(t)h_m(xt)dt + \int_{1/x}^\infty f_n(t)h_m(xt)dt. \tag{30}$$

Perform the change of variable $t \to t/x$ in the last integral, use $f_n(t) = f_{n-1}(t) - a_{n-1}t^{-\alpha_{n-1}}$ in the first integral in the right-hand side, and use $h_m(t) = h_{m-1}(t) - b_{m-1}t^{\beta_{m-1}}$ in the last integral. Using the inequalities $\alpha_{n-1} - \beta_m < 1 < \alpha_n - \beta_{m-1}$, $|f_n(t)| \le F_n t^{-\alpha_n}$, and $|h_m(t)| \le H_m t^{\beta_m}$ and straightforward operations we obtain

$$\left|\int_0^\infty f_n(t)h_m(xt)dt\right| \le [C_{n,m}^3 + F_n H_m |\log x|]x^{\beta_m} \quad \text{if} \quad \beta_m = \alpha_n - 1, \tag{31}$$

with

$$C_{n,m}^3 \equiv F_n\frac{|b_{m-1}| + H_{m-1}}{\alpha_n - \beta_{m-1} - 1} + H_m\frac{|a_{n-1}| + F_{n-1}}{\beta_m + 1 - \alpha_{n-1}}. \tag{32}$$

In [9] we introduced certain families of functions f and h quite common in practice which satisfy the bounds $|f_n(t)| \le F_n t^{-\alpha_n}$ and $|h_m(t)| \le H_m t^{\beta_m}\ \forall\ t \in (0,\infty)$.

Moreover, for those families of functions f and h, the constants F_n and H_m can be easily obtained from f and h.

FAMILY 1. *Let $f \in \mathcal{F}$ be a real function with $\alpha_n = n + \alpha$. We consider the function $\tilde{f}(u) \equiv u^{-\alpha} f(u^{-1})$. If $\tilde{f}(w)$ is a bounded analytic function in the region U of the complex w-plane consisting of all points w located at a distance $< r_0$ from the positive real axis, $U_{r_0} = \{w \in \mathbb{C}, |\Im w| < r_0 \text{ if } \Re w \geq 0 \text{ and } |w| < r_0 \text{ if } \Re w < 0\}$, then*

$$|f_n(t)| \leq 2Mr^{-n}t^{-n-\alpha}, \tag{33}$$

where M is a bound of $|\tilde{f}(w)|$ in U_{r_0} and $0 < r < r_0$.

FAMILY 2. *Let $h \in \mathcal{H}$ be a real function with $\beta_m = m + \beta$. We consider the function $\tilde{h}(u) \equiv u^{\beta} h(u)$. If $\tilde{h}(w)$ is a bounded analytic function in the region of the complex w-plane consisting of all points w located at a distance $< \tilde{r}_0$ from the positive real axis, $U_{\tilde{r}_0}$, then*

$$|h_m(t)| \leq 2\tilde{M}\tilde{r}^{-m}t^{m+\beta},$$

where $\tilde{M}$ is a bound of $|\tilde{h}(w)|$ in $U_{\tilde{r}_0}$ and $0 < \tilde{r} < \tilde{r}_0$.

FAMILY 3. *If $f(t)$ is real and the expansion* (7) *satisfies the error test, that is, if $sign(f_n(t)) = -sign(f_{n+1}(t))$ $\forall\ n = 0, 1, 2, \ldots$ and $\forall\ t \in (0, \infty)$, then*

$$|f_n(t)| \leq |a_n| t^{-\alpha_n} \qquad \text{and} \qquad |f_n(t)| \leq |a_{n-1}| t^{-\alpha_{n-1}}.$$

FAMILY 4. *If $h(t)$ is real and the expansion* (8) *satisfies the error test, then*

$$|h_m(t)| \leq |b_m| t^{\beta_m} \qquad \text{and} \qquad |h_m(t)| \leq |b_{m-1}| t^{\beta_{m-1}}.$$

From these bounds, the following observation is obvious.

Remark 3. If f belongs to Family 1, then the bounds (27) and (31) hold with $F_n = 2Mr^{-n}$. If h belongs to Family 2, then the bounds (27) and (31) hold with $H_m = 2\tilde{M}\tilde{r}^{-m}$. If f is real and the expansion (7) satisfies the error test, then the bounds (27) and (31) hold replacing F_n by $|a_n|$. If g is real and the expansion (8) satisfies the error test, then the bounds (27) and (31) hold replacing H_m by $|b_m|$.

5. Universality. Many classical techniques, the MT techniques, M&W theory, and the AC method are easy corollaries of Theorems 1 and 2.

COROLLARY 1 (classical methods I). *If $t^n f(t) \in L^1(0, \infty)$ $\forall\ n \geq 0$, then, in Definition* 1, *$a_k = 0$ $\forall\ k$,*

$$M[f; 1 + \beta_k] = \int_0^\infty f(t)t^{\beta_k}dt, \qquad \text{and} \qquad f_n(t) = f(t).$$

Then, from Theorem 1,

$$\int_0^\infty h(xt)f(t)dt = \sum_{k=0}^{m-1}\left[b_k \int_0^\infty f(t)t^{\beta_k}dt\right]x^{\beta_k} + \int_0^\infty f(t)h_m(xt)dt.$$

An important example is Watson's lemma: set $f(t) = e^{-t}$ and $x = \tilde{x}^{-1}$ in the above formula with $\tilde{x} \to \infty$, to get

$$\tilde{x}\int_0^\infty h(t)e^{-\tilde{x}t}dt = \int_0^\infty h(xt)e^{-t}dt = \sum_{k=0}^{m-1}\frac{b_k\Gamma(\beta_k + 1)}{\tilde{x}^{\beta_k}} + \tilde{x}\int_0^\infty e^{-\tilde{x}t}h_m(t)dt.$$

COROLLARY 2 (classical methods II). *If $t^{-n}h(t) \in L^1(0,\infty)\ \forall\ n \geq 0$, then, in Definition 2, $b_k = 0\ \forall\ k$,*

$$M[h; 1-\alpha_k] = \int_0^\infty h(t)t^{-\alpha_k}dt, \qquad \text{and} \qquad h_m(t) = h(t).$$

Then, from Theorem 1,

$$\int_0^\infty h(xt)f(t)dt = \sum_{k=0}^{n-1}\left[a_k\int_0^\infty h(t)t^{-\alpha_k}dt\right]x^{\alpha_k-1} + \int_0^\infty f_n(t)h(xt)dt.$$

COROLLARY 3 (MT techniques). *Formula (12) is just (18) with a different expression for the remainder. Apart from the conditions required for f and h in Theorem 1 above, the MT technique requires also the integrability of $M[f; 1-c-y\cdot]$ or of $M[h; c+y\cdot]$ in order to write $I(x)$ in the form (11). Then expansion (12) follows from calculating the poles and residues of $M[f; 1-z]$ and $M[h; z]$. But what we see in Theorem 1 is that it is not necessary to write $I(x)$ in the form (11) and, therefore, the integrability of $M[f; 1-c-y\cdot]$ or of $M[h; c+y\cdot]$ is not necessary. In fact, the location of the poles and the value of the residue of $M[f; 1-z]$ and $M[h; z]$ are of fundamental importance to derive the asymptotic expansion of $I(x)$ in the MT technique. But from the second lines of formulas (9) and (10) we see that that information is already contained in the expansions (7) and (8) of $f(t)$ and $h(t)$ and the expansion of $I(x)$ follows directly from (1).*

COROLLARY 4 (M&W method). *If $h \in \mathcal{S}[0,\infty) \subset \mathcal{C}^{(\infty)}[0,\infty)$ and $\alpha_k = k+\alpha$, then $\beta_k = k$ ($k = 0,1,2,\ldots$), $m = n$ in Lemma 1, and*

$$b_k = \frac{h^{(k)}(0)}{k!}.$$

Integrating by parts we have that

$$\int_0^\infty f_n(t)h_n(xt) = (-x)^n\int_0^\infty f_{n,n}(t)h_n^{(n)}(xt)dt = (-x)^n\int_0^\infty f_{n,n}(t)h^{(n)}(xt)dt,$$

where $f_{n,n}(t)$ is defined in (14). On the other hand, if $0 < \alpha < 1$,

$$M[h; 1-k-\alpha] = \int_0^\infty h_k(t)t^{-k-\alpha}dt = \frac{1}{(\alpha)_k}\int_0^\infty t^{-\alpha}h_k^{(k)}(t)dt = \left\langle \mathbf{t}_+^{-k-\alpha}, h\right\rangle.$$

(Observe that the definition of $\left\langle \mathbf{t}_+^{-k-\alpha}, h\right\rangle$ by means of integration by parts in M&W's theory coincides with the definition of analytic continuation given by the Mellin transform of h.) Therefore, from (18),

(34)
$$\int_0^\infty h(xt)f(t)dt = \sum_{k=0}^{n-1}\frac{a_k}{(\alpha)_k}x^{k+\alpha-1}\int_0^\infty t^{-\alpha}h^{(k)}(t)dt + \sum_{k=0}^{n-1}\frac{M[f; k+1]}{k!}h^{(k)}(0)x^k$$
$$+ (-1)^n x^{n-1}\int_0^\infty f_{n,n}\left(\frac{t}{x}\right)h^{(n)}(t)dt,$$

which is a generalization of the expansions given in [24, Chap. 6] *for $0 < \alpha < 1$. If $\alpha = 1$ ($\alpha_k - \beta_k = 1\ \forall\ k$), then from (19),*

$$\lim_{z\to 0}\left\{x^k\left[a_k x^{-z}M[h; z-k] + b_k M[f; z+k+1]\right]\right\}$$
$$= \lim_{z\to 0}\left\{x^k\left[\frac{a_k}{(-z)_{k+1}}\int_0^\infty t^z h^{(k+1)}(t)dt + \frac{M[f; z+k+1]}{k!}h^{(k)}(0)\right]\right\}.$$

It is obvious that the first term in the last line above has a pole at $z=0$*:*

$$\frac{t^z}{(-z)_{k+1}}=\frac{1}{k!}\left[\frac{1}{z}+\log t+\psi(k+1)+\gamma\right]+\mathcal{O}(z)\quad as\quad z\to 0.$$

Moreover, the residue of that term at $z=0$ *is* $a_k h^{(k)}(0)/k!$*:*

$$\begin{aligned}\frac{a_k}{(-z)_{k+1}}\int_0^\infty t^z h^{(k+1)}(t)dt =&\frac{a_k}{k!}\left\{\left[\frac{1}{z}+\psi(k+1)+\gamma\right]h^{(k)}(0)\right.\\ &\left.-\int_0^\infty h^{(k+1)}(t)\log t dt\right\}+\mathcal{O}(z)\quad \text{as}\quad z\to 0.\end{aligned}$$

But also, from the second line of (9)*, the Mellin transform* $M[f;z+k+1]$ *has a pole at* $z=0$ *with residue* a_k*. Using this last identity and taking the limit* $z\to 0$ *we obtain*

$$\begin{aligned}\int_0^\infty h(xt)f(t)dt =& -\sum_{k=0}^{n-1}\frac{a_k}{k!}x^k\int_0^\infty h^{(k+1)}(t)\log t dt\\ &+\sum_{k=0}^{n-1}\left[a_k(\psi(k+1)+\gamma-\log x)+\lim_{z\to k+1}\left(M[f;z]+\frac{a_k}{z-k-1}\right)\right]\frac{h^{(k)}(0)}{k!}x^k \qquad (35)\\ &+(-1)^n x^{n-1}\int_0^\infty f_{n,n}\left(\frac{t}{x}\right)h^{(n)}(t)dt,\end{aligned}$$

which is a generalization of the expansions given in [24, Chap. 6] *for* $\alpha=1$.

The method of regularization given in [24, Chap. 6, sec. 7]*, is also a corollary of Theorem* 1*. The expansions derived in* [23], [24, Chap. 6, sec. 7] *may be cast in the form* (18)–(19)*. The conditions required for* f *and* h *there are more stringent than those of Theorem* 1 *above: in that theory both* f *and* h *must belong to* $\mathcal{F}$ *and* $\mathcal{H}$ *simultaneously.*

Corollary 5 (AC techniques). *The expansions derived in* [17, Theorems 1 and 2] *by means of analytic continuation are nothing more than expansions* (34) *and* (35)*. The conditions required for* $f(t)$ *and* $h(t)$ *in* [17] *are more stringent than those of Theorem* 1 *above:* $f\in\mathcal{F}$, $h\in\mathcal{C}^{(\infty)}[0,\infty)\subset\mathcal{H}$*. Moreover, the expansions derived in* [22] *and* [16] *are just the expansion given in Theorem* 1 *for the particular case* $\alpha_k=k+\alpha$ *and* $\beta_k=k+\beta$.

6. Examples. Formula (18) can be applied straightforwardly to many examples of important integral transforms and special functions. Some examples previously considered by using the much more complicated distributional techniques, Mellin transforms, or AC methods are the Laplace, Stieltjes, and Lambert transforms for large or small argument, fractional integrals for large argument, the Poisson transform for large time, hypergeometric functions for large variables, elliptic integrals for large or small parameters, the Appell function F_1 or the Lauricella functions near the infinity, and thermonuclear reaction rates for small Sommerfeld parameter [3], [8], [9], [10], [11], [12], [14], [15], [24, Chap. 6]. The asymptotic expansions obtained there can be derived trivially as simple corollaries of Theorems 1 and 2. Other interesting examples not yet considered are the Appell function F_2 for large variables, the Appell functions F_1 and F_2 near their branch points, or the Hurwitz–Lerch zeta function near its singular point.

We give details here of the example of the exponential integral near the origin. The exponential integral $E_1(z)$ is defined in [1, eq. (5.1.1)]. After a straightforward change of the integration variable it reads

$$E_1(z) = e^{-z} \int_0^\infty f(t)h(xt)dt, \qquad \Re z > 0, \tag{36}$$

with $x = |z|$,

$$f(t) = \frac{1}{1+t}, \qquad h(t) = e^{-\gamma t},$$

and $\gamma \equiv e^{i\mathrm{Arg}(z)}$. We can apply Theorem 1 to the integral in the right-hand side of (36) with $\alpha_k = k+1$, $\beta_k = k$, $m = n$, $x = |z|$, $a_k = (-1)^k$, $b_k = (-\gamma)^k/k!$,

$$M[f;w] = \frac{\pi}{\sin(\pi w)} \qquad \text{and} \qquad M[h;w] = \frac{\Gamma(w)}{\gamma^w}.$$

Then (18) and (19) give the following asymptotic expansion for small z:

$$e^z E_1(z) = \sum_{k=0}^{n-1} \frac{z^k}{k!} \left[\psi(k+1) - \log z\right] + R_n(z), \tag{37}$$

with

$$R_n(z) = \int_0^\infty f_n(t)h_n(xt)dt.$$

Two consecutive derivatives of the function $f(t)$ have opposite sign for any $t > 0$. Using the Lagrange formula for the remainder of the Taylor expansion of $\tilde{f}(u) = f(u^{-1})$ at $u = 0$ we see that the expansion of the function $f(t)$ in inverse powers of t satisfies the error test (belongs to Family 3) and then $|f_n(t)| \le t^{-n-1}$ for $t > 0$. The function $f(t)$ also belongs to Family 1 and then, from (33), we can obtain another bound for $f_n(t)$, although less competitive: as a function of the complex variable u, the function $\tilde{f}(u)$ is analytic in the region U_{r_0} considered in Family 1 with $r_0 = 1$. We choose $r < r_0$ in (33) in such a way that the maximum M of $|\tilde{f}(u)|$ in U_{r_0} satisfies the condition that Mr^{-n} attains its smallest possible value: $M = n-1$, $r = (n-1)/n$, and $Mr^{-n} = n\left(\frac{n}{n-1}\right)^{n-1}$. Then, another bound (less competitive than the previous one) is $|f_n(t)| \le 2n\left(\frac{n}{n-1}\right)^{n-1} t^{-n-1}$ for $t > 0$.

From the Lagrange formula for the Taylor remainder $h_n(t)$ of the expansion of $h(t)$ at $t = 0$ we have that $h_n(t) = h^{(n)}(\xi)t^n/n!$, with $\xi \in (0,t)$. Then, for $\Re z > 0$ we have $|h_n(t)| \le t^n/n!$ for $t > 0$.

In this example $\beta_n = \alpha_n - 1$ and then in (31) and (32) we set $F_n = 1$ and $H_n = 1/n!$. Formula (31) reads

$$|R_n(z)| \le \frac{1}{(n-1)!}|z|^n \left[2 + \frac{1}{n}\left(2 + |\log z|\right)\right].$$

From this bound we see that expansion (37) is convergent $\forall\, z$ and then

$$E_1(z) = e^{-z} \sum_{k=0}^\infty \frac{z^k}{k!}\psi(k+1) - \log z,$$

to be compared with the expansion in [1, eq. (5.1.11)].

7. Concluding Remarks. When the positive moments of $f(t)$ or the negative moments of $h(t)$ exist, asymptotic expansions of integral transforms $\int_0^\infty f(t)h(xt)dt$ for small x may be obtained by means of classical techniques [24, Chaps. 1 and 2]. When neither the positive moments of $f(t)$ nor the negative moments of $h(t)$ exist, we may try other asymptotic methods: MT techniques [24, Chap. 3], M&W theory [24, Chaps. 5 and 6], or AC methods [22], [17], [16].

Theorem 1 above offers a new asymptotic method for $\int_0^\infty f(t)h(xt)dt$, which has the following features.

(i) It is very general. It is valid whether the positive moments of $f(t)$ or the negative moments of $h(t)$ exist or not and it contains some classical methods, MT techniques, M&W theory, and the AC method as straightforward corollaries.

(ii) It is trivial. It does not require the complexity of the theory of distributions or the transformation to the Mellin space used in MT techniques or M&W theory, nor the analytic continuation techniques used in the AC method. It just follows from an expansion of the integrand and an interchange of sum and integral (in an appropriate order).

(iii) It unifies somehow the asymptotic theory. We see that the asymptotic principle under Theorem 1 is quite pragmatic:

  (a) When the positive moments of $f(t)$ exist, expand $h(t)$ at $t = 0$ and interchange sum and integral.

  (b) When the negative moments of $h(t)$ exist, expand $f(t)$ at $t = \infty$ and interchange sum and integral.

  (c) When both the positive moments of $f(t)$ and the negative moments of $h(t)$ do not exist, interlace the expansions of $h(t)$ at $t = 0$ and of $f(t)$ at $t = \infty$ and interchange sum and integral in such a way that only finite moments appear.

Theorem 1 above shows that the asymptotic *principia* under classical methods I and II, M&W theory, MT techniques, and the AC method is indeed quite simple: expand the integrand and exchange sum and integral (in a proper way). Just one asymptotic sequence appears when classical methods apply because just one, f or h, needs to be expanded. On the other hand, two asymptotic sequences appear naturally in M&W theory and MT and AC techniques if both f and h need to be expanded in order to avoid the appearance of divergent coefficients. Theorem 1 combines all these situations in a single formulation. If the positive moments of f exist, then $a_k = 0$. If the negative moments of h exist, then $b_k = 0$. In any case, just one asymptotic sequence appears in (18). If none of these moments exist, then two asymptotic sequences appear in (18).

The presence of double poles in the integrand in (11) in MT theory corresponds with the case $\alpha_n = \beta_m + 1$ in our theory or in M&W's method or in the AC technique. This situation translates into the appearance of $\log x$ terms in the expansion (18) and the coincidence of exponents of x in the two asymptotic sequences of that formula.

MT techniques and M&W theory deal also with the possibility of $f(t)$ having an oscillatory asymptotic expansion at infinity of the form $f(t) = e^{ict}\sum_{k=0}^{n-1} a_k t^{-\alpha_k} + f_n(t)$ instead of (7) (and similarly for $h(t)$ at $t = 0^+$). Theorems 1 and 2 may be generalized to this kind of function. Moreover, they may be generalized to complex values of α_k, β_k, and x. This is the subject of further investigation.

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