Uniform Asymptotic Expansions of Symmetric Elliptic Integrals

José L. López[1]

[1] *Departamento de Matemática e Informática, Universidad Pública de Navarra, 31006-Pamplona, Spain. e-mail: jl.lopez@unavarra.es.*

ABSTRACT

Symmetric standard elliptic integrals are considered when two or more parameters are larger than the others. Distributional approach is used for deriving seven expansions of these integrals in inverse powers of the asymptotic parameters. Some of these expansions involve also logarithmic terms in the asymptotic variables. These expansions are uniformly convergent when the asymptotic parameters are greater than the remaining ones. The coefficients of six of these expansions involve hypergeometric functions with less parameters than the original integrals. The coefficients of the seventh expansion involve again elliptic integrals, but with less parameters than the original integrals. Convergence speed of any of these expansions increases for increasing difference between the asymptotic variables and the remaining ones. All the expansions are accompanied by an error bound at any order of the approximation.

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1. Introduction

Elliptic integrals (EI) are integrals of the type $\int R(x,y)dx$, where $R(x,y)$ is a rational function of x and y, with y^2 a polynomial of the third or fourth degree in x. When the polynomial y^2 has not a repeated factor and $R(x,y)$ contains some odd power of y, EI cannot, in general, be expressed in terms of elementary functions. Legendre showed that all EI can be expressed in terms of three standard EI (Legendre's normal EI) [13].

The three complete EI of the first, second and third kind are particularly important cases of the respective three standard EI. These integrals and the three standard EI are special non-elementary functions that play an important role in several mathematical and physical problems.

A survey of properties of the standard EI can be found, for example, in [[1], chap. 17], [2] or [[16], chap. 12]. However, as it has been shown by Carlson [5]-[9], for numerical computations it is more convenient to use symmetric standard EI instead of Legendre's normal EI. (Legendre's normal EI are connected with the symmetric standard EI by means of simple formulas [[16], eq. 12.33].) A very complete table of the three symmetric standard EI can be found in [5]-[9]. They are defined as follows,

$$R_F(x,y,z) = \frac{1}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}},$$

$$R_D(x,y,z) = \frac{3}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)^3}},$$

$$R_J(x,y,z,p) = \frac{3}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}(t+p)},$$

where we assume that the parameters x, y, z are nonnegative. We assume also that they are distinct (otherwise these integrals reduce to elementary functions). If the fourth argument of R_J is negative, the Cauchy principal value of R_J can be written in terms of R_F and R_J with all the arguments nonnegative [10]. Therefore, we will consider $p > 0$ and $p \neq x$, y, z (otherwise R_J reduces to R_D).

On the other hand, the asymptotic approximation of EI has not been exhaustively investigated: classical methods for approximation of integrals cannot be applied. Some result concerning approximations of EI can be found for example in [2] and [12]. Although the more recent results about the asymptotic behavior of these integrals have been obtained by Carlson, Gustafson and Wong: R_F, R_D and R_J may be written as a convolution and the method of regularization [[18], chap. 6, sec. 7] can be applied.

When one of the parameters of the integrals tends to zero or infinity, the first (and sometimes the second too) term of the asymptotic expansion of R_F, R_D and R_J, as well as a quite accurate bound for the first error term has been obtained by Gustafson [11]. Higher terms of the expansion and higher error bounds are not explicitly derived in that work because of the complexity of the Mellin transforms involved in their calculation. Using a very clever analytical trick [10], Carlson and Gustafson have sharpened the bounds for the first error terms obtained in [11] in the case of one parameter going to infinity. Besides, they supply in [10] very accurate bounds for the first error term of the totally symmetric elliptic integral of the second kind. Moreover, for all the symmetric EI, they consider also the case of several parameters going to infinity.

Complete convergent expansions of R_F, R_D and R_J (and not only first terms) have been obtained by Carlson using also Mellin transforms techniques [3]. Although these expansions have an attractively simple structure, explicit computation of the terms of the expansions is not straightforward and the upper bound on the truncation error is not quite satisfactory [[3], sec. 5]. Carlson and Gustafson have solved this problem for $R_F(x,y,z)$ in [4], where an algorithm for computing the coefficients of the convergent expansion of $R_F(x,y,z)$ for large z in terms of Legendre functions and their derivatives is derived. Moreover, accurate error bounds are given too at any order of the approximation.

This problem has also been solved for $R_D(x,y,z)$ and $R_J(x,y,z,p)$ when only one of the parameters is large [14].

In this paper we try to solve the problem for the remaining cases not considered in [4], [14]. That is, we consider complete convergent expansions for R_F, R_D and R_J when two or more of their parameters x, y, z or p is large. Then, we face the challenge of obtaining easy algorithms for computing the coefficients of these expansions and simple expressions for the error bounds at any order of the approximation.

We use here the same principles used in [14]: distributional approach for obtaining the expansions and error test for finding error bounds. In section 2, we make a review of the asymptotic expansions of Stieltjes [[18], chap. 6, sec. 2] and generalized Stieltjes transforms (see [[17], theorem 2 and example 1]): distributional approach is used in lemmas 1-4 and theorems 1-4 for deriving complete expansions of a certain family of integrals which contains R_F, R_D and R_J. On the other hand, using lemmas 5 and 6, we obtain simple expressions for the error bounds in the expansions of this family of integrals in propositions 1-4. In section 3 we apply the results of section 2 for deriving complete convergent expansions of $R_F(x,az,bz)$, $R_D(x,az,bz)$, $R_D(az,bz,x)$, $R_J(x,az,bz,p)$, $R_J(x,y,az,bz)$, $R_J(az,bz,cz,x)$ and $R_J(x,az,bz,cz)$ for large z. They are presented in corollaries 1-7 accompanied by error bounds at any order of the approximation. Numerical examples are shown as an illustration. A brief summary and a few comments are postponed to section 4.

2. Distributional approach

The procedure for deriving convergent expansions of the integrals R_F, R_D and R_J are based on the distributional approach. It requires the concepts of rapidly decreasing functions and tempered distributions.

Definition 1. *We denote by $\mathcal{S}$ the space of rapidly decreasing functions (infinitely differentiable functions $\varphi(t)$ defined on $[0,\infty)$ that, together with their derivatives, approach zero more rapidly than any power of t^{-1} as $t\to\infty$).*

Definition 2. *We denote by $<\Lambda,\varphi>$ the image of a tempered distribution Λ (a continuous linear functional defined over $\mathcal{S}$) acting over a function $\varphi\in\mathcal{S}$. Recall that we can associate to any locally integrable function $g(t)$ on $[0,\infty)$ a tempered distribution*

Λ_g defined by

$$< \Lambda_g, \varphi >= \int_0^\infty g(t)\varphi(t)dt.$$

Definition 3. *For a locally integrable function $f(t)$ on $(0,\infty)$, we denote by $M[f;w]$ the Mellin transform of $f(t)$ or its analytic continuation. It is defined by*

$$M[f;w] = \int_0^\infty t^{w-1} f(t)dt \tag{1}$$

when the integral converges.

Convergent expansions of $R_F(x,az,bz)$, $R_D(x,az,bz)$, $R_D(az,bz,x)$, $R_J(x,az,bz,cz)$, $R_J(x,az,bz,p)$, $R_J(az,bz,cz,p)$ and $R_J(x,y,az,bz)$ for large positive z and uniformly valid for positive a, b and c can be derived from [[17], theorem 2] (see also example 1 there) and [[18], chap. 6, sec. 2]. The results obtained in [17] have been proved by using Mellin transform techniques, whereas the results of [[18], chap. 6, sec. 2] have been obtained by using the distributional approach. But, as it is suggested by Wong [[17], example 1], all the expansions derived in [17] can also be obtained by means of the distributional approach. We carry out Wong's proposal in the following four lemmas and theorems. The first two lemmas are proved in [[18], chap. 6, lemmas 1 and 2].

Lemma 1. *Let $f(t)$ a locally integrable function on $[0,\infty)$, $\{a_k\}$ a sequence of complex numbers and let $f(t)$ satisfy, for $n = 1, 2, 3, ...,$*

$$f(t) = \sum_{k=0}^{n-1} \frac{a_k}{t^{k+\alpha}} + f_n(t),$$

where $f_n(t) = \mathcal{O}(t^{-n-\alpha})$ as $t \to \infty$ and $0 < \alpha < 1$. Define

$$f_{n,n}(t) = \frac{(-1)^n}{(n-1)!} \int_t^\infty (u-t)^{n-1} f_n(u)du. \tag{2}$$

Then, for any integer $n \geq 1$ and for any function $\varphi \in \mathcal{S}$ we have

$$< f, \varphi >= \sum_{k=0}^{n-1} \frac{a_k}{(\alpha)_k} < t^{-\alpha}, \varphi^{(k)} > + \sum_{k=0}^{n-1} \frac{M[f;k+1]}{k!} < \delta, \varphi^{(k)} > +(-1)^n < f_{n,n}, \varphi^{(n)} >,$$

where f, $f_{n,n}$ and $t^{-\alpha}$ denote the tempered distributions associated to the locally integrable functions $f(t)$, $f_{n,n}(t)$ and $t^{-\alpha}$ respectively and δ is the delta distribution in the origin.

Lemma 2. *Let $f(t)$ as in lemma 1 but with $\alpha = 1$. Then, for any integer $n \geq 1$ and for any function $\varphi \in \mathcal{S}$ we have*

$$< f, \varphi >= -\sum_{k=0}^{n-1} \frac{a_k}{k!} < \log(t), \varphi^{(k+1)} > + \sum_{k=0}^{n-1} \frac{b_k}{k!} < \delta, \varphi^{(k)} > +(-1)^n < f_{n,n}, \varphi^{(n)} >,$$

where f, $f_{n,n}$ *and* $\log(t)$ *denote the tempered distributions associated to the locally integrable functions* $f(t)$, $f_{n,n}(t)$ *and* $\log(t)$ *respectively,* δ *is the delta distribution in the origin and*

$$\begin{aligned} b_k =& a_k \sum_{j=1}^{k} \frac{1}{j} + \lim_{w \to k+1} \left\{ M[f; w] + \frac{a_k}{w-k-1} \right\} \\ =& a_k \sum_{j=1}^{k} \frac{1}{j} + \int_0^1 t^k f_k(t) dt + \int_1^\infty t^k f_{k+1}(t) dt, \end{aligned} \tag{3}$$

empty sums being understood as zero.

Lemma 3. *Let* $f(t)$ *as in lemma 1 with* $0 < \alpha \leq 1$. *Define, for* $t \in [0, \infty)$, $z > 0$, $\eta > 0$ *and* $\alpha + \rho > 1$

$$\varphi_\eta(t) = \frac{e^{-\eta t}}{(t+z)^\rho} \in \mathcal{S}.$$

Then, for $k = 0, 1, 2, ...$ *and* $n = 1, 2, 3, ...,$ *the following identities hold,*

$$\lim_{\eta \to 0} < f, \varphi_\eta >= \int_0^\infty \frac{f(t)}{(t+z)^\rho} dt,$$

$$\lim_{\eta \to 0} < \delta, \varphi_\eta^{(k)} >= \frac{(-1)^k (\rho)_k}{z^{k+\rho}},$$

where $(\rho)_k$ *denotes the Pochhammer's symbol,*

$$\lim_{\eta \to 0} < t^{-\alpha}, \varphi_\eta^{(k)} >= \frac{(-1)^k \Gamma(k+\rho+\alpha-1)\Gamma(1-\alpha)}{\Gamma(\rho) z^{k+\rho+\alpha-1}}, \qquad \text{for } \ 0 < \alpha < 1,$$

$$\lim_{\eta \to 0} < \log(t), \varphi_\eta^{(k+1)} >= \frac{(-1)^{k+1}}{z^{k+\rho}} (\rho)_k \big(\log(z) - \gamma - \psi(k+\rho)\big),$$

where γ *is the Euler constant and* ψ *the digamma function and*

$$\lim_{\eta \to 0} < f_{n,n}, \varphi_\eta^{(n)} >= (-1)^n (\rho)_n \int_0^\infty \frac{f_{n,n}(t)}{(t+z)^{n+\rho}} dt.$$

Proof. The first identity is trivial by using the dominated convergence theorem. The second one follows after a simply computation. On the other hand,

$$< t^{-\alpha}, \varphi_\eta^{(k)} >= \frac{(-1)^k}{(\alpha)_k} \sum_{j=0}^{k} \binom{k}{j} \eta^j (\rho)_{k-j} \int_0^\infty \frac{e^{-\eta t}}{t^\alpha (t+z)^{k+\rho-j}} dt.$$

For $0 < \alpha < 1$, the integrand of each integral in the right hand side of the above equation is absolutely dominated by the integrable function $t^{-\alpha}(t+z)^{j-k-\rho}$ $\forall\ \eta, t \geq 0$ and then, finite. Therefore, using the dominated convergence theorem and after straight forward operations we obtain the third identity. The remaining identities may be proved in a similar way (see [[14], lemma 2]). □

Lemma 4. *Let $f(t)$ as in lemma 1 with $0 < \alpha \leq 1$. Define, for $t \in [0, \infty)$, $z > 0$, $\eta, \rho, \sigma, a, b > 0$ and $\alpha + \rho + \sigma > 1$*

$$\varphi_\eta(t) = \frac{e^{-\eta t}}{(t+az)^\rho (t+bz)^\sigma} \in \mathcal{S}.$$

Then, for $k = 0, 1, 2, ...$ and $n = 1, 2, 3, ...$, the following identities hold,

$$\lim_{\eta \to 0} < f, \varphi_\eta >= \int_0^\infty \frac{f(t)}{(t+az)^\rho (t+bz)^\sigma} dt,$$

$$\lim_{\eta \to 0} < \delta, \varphi_\eta^{(k)} >= \frac{(-1)^k}{z^{k+\rho+\sigma}} \sum_{j=0}^{k} \binom{k}{j} \frac{(\rho)_j (\sigma)_{k-j}}{a^{\rho+j} b^{\sigma+k-j}},$$

$$\lim_{\eta \to 0} < t^{-\alpha}, \varphi_\eta^{(k)} >= \frac{(-1)^k \Gamma(1-\alpha)\Gamma(k+\rho+\sigma+\alpha-1)}{\Gamma(k+\rho+\sigma) z^{k+\rho+\sigma+\alpha-1}} \times$$
$$\sum_{j=0}^{k} \binom{k}{j} \frac{(\rho)_j (\sigma)_{k-j}}{a^{\rho+j+\alpha-1} b^{\sigma+k-j}} F\left(\begin{matrix} 1-\alpha, k+\sigma-j \\ k+\rho+\sigma \end{matrix} \middle| 1 - \frac{a}{b} \right), \qquad \text{for } 0 < \alpha < 1,$$

where $F\left(\begin{matrix} \alpha, \beta \\ \delta \end{matrix} \middle| z \right)$ is the Gauss hypergeometric function,

$$\lim_{\eta \to 0} < \log(t), \varphi_\eta^{(k+1)} >= \frac{(-1)^{k+1}}{(k+\rho+\sigma) z^{k+\rho+\sigma}} \sum_{j=0}^{k+1} \binom{k+1}{j} \frac{(\rho)_j (\sigma)_{k+1-j}}{a^{\rho+j-1} b^{\sigma+k+1-j}} \times$$
$$\left[(\log(az) - \gamma - \psi(k+\rho+\sigma)) F\left(\begin{matrix} 1, k+1+\sigma-j \\ k+1+\rho+\sigma \end{matrix} \middle| 1 - \frac{a}{b} \right) + \right.$$
$$\left. F'\left(\begin{matrix} 1, k+1+\sigma-j \\ k+1+\rho+\sigma \end{matrix} \middle| 1 - \frac{a}{b} \right) \right],$$

where $F'\left(\begin{matrix} \alpha, \beta \\ \delta \end{matrix} \middle| z \right)$ is the derivative of the Gauss hypergeometric function with respect to the parameter α and

$$\lim_{\eta \to 0} < f_{n,n}, \varphi_\eta^{(n)} >= (-1)^n \sum_{j=0}^{n} \binom{n}{j} (\rho)_j (\sigma)_{n-j} \int_0^\infty \frac{f_{n,n}(t)}{(t+az)^{j+\rho} (t+bz)^{n-j+\sigma}} dt.$$

Proof. The proof of the first, second and last equalities is similar to the proof of the corresponding equalities in lemma 3. The proof of the third equality is also similar, but considering the integrable function $t^{-\alpha}(t+az)^{-i-\rho}(t+bz)^{i-j-\sigma}$ with $i \leq j = 0, 1, 2, ...k$ instead of $t^{-\alpha}(t+z)^{j-k-\rho}$ and using formula [[15], p. 303, eq. 24]. The proof of the fourth equality is similar to the proof of the fourth equality in lemma 3 (third equality in [[14], lemma 2]) using the bound $(t+az)^{\rho}(t+bz)^{\sigma} \geq (t+cz)^{\rho+\sigma}$, where $c =\min\{a,b\}$. But it uses the derivative with respect to α of formula [[15], p. 303, eq. 24] instead of [[15], p. 489, eq. 7], which is used in [[14], lemma 2]. □

Theorem 1. *Let $f(t)$ a locally integrable function on $[0,\infty)$, $\{a_k\}$ a sequence of complex numbers and let $f(t)$ satisfy, for $n = 1,2,3,...,$*

$$f(t) = \sum_{k=0}^{n-1} \frac{a_k}{t^{k+\alpha}} + f_n(t), \tag{4}$$

where $f_n(t) = \mathcal{O}(t^{-n-\alpha})$ as $t \to \infty$ and $0 < \alpha < 1$. Then, for $z > 0$, $\alpha + \rho > 1$ and $n = 1,2,3,...,$

$$\begin{aligned}\int_0^{\infty} \frac{f(t)}{(t+z)^{\rho}} dt = \sum_{k=0}^{n-1} \frac{(-1)^k}{z^{k+\rho}} \Bigg[& \frac{\pi\Gamma(k+\rho+\alpha-1)a_k z^{1-\alpha}}{\Gamma(k+\alpha)\Gamma(\rho)\sin(\pi\alpha)} + \\ & \frac{(\rho)_k M[f;k+1]}{k!}\Bigg] + R_n(\rho;z),\end{aligned} \tag{5}$$

where the remainder term satisfies

$$R_n(\rho;z) = (\rho)_n \int_0^{\infty} \frac{f_{n,n}(t)dt}{(t+z)^{n+\rho}} \tag{6}$$

and $f_{n,n}(t)$ is defined in (2).

Proof. It follows from lemmas 1 and 3 using the reflection formula of the gamma function. □

Theorem 2. *Let $f(t)$ a locally integrable function on $[0,\infty)$, $\{a_k\}$ a sequence of complex numbers and let $f(t)$ have the following asymptotic expansion for large t and $n = 1,2,3,...,$*

$$f(t) = \sum_{k=0}^{n-1} \frac{a_k}{t^{k+1}} + f_n(t), \tag{7}$$

where $f_n(t) = \mathcal{O}(t^{-n-1})$ as $t \to \infty$. Then, for $z,\rho > 0$ and $n = 1,2,3,...,$

$$\int_0^{\infty} \frac{f(t)}{(t+z)^{\rho}} dt = \sum_{k=0}^{n-1} \frac{(-1)^k}{k! z^{k+\rho}} (\rho)_k \left[a_k\big(\log(z) - \gamma - \psi(k+\rho)\big) + b_k\right] + R_n(\rho;z), \tag{8}$$

where, for $k = 0,1,2,...,$ the coefficients b_k are given by

$$\begin{aligned} b_k =& a_k \sum_{j=1}^{k} \frac{1}{j} + \lim_{w \to k+1} \left\{ M[f;w] + \frac{a_k}{w-k-1} \right\} \\ =& a_k \sum_{j=1}^{k} \frac{1}{j} + \lim_{T \to \infty} \left\{ \int_0^T t^k f(t) dt - \sum_{j=0}^{k-1} a_j \frac{T^{k-j}}{k-j} - a_k \log T \right\}, \end{aligned} \tag{9}$$

empty sums being understood as zero. The remainder term is given by (6).

Proof. From lemmas 2 and 3 we obtain immediately formulas (6), (8) and the first line in (9). Introducing

$$f_k(t) = f(t) - \sum_{j=0}^{k-1} \frac{a_j}{t^{j+1}}$$

in the second line of (3) and after simple manipulations we obtain the second line in (9). □

Theorem 3. *Let $f(t)$ as in theorem 1. Then, for $a, b, \rho, \sigma, z > 0$, $\alpha + \rho + \sigma > 1$ and $n = 1, 2, 3, ...,$*

$$\begin{aligned} \int_0^\infty \frac{f(t)}{(t+az)^\rho (t+bz)^\sigma} dt = & \frac{\pi}{\sin(\alpha\pi)} \sum_{k=0}^{n-1} \frac{(-1)^k A_k \Gamma(k+\rho+\sigma+\alpha-1)}{\Gamma(k+\alpha)\Gamma(k+\rho+\sigma) z^{k+\alpha+\rho+\sigma-1}} + \\ & \sum_{k=0}^{n-1} \frac{(-1)^k B_k}{k!} \frac{M[f;k+1]}{z^{k+\rho+\sigma}} + R_n(\rho,\sigma;z), \end{aligned} \tag{10}$$

where the coefficients A_k and B_k are defined by

$$A_k \equiv a_k \sum_{j=0}^{k} \binom{k}{j} \frac{(\rho)_j (\sigma)_{k-j}}{a^{\rho+\alpha+j-1} b^{k+\sigma-j}} F\left(\begin{matrix} 1-\alpha, k+\sigma-j \\ k+\rho+\sigma \end{matrix} \middle| 1 - \frac{a}{b} \right), \tag{11}$$

$$B_k \equiv \sum_{j=0}^{k} \binom{k}{j} \frac{(\rho)_j (\sigma)_{k-j}}{a^{\rho+j} b^{k+\sigma-j}} \tag{12}$$

and the remainder term satisfies

$$R_n(\rho,\sigma;z) = \sum_{j=0}^{n} \binom{n}{j} (\rho)_j (\sigma)_{n-j} \int_0^\infty \frac{f_{n,n}(t) dt}{(t+az)^{j+\rho} (t+bz)^{n+\sigma-j}}, \tag{13}$$

where $f_{n,n}(t)$ is defined in (2).

Proof. It follows from lemmas 1 and 4 after straightforward computations and using formula [[15], p. 303, eq. 24]. □

Theorem 4. *Let $f(t)$ as in theorem 2. Then, for $a, b, z, \rho, \sigma > 0$ and $n = 1, 2, 3, ...,$*

$$\int_0^\infty \frac{f(t)}{(t+az)^\rho (t+bz)^\sigma} dt = \sum_{k=0}^{n-1} \frac{(-1)^k}{k! z^{k+\rho+\sigma}} \left[(A_k \log(az) + A'_k) + B_k \right] + R_n(\rho,\sigma;z), \tag{14}$$

where

$$B_k = b_k \sum_{j=0}^{k} \binom{k}{j} \frac{(\rho)_j (\sigma)_{k-j}}{a^{\rho+j} b^{k+\sigma-j}},$$

the coefficients b_k being defined in (9), and where, for $k = 0, 1, 2, ...$, the coefficients A_k and A'_k are defined by

$$A_k \equiv a_k \sum_{j=0}^{k+1} \binom{k+1}{j} \frac{(\rho)_j (\sigma)_{k+1-j}}{(k+\rho+\sigma) a^{\rho+j-1} b^{k-j+\sigma+1}} F\left(\begin{array}{c} 1, k+1+\sigma-j \\ k+\rho+\sigma+1 \end{array} \middle| 1 - \frac{a}{b} \right),$$

$$\begin{aligned} A'_k \equiv & a_k \sum_{j=0}^{k+1} \binom{k+1}{j} \frac{(\rho)_j (\sigma)_{k+1-j}}{(k+\rho+\sigma) a^{\rho+j-1} b^{k-j+\sigma+1}} \left[F'\left(\begin{array}{c} 1, k+1+\sigma-j \\ k+\rho+\sigma+1 \end{array} \middle| 1 - \frac{a}{b} \right) - \right. \\ & \left. (\psi(k+\rho+\sigma)+\gamma) F\left(\begin{array}{c} 1, k+1+\sigma-j \\ k+\rho+\sigma+1 \end{array} \middle| 1 - \frac{a}{b} \right) \right]. \end{aligned}$$

The remainder term is given by (13) in theorem 3.

Proof. The proof is similar to the proof of theorem 2, but using lemma 4 instead of lemma 3. □

A bound for the error term in the expansions given in the above theorems will be obtained in the following propositions when the function $f(t)$ has the form

$$f(t) = \prod_{k=1}^{m} \frac{1}{(t+x_k)^{\mu_k}}, \tag{15}$$

where $m \in \mathbb{N}$, $x_1,...,x_m$ are nonnegative parameters at least one different from zero and $\mu_1,...,\mu_m > 0$. Define

$$\mu = \sum_{k=1}^{m} \mu_k > 0.$$

For $\mu \notin \mathbb{N}$, the asymptotic expansion of $f(t)$ in $t = \infty$ is given, for $n = 1, 2, 3, ...$, by (4) with $\alpha \equiv \mu - \lfloor \mu \rfloor$,

$$f(t) = \sum_{k=0}^{n-1} \frac{a_k}{t^{k+\mu-\lfloor \mu \rfloor}} + f_n(t), \tag{16}$$

where

$$a_0 = a_1 = ... = a_{\lfloor \mu \rfloor - 1} = 0 \qquad \text{if} \ \ \lfloor \mu \rfloor \geq 1,$$

$$a_{k+\lfloor \mu \rfloor} = \lim_{u \to 0} \frac{1}{k!} \frac{d^k}{du^k} \left(u^{-\mu} f(u^{-1}) \right) \qquad \text{for} \ \ k = 0, 1, 2, ... \tag{17}$$

and $f_n(t) = \mathcal{O}(t^{-n-\mu+\lfloor \mu \rfloor})$ as $t \to \infty$. Then , we have the following lemma, proved in [[14], lemma 3].

Lemma 5. *For $\mu \notin \mathbb{N}$ and $\forall\ t \in [0, \infty)$, the remainder term $f_n(t)$ and the coefficients a_n in the expansion (16)-(17) of the function $f(t)$ defined in (15) verify*

$$|f_n(t)| \leq \frac{|a_n|}{t^{n+\mu-\lfloor \mu \rfloor}} \quad \text{for} \ \ n \geq \lfloor \mu \rfloor, \qquad |f_n(t)| \leq \frac{|a_{n-1}|}{t^{n+\mu-\lfloor \mu \rfloor - 1}} \quad \text{for} \ \ n \geq \lfloor \mu \rfloor + 1 \tag{18}$$

and sign($f_n(t)$) =sign(a_n) =sign($(-1)^{n-\lfloor\mu\rfloor}$) for $n \geq \lfloor\mu\rfloor$.

On the other hand, for $\mu \in \mathbb{N}$, the asymptotic expansion in $t = \infty$ of the function $f(t)$ defined in (15) is given, for $n = 1, 2, 3, ...$, by (4) with $\alpha \equiv 1$,

$$f(t) = \sum_{k=0}^{n-1} \frac{a_k}{t^{k+1}} + f_n(t), \tag{19}$$

where

$$a_0 = a_1 = ... = a_{\mu-2} = 0 \qquad \text{if } \ \mu \geq 2,$$

$$a_{k+\mu-1} = \lim_{u\to 0} \frac{1}{k!} \frac{d^k}{du^k} \left(u^{-\mu} f(u^{-1})\right) \qquad \text{for } \ k = 0, 1, 2, ... \tag{20}$$

and $f_n(t) = \mathcal{O}(t^{-n-1})$ as $t \to \infty$. Then, we have the following lemma, proved in [[14], lemma 4].

Lemma 6. *For $\mu \in \mathbb{N}$ and $\forall\ t \in [0, \infty)$, the remainder term $f_n(t)$ and the coefficients a_n in the expansion (19)-(20) of the function $f(t)$ defined in (15) verify*

$$|f_n(t)| \leq \frac{|a_n|}{t^{n+1}} \qquad \text{for } \ n \geq \mu - 1, \qquad |f_n(t)| \leq \frac{|a_{n-1}|}{t^n} \qquad \text{for } \ n \geq \mu \tag{21}$$

and sign($f_n(t)$) =sign(a_n) =sign($(-1)^{n-\mu+1}$) for $n \geq \mu - 1$.

Proposition 1. *If the function $f(t)$ of theorem 1 has the form (15) with $\mu \notin \mathbb{N}$ then, $\forall\ z > 0$ and $n \geq \lfloor\mu\rfloor$, the error term $R_n(\rho; z)$ in the expansion (5) (which holds for $\alpha \equiv \mu - \lfloor\mu\rfloor$) satisfies*

$$0 \leq (-1)^{\lfloor\mu\rfloor} R_n(\rho; z) \leq \frac{\pi |a_n| \Gamma(n + \rho + \mu - \lfloor\mu\rfloor - 1)}{\Gamma(\rho)\Gamma(n + \mu - \lfloor\mu\rfloor)|\sin(\pi\mu)| z^{n+\rho+\mu-\lfloor\mu\rfloor-1}}, \tag{22}$$

providing the expansion (5) of an asymptotic character for large z.

Proof. The parameter α in theorem 1 equals $\mu - \lfloor\mu\rfloor$ in lemma 5. Using sign($f_n(u)$) =sign($(-1)^{n-\lfloor\mu\rfloor}$) $\forall u \in [0, \infty)$ in (2) and (6) we obtain $(-1)^{\lfloor\mu\rfloor} R_n(z) \geq 0$. Introducing the first bound of (18) in the right hand side of (2) and performing the change of variable $u \to t/u$ we obtain

$$|f_{n,n}(t)| \leq \frac{\Gamma(\mu - \lfloor\mu\rfloor)}{\Gamma(n + \mu - \lfloor\mu\rfloor)} \frac{|a_n|}{t^{\mu-\lfloor\mu\rfloor}} \qquad \forall \quad t \in [0, \infty). \tag{23}$$

Introducing this bound in (6) and after the change of variable $t \to z(t^{-1} - 1)$ we obtain (22). $\square$

Proposition 2. *If the function $f(t)$ of theorem 2 has the form (15) with $\mu \in \mathbb{N}$ then, $\forall$ $z > 0$ and $n \geq \mu$, the error term $R_n(\rho; z)$ in the expansion (8) satisfies the bounds*

$$0 \leq -(-1)^{\mu} R_n(\rho; z) \leq \frac{\pi \Gamma(n + \rho - 1/2)}{\Gamma(\rho)\Gamma(n + 1/2)} \frac{\bar{a}_n}{z^{n+\rho-1/2}}, \tag{24}$$

where $\bar{a}_n$ =*max*$\{|a_n|, |a_{n-1}|\}$*, and*

$$|R_n(\rho;z)| \leq \left[n\epsilon|a_{n-1}| + |a_n|\left(S_n(z,\epsilon,\rho) + T_n(z,\epsilon,\rho)\right)\right] \frac{(\rho)_n}{n! z^{n+\rho}}, \tag{25}$$

where ϵ *is an arbitrary positive number,*

$$S_n(z,\epsilon,\rho) = \min\left\{\frac{nz\left[(\epsilon+z)^{n+\rho-1} - z^{n+\rho-1}\right]}{\epsilon(n+\rho-1)(\epsilon+z)^{n+\rho-1}}, \psi(n+1) + \gamma\right\} \tag{26}$$

and

$$\begin{aligned} T_n(z,\epsilon,\rho) &= \frac{z^{n+\rho}}{(n+\rho)(\epsilon+z)^{n+\rho}} F\left(\begin{array}{c} n+\rho, 1 \\ n+\rho+1 \end{array} \middle| \frac{z}{\epsilon+z}\right) \\ &\leq \left(\frac{z}{\epsilon+z}\right)^{\rho} \left(\log\left(1+\frac{z}{\epsilon}\right) - \sum_{k=1}^{n-1} \frac{z^k}{k(z+\epsilon)^k}\right). \end{aligned} \tag{27}$$

For large z *and fixed* n*, the optimum value for* ϵ *is given by*

$$\epsilon = \frac{|a_n|}{n|a_{n-1}|}. \tag{28}$$

Any of these bounds provide the expansion (8) of an asymptotic character for large z*.*

Proof. Similar to the proof of proposition 1, but using lemma 6 instead of lemma 5 (see [[14], proposition 2]). □

Proposition 3. *If the function* $f(t)$ *of theorem 3 has the form (15) with* $\mu \notin \mathbb{N}$ *then,* $\forall\ z > 0$ *and* $n \geq \lfloor\mu\rfloor$*, the error term* $R_n(\rho,\sigma;z)$ *in the expansion (10) (which holds for* $\alpha \equiv \mu - \lfloor\mu\rfloor$*) satisfies*

$$\begin{aligned} 0 \leq (-1)^{\lfloor\mu\rfloor} R_n(\rho,\sigma;z) &\leq \frac{\pi|a_n|\Gamma(n+\rho+\sigma+\mu-\lfloor\mu\rfloor-1)}{\Gamma(n+\rho+\sigma)\Gamma(n+\mu-\lfloor\mu\rfloor)|\sin(\pi\mu)|z^{n+\rho+\sigma+\mu-\lfloor\mu\rfloor-1}} \times \\ &\sum_{j=0}^{n} \binom{n}{j} \frac{(\rho)_j(\sigma)_{n-j}}{a^{\rho+j+\mu-\lfloor\mu\rfloor-1} b^{n+\sigma-j}} F\left(\begin{array}{c} \lfloor\mu\rfloor-\mu+1, n+\sigma-j \\ \rho+\sigma+n \end{array} \middle| 1-\frac{a}{b}\right) \leq \\ &\frac{\pi|a_n|\Gamma(n+\rho+\sigma+\mu-\lfloor\mu\rfloor-1)}{\Gamma(n+\rho+\sigma)\Gamma(n+\mu-\lfloor\mu\rfloor)|\sin(\pi\mu)|} \frac{(\rho+\sigma)_n}{(cz)^{n+\rho+\sigma+\mu-\lfloor\mu\rfloor-1}}, \end{aligned} \tag{29}$$

where c =*min*$\{a,b\}$*, providing the expansion (10) of an asymptotic character for large* z*.*

Proof. The proof of the first inequality is the same as the proof of proposition 1: introduce (23) in (13) and use [[15], p. 303, eq. 24]. On the other hand, if instead of using [[15], p. 303, eq. 24], we bound the integrand in (13) by using $t+a \geq t+c$ and $t+b \geq t+c$ and use the equality

$$\sum_{k=0}^{n} \binom{n}{k} (\rho)_k(\sigma)_{n-k} = (\rho+\sigma)_n, \tag{30}$$

we obtain the second inequality in (29). □

Proposition 4. *If the function $f(t)$ of theorem 4 has the form (15) with $\mu\in\mathbb{N}$ then, $\forall$ $z>0$ and $n\ge\mu$, the error term $R_n(\rho,\sigma;z)$ in the expansion (14) satisfies the bound*

$$0\le -(-1)^\mu R_n(\rho,\sigma;z)\le\frac{\pi|\bar a_n|\Gamma(n+\rho+\sigma-1/2)}{\Gamma(n+\rho+\sigma)\Gamma(n+1/2)z^{n+\rho+\sigma-1/2}}\times$$
$$\sum_{j=0}^{n}\binom{n}{j}\frac{(\rho)_j(\sigma)_{n-j}}{a^{\rho+j-1/2}b^{n+\sigma-j}}F\left(\begin{array}{c}\frac{1}{2},n+\sigma-j\\ \rho+\sigma+n\end{array}\middle|\,1-\frac{a}{b}\right)\le \tag{31}$$
$$\frac{\pi|\bar a_n|\Gamma(n+\rho+\sigma-1/2)}{\Gamma(n+\rho+\sigma)\Gamma(n+1/2)}\frac{(\rho+\sigma)_n}{(cz)^{n+\rho+\sigma-1/2}},$$

where $\bar a_n$ =max$\{|a_n|,|a_{n-1}|\}$ and c =min$\{a,b\}$. $R_n(\rho,\sigma;z)$ satisfies also the bound

$$|R_n(\rho,\sigma;z)|\le[n\epsilon|a_{n-1}|+|a_n|(S_n(cz,\epsilon,\rho+\sigma)+T_n(cz,\epsilon,\rho+\sigma))]\frac{(\rho+\sigma)_n}{n!(cz)^{n+\rho+\sigma}} \tag{32}$$

where S_n and T_n are given in (26) and (27) respectively and ϵ is an arbitrary positive number. For large z and fixed n, the optimum value for ϵ is given by (28).

Any of these bounds provide the expansion (14) of an asymptotic character for large z.

Proof. The proof of the first two inequalities in (31) is the same as the proof of the inequalities (24) in proposition 2, but using (13) instead of (6). On the other hand, using $|f_n(t)|\le\bar a_n t^{-n-1/2}$ (deduced in the proof of proposition 2 in [14]), and repeating then the proof for the second bound in proposition 3 with $\mu=3/2$, we obtain the second inequality in (31). For deriving the bound (32) we use in (13) $(t+az)^{j+\rho}(t+bz)^{n+\sigma-j}\ge(t+cz)^{n+\sigma+\rho}$ for $j=0,1,2,...n$ and $\forall$ $t\ge 0$. Then, we obtain

$$|R_n(\rho,\sigma;z)|\le\sum_{j=0}^{n}\binom{n}{j}(\rho)_j(\sigma)_{n-j}R_n(\rho+\sigma;cz),$$

where $R_n(\rho;z)$ has been defined in (6). Using (30) and (25)-(27) we obtain (32). □

3. Uniform expansions of the symmetric standard elliptic integrals

Convergent expansions of R_F, R_D and R_J for large values of two or three of their parameters may be obtained as corollaries of theorems 1-4. Error bounds for the remainder terms in these expansions follow from propositions 1-4. We derive the explicit expansions and error bounds for the remainders in the following corollaries.

Corollary 1. *A uniformly convergent expansion of $R_F(x,az,bz)$ for $0\le x\le az\le bz$ and $0<az$, is given, for $n=1,2,3,...,$ by*

$$R_F(x,az,bz)=\frac{1}{2}\sqrt{\frac{\pi}{abx}}\sum_{k=0}^{n-1}\left[\frac{(k-1)!A_k^F(a,b)x^k}{\Gamma(k+1/2)z^k}+\right.$$
$$\left.\frac{(1/2)_k\sqrt{\pi b}x^{k+1/2}}{k!a^kz^{k+1/2}}F\left(\begin{array}{c}k+1/2,1/2\\ 1\end{array}\middle|\,1-\frac{b}{a}\right)\right]+R_n^F(x,az,bz), \tag{33}$$

where $A_0^F(a,b)=0$ *and, for* $k=1,2,3,...,$

$$A_k^F(a,b)=-\sum_{j=0}^{k-1}\frac{(1/2)_j(1/2)_{k-1-j}}{j!(k-1-j)!a^jb^{k-1-j}}. \qquad (34)$$

For $n=1,2,3,...,$ *the remainder term* $R_n^F(x,az,bz)$ *verifies*

$$0\le -R_n^F(x,az,bz)\le\frac{\sqrt{\pi}(n-1)!|A_n^F(a,b)|x^n}{2\sqrt{abx}\Gamma(n+1/2)z^n}. \qquad (35)$$

Proof. After the change of variable $t\to zt^{-1}$, the integral $2\sqrt{abx}R_F(x,az,bz)$ has the form considered in theorem 1 with $\alpha=1/2$,

$$f(t)\equiv f^F(t)=\frac{1}{\sqrt{t(t+a^{-1})(t+b^{-1})}}=\sum_{k=0}^{n-1}\frac{(-1)^kA_k^F}{t^{k+1/2}}+f_n^F(t), \qquad (36)$$

where $f_n^F(t)=\mathcal{O}(t^{-n-1/2})$ as $t\to\infty$, $\rho=1/2$ and z replaced by z/x. Therefore, the asymptotic expansion of $2\sqrt{abx}R_F(x,az,bz)$ for large z follows from eq. (5) in theorem 1. Coefficients $a_k\equiv(-1)^kA_k^F(a,b)$ in eq. (4) are trivially given by formula (34). On the other hand, the analytic extension of the Mellin transform of $f^F(t)$ can be obtained from [[15], p. 303, eq. 24]. Introducing then $M[f^F(t);k+1]$ in (5) we obtain (33).

Function $f^F(t)$ satisfies the conditions of proposition 1 with $\mu=3/2$. Therefore, $R_n^F(x,az,bz)\le 0$ and, for $n=1,2,3,...,$ the bound (22) holds for $2\sqrt{abx}R_F(x,az,bz)$ setting $\rho\equiv 1/2$, $\mu=3/2$ and $a_n\equiv(-1)^kA_n^F(a,b)$ given in (34).

Introducing the bound $|A_{n+1}^F|\le a^{-n}$ in (35) we obtain, for $n\ge 1$,

$$|R_n^F(x,az,bz)|\le C(a,z)\frac{x^n}{(az)^n\sqrt{n}}, \qquad (37)$$

where $C(a,z)$ is independent of n. Therefore, expansion (33) is uniformly convergent for $x\le az$. □

z	$R_F(1,z,2z)$	1st order aprox.	Relative error	Relative error bound	2nd order aprox.	Relative error	Relative error bound
10	.3561342012	.4145837013	.164	.199	.3589736808	.00797	.00993
20	.2623854105	.2931549466	.117	.135	.2631384963	.00287	.00337
50	.1724885762	.1854074678	.0749	.0820	.1726159759	.000739	.000820
100	.1244765346	.1311028777	.0532	.0568	.1245093346	.000263	.000284

Table 1. *Numerical example of the approximation (33). Second, third and sixth columns represent* $R_F(1,z,2z)$, *approximation (33) for* $n=1$ *and approximation (33) for* $n=2$ *respectively. Fourth and seventh columns represent the respective relative errors* $-R_n^F(1,z,2z)/R_F(1,z,2z)$ *in (33). Fifth and last columns represent the respective error bounds given by eq. (35).*

Corollary 2. *A uniformly convergent expansion of* $R_D(x,az,bz)$ *for* $0\le x<az$ *and* $0\le x<bz$ *is given, for* $n=1,2,3,...,$ *by*

$$R_D(x,az,bz)=\frac{3}{2z}\sqrt{\frac{\pi}{ab^3x}}\sum_{k=0}^{n-1}\frac{x^k}{z^k}\left[\frac{(k-1)!A_k^D(a,b)}{\Gamma(k+1/2)}+\right.$$
$$\left.\frac{\Gamma(k+3/2)\sqrt{xb^3}}{k!a^{k+1}z^{1/2}}F\left(\begin{matrix}k+3/2,3/2\\2\end{matrix}\middle|1-\frac{b}{a}\right)\right]+\bar{R}_n^D(x,az,bz), \tag{38}$$

where $A_0^D(a,b)=0$ *and, for* $k=1,2,3,...,$

$$A_k^D(a,b)=-\sum_{j=0}^{k-1}\frac{(1/2)_j(3/2)_{k-1-j}}{j!(k-1-j)!a^jb^{k-1-j}}. \tag{39}$$

For $n=1,2,3,...,$ *the remainder term* $\bar{R}_n^D(x,az,bz)$ *verifies*

$$0\le-\bar{R}_n^D(x,az,bz)\le\frac{3\sqrt{\pi}(n-1)!|A_n^D(a,b)|x^{n-1/2}}{2\sqrt{ab^3}\Gamma(n+1/2)z^{n+1}}. \tag{40}$$

Proof. After the change of variable $t\to zt^{-1}$, the integral $(2\sqrt{ab^3x}z/3)R_D(x,az,bz)$ has the form considered in theorem 1 with $\alpha=1$,

$$f(t)\equiv\bar{f}^D(t)=\frac{\sqrt{t}}{\sqrt{(t+a^{-1})(t+b^{-1})^3}}=\sum_{k=0}^{n-1}\frac{(-1)^kA_k^D}{t^{k+1/2}}+\bar{f}_n^D(t), \tag{41}$$

where $\bar{f}_n^D(t)=\mathcal{O}(t^{-n-1/2})$ as $t\to\infty$, $\rho=1/2$ and z replaced by z/x. Therefore, the asymptotic expansion of $(2\sqrt{ab^3x}z/3)R_D(x,az,bz)$ for large z follows from eq. (5) in theorem 1. Coefficients $a_k\equiv(-1)^kA_k^D(a,b)$ in eq. (4) are trivially given by formula (39). On the other hand, the analytic extension of the Mellin transform of $\bar{f}^D(t)$ can be obtained from [[15], p. 303, eq. 24]. Introducing then $M[\bar{f}^D(t);k+1]$ in (5) we obtain (38).

Function $\bar{f}^D(t)$ satisfies the conditions of proposition 1 with $\mu=3/2$. Therefore, $\bar{R}_n^D(x,az,bz)\le 0$ and, for $n=1,2,3,...$, the bound (22) holds for $(2\sqrt{ab^3x}z/3)R_D(x,az,bz)$ replacing z by z/x and setting $\rho\equiv1/2$, $\mu=3/2$ and $a_n\equiv(-1)^nA_n^D(a,b)$ given in (39).

Introducing the bound $|A_n^D|\le ns^{1-n}$, where $s=\min\{a,b\}$, in (40) we obtain, for $n\ge1$,

$$|\bar{R}_n^D(x,az,bz)|\le C(s,z)\frac{x^n\sqrt{n}}{(sz)^n}, \tag{42}$$

where $C(s,z)$ is independent of n. Therefore, expansion (38) is uniformly convergent for $x<sz$. □

z	$R_D(1,z,2z)$	2^{st} order aprox.	Relative error	Relative error bound	3^{nd} order aprox.	Relative error	Relative error bound
10	.0255837279	.0262739405	.0270	.0345	.0256376739	.00211	.00269
20	.0097457248	.0098379248	.00946	.0113	.0097493258	.000369	.000442
50	.0026456597	.0026519429	.00237	.00267	.0026457578	.0000371	.0000417
100	.0009708928	.0009717047	.000836	.000910	.0009708991	.00000653	.00000710

Table 2. *Numerical example of the approximation (38). Second, third and sixth columns represent* $R_D(1,z,2z)$*, approximation (38) for* $n=2$ *and approximation (38) for* $n=3$ *respectively. Fourth and seventh columns represent the respective relative errors* $-R_n^D(1,z,2z)/R_D(1,z,2z)$ *in (38). Fifth and last columns represent the respective error bounds given by eq. (40).*

Corollary 3. *A uniformly convergent expansion of* $R_D(az,bz,x)$ *for* $0<x<az\le bz$ *is given, for* $n=1,2,3,\ldots$*, by*

$$R_D(az,bz,x) = -\,3\sqrt{\frac{\pi}{abx}}\sum_{k=0}^{n-1}\left[\frac{k!A_{k+1}^F(a,b)x^k}{\Gamma(k+1/2)z^{k+1}}+\right.$$
$$\left.\frac{(3/2)_k\sqrt{\pi b}x^{k+1/2}}{2k!a^{k+1}z^{k+3/2}}F\left(\begin{array}{c} k+\frac{3}{2},\frac{1}{2} \\ 1\end{array}\middle|\, 1-\frac{b}{a}\right)\right]+R_n^D(az,bz,x), \tag{43}$$

where, for $k=0,1,2,\ldots$, $A_k^F(a,b)$ *are given in (34). For* $n=1,2,3,\ldots$*, the remainder term* $R_n^D(az,bz,x)$ *verifies*

$$0\le R_n^D(az,bz,x)\le\frac{3\sqrt{\pi}n!|A_{n+1}^F(a,b)|x^n}{\sqrt{abx}\Gamma(n+1/2)z^{n+1}}. \tag{44}$$

Proof. After the change of variable $t\to zt^{-1}$, the integral $(2\sqrt{abx^3}/3)R_D(az,bz,x)$ has the form considered in theorem 1 with $\alpha=1/2$,

$$f(t)\equiv f^D(t)=\frac{\sqrt{t}}{\sqrt{(t+a^{-1})(t+b^{-1})}}=-\sum_{k=0}^{n-1}\frac{(-1)^kA_{k+1}^F}{t^{k+1/2}}+f_n^D(t), \tag{45}$$

where $f_n^D(t)=\mathcal{O}(t^{-n-1/2})$ as $t\to\infty$, $\rho=3/2$ and z replaced by z/x. Therefore, the asymptotic expansion of $(2\sqrt{abx^3}/3)R_D(az,bz,x)$ for large z follows from eq. (5) in theorem 1. Coefficients a_k in eq. (4) are trivially given by $a_k\equiv-(-1)^kA_{k+1}^F(a,b)$. On the other hand, the analytic extension of the Mellin transform of $f^D(t)$ can be obtained from [[15], p. 303, eq. 24]. Introducing then $M[f^D(t);k+1]$ in (5) we obtain (43).

Function $f^D(t)$ satisfies the conditions of proposition 1 with $\mu=1/2$. Therefore, $R_n^D(az,bz,x)\ge 0$ and, for $n=1,2,3,\ldots$, the bound (22) holds for $(2\sqrt{abx^3}/3)R_D(az,bz,x)$ setting $\rho\equiv 3/2$, $\mu=1/2$ and $a_n\equiv(-1)^{n+1}A_{n+1}^F(a,b)$.

Introducing the bound $|A_{n+1}^F|\le a^{-n}$ in (44) we obtain, for $n\ge 1$,

$$R_n^D(az,bz,x)\le C(a,z)\frac{x^n\sqrt{n}}{(az)^n}, \tag{46}$$

where $C(a,z)$ is independent of n. Therefore, expansion (43) is uniformly convergent for $x < az$. □

z	$R_D(z,2z,1)$	1st order aprox.	Relative error	Relative error bound	2nd order aprox.	Relative error	Relative error bound
10	.1454172243	.1215280884	-.164	.219	.1428666794	-.0175	.0231
20	.0804778214	.0740326848	-.0801	.0988	.0801348017	-.00426	.00522
50	.0354313796	.0343225435	-.0313	.0359	.0354078421	-.000664	.000758
100	.0186361617	.0183480551	-.0155	.0171	.0186331086	-.000164	.000180

Table 3. *Numerical example of the approximation (43). Second, third and sixth columns represent* $R_D(z,2z,1)$, *approximation (43) for* $n=1$ *and approximation (43) for* $n=2$ *respectively. Fourth and seventh columns represent the respective relative errors* $-R_n^D(z,2z,1)/R_D(z,2z,1)$ *in (43). Fifth and last columns represent the respective error bounds given by eq. (44).*

Corollary 4. *A uniformly convergent expansion of* $R_J(x,az,bz,cz)$ *for* $0 \le x < az \le bz$ *and* $x < cz$ *is given, for* $n = 1,2,3,...,$ *by*

$$R_J(x,az,bz,cz) = \frac{3\sqrt{\pi}}{2c\sqrt{abx}z}\sum_{k=0}^{n-1}\frac{x^k}{z^k}\left[\frac{(k-1)!A_k^J(a,b,c)}{\Gamma(k+1/2)} + \frac{\Gamma(k+1/2)\sqrt{x}}{k!\pi c^k\sqrt{z}}B_k(a,b,c)\right] + R_n^J(x,az,bz,cz), \tag{47}$$

where $A_0^J(a,b,c) = 0$ *and, for* $k = 1,2,3,...,$

$$A_k^J(a,b,c) = \sum_{j=1}^{k} c^{j-k}A_j^F(a,b), \tag{48}$$

where $A_j^F(a,b)$ *are given in (34) and the coefficients* $B_k(a,b,c)$ *verify the recurrence*

$$B_k(a,b,c) = B_{k-1}(a,b,c) + \frac{\pi\sqrt{b}c^k}{a^k}F\left(\begin{array}{c} k+1/2, 1/2 \\ 1 \end{array}\middle| 1-\frac{b}{a}\right), \tag{49}$$

where

$$B_0(a,b,c) = \frac{2}{3}c\sqrt{ab}R_J(0,a,b,c). \tag{50}$$

For $n = 1,2,3,...,$ *the remainder term* $R_n^J(x,az,bz,cz)$ *verifies*

$$0 \le -R_n^J(x,az,bz,cz) \le \frac{3\sqrt{\pi}(n-1)!|A_n^J(a,b,c)|x^{n-1/2}}{2c\sqrt{ab}\Gamma(n+1/2)z^{n+1}}. \tag{51}$$

Proof. After the change of variable $t \to zt^{-1}$, the integral $(2\sqrt{abx}cz/3)R_J(x,az,bz,cz)$ has the form considered in theorem 1 with $\alpha = 1/2$,

$$f(t) \equiv f_1^J(t) = \frac{\sqrt{t}}{\sqrt{(t+a^{-1})(t+b^{-1})(t+c^{-1})}} = \sum_{k=0}^{n-1}\frac{(-1)^k A_k^J}{t^{k+1/2}} + f_n^J(t), \tag{52}$$

where $f_n^J(t) = \mathcal{O}(t^{-n-1/2})$ as $t \to \infty$, $\rho = 1/2$ and z replaced by z/x. Therefore, the asymptotic expansion of $(2\sqrt{abx}cz/3)R_J(x, az, bz, cz)$ for large z follows from eq. (5) in theorem 1. Coefficients $a_k \equiv (-1)^k A_k^J(a, b, c)$ in eq. (4) are trivially given by formula (48). On the other hand, after straightforward operations we obtain that the analytic extension of the Mellin transform of $f_1^J(t)$ verify the recurrence

$$M[f_1^J; k+1] = M[\sqrt{t(t+a^{-1})^{-1}(t+b^{-1})^{-1}}; k] - c^{-1}M[f_1^J; k].$$

Then, using formula [[15], p. 303, eq. 24] and defining $B_k(a, b, c) \equiv (-c)^k M[f_1^J; k+1]$ we obtain the recurrence (49) and therefore expansion (47).

Function $f_1^J(t)$ satisfies the conditions of proposition 1 with $\mu = 3/2$. Therefore, $R_n^J(x, az, bz, cz) \leq 0$ and then, replacing z by z/x, the bound (22) holds for $(2\sqrt{abx}cz/3)R_J(x, az, bz, cz)$ for $n = 1, 2, 3, ...$, setting $\rho \equiv 1/2$, $\mu = 3/2$ and $a_n \equiv (-1)^n A_n^J(a, b, c)$ given in (48). From this we obtain (51).

Introducing the bound $|A_n^J| \leq ns^{1-n}$, where s =min$\{a, c\}$, in (51) we obtain, for $n \geq 1$,

$$|R_n^J(x, az, bz, cz)| \leq C(s, z)\frac{x^n\sqrt{n}}{(sz)^n}, \tag{53}$$

where $C(s, z)$ is independent of n. Therefore, expansion (47) is uniformly convergent for $x < sz$. □

z	$R_J(1,z,3z,2z)$	2st order aprox.	Relative error	Relative error bound	3nd order aprox.	Relative error	Relative error bound
10	.0220288446	.0225580167	.0240	.0306	.0220681536	.00178	.00227
20	.0083692419	.0084398219	.00843	.0101	.0083718630	.000313	.000374
50	.0022659947	.0022707974	.00212	.00238	.0022660661	.0000315	.0000353
100	.0008303642	.0008309844	.000747	.000811	.0008303689	.00000554	.00000603

Table 4. *Numerical example of the approximation (47). Second, third and sixth columns represent $R_J(1, z, 3z, 2z)$, approximation (47) for $n = 2$ and approximation (47) for $n = 3$ respectively. Fourth and seventh columns represent the respective relative errors $-R_n^J(1, z, 3z, 2z)/R_J(1, z, 3z, 2z)$ in (47). Fifth and last columns represent the respective error bounds given by (51).*

Corollary 5. *A uniformly convergent expansion of $R_J(x, az, bz, p)$ for $0 < p < az \leq bz$ and $0 \leq x < az$ is given, for $n = 1, 2, 3, ...$, by*

$$\begin{aligned} R_J(x, az, bz, p) =& \frac{3}{2}\sum_{k=0}^{n-1}\left[\frac{A_k^J(x,p)B_k(a,b)}{z^{k+1/2}} + \frac{2(-1)^k x^{k+1/2}C_k(a,b)\Gamma(1/2-k)}{p\sqrt{\pi ab}z^{k+1}}\times\right. \\ & \left. F\left(\begin{matrix} k+1, 1 \\ 3/2 \end{matrix}\middle| 1-\frac{x}{p}\right)\right] + R_n^J(x, az, bz, p), \end{aligned} \tag{54}$$

where $A_0^J(x, p) = 0$ and, for $k = 1, 2, 3, ...$, the coefficients $A_k^J(x, p)$, $B_k(a, b)$ and $C_k(a, b)$ are given by

$$A_k^J(x, p) = -\sum_{j=0}^{k-1}\frac{(1/2)_j}{j!}x^j p^{k-j-1}, \tag{55}$$

p	$R_J(1,z,2z,2)$	2nd order aprox.	Relative error	Relative error bound	3rd order aprox.	Relative error	Relative error bound
10	.1046014390	.1169052834	.118	.167	.1067522930	.0206	.0291
20	.0591144006	.0614960026	.0403	.0522	.0593224077	.00352	.00455
50	.0265906526	.0268538035	.00990	.0118	.0265998397	.000346	.000410
100	.0141558976	.0142046739	.00345	.00390	.0141567487	.0000601	.0000680

Table 5. *Numerical example of the approximation (54). Second, third and sixth columns represent $R_J(1,z,2z,2)$, approximation (54) for $n=2$ and approximation (54) for $n=3$ respectively. Fourth and seventh columns represent the respective relative errors $-R_n^J(1,z,2z,2)/R_J(1,z,2z,2)$ in (54). Fifth and last columns represent the respective error bounds given by eq. .*

$$B_k(a,b)=\sum_{j=0}^{k}\frac{\Gamma(j+1/2)\Gamma(k-j+1/2)}{j!(k-j)!a^j b^{k-j+1/2}}F\left(\begin{array}{c}1/2,k-j+1/2\\ k+1\end{array}\middle|1-\frac{a}{b}\right) \tag{56}$$

and

$$C_k(a,b)=\sum_{j=0}^{k}\binom{k}{j}\frac{(1/2)_j(1/2)_{k-j}}{a^j b^{k-j}}. \tag{57}$$

For $n=1,2,3,...$, the remainder term $R_n^J(x,az,bz,p)$ verifies

$$\begin{aligned}0\le -R_n^J(x,az,bz,p)\le&\frac{3\pi|A_n^J(x,p)|}{2n!z^{n+1/2}}\sum_{j=0}^{n}\binom{n}{j}\frac{(1/2)_j(1/2)_{n-j}}{a^j b^{n-j+1/2}}\times\\ &F\left(\begin{array}{c}1/2,n-j+1/2\\ n+1\end{array}\middle|1-\frac{a}{b}\right)\le\frac{3\pi|A_n^J(x,p)|}{2(cz)^{n+1/2}},\end{aligned} \tag{58}$$

where $c=\min\{a,b\}$.

Proof. The integral $(2/3)R_J(x,az,bz,p)$ has the form considered in theorem 3 with $\alpha=1/2$,

$$f(t)\equiv f_2^J(t)=\frac{1}{\sqrt{t+x}(t+p)}=\sum_{k=0}^{n-1}\frac{(-1)^k A_k^J}{t^{k+1/2}}+f_n^J(t), \tag{59}$$

where $f_n^J(t)=\mathcal{O}(t^{-n-1/2})$ as $t\to\infty$ and $\rho=\sigma=1/2$. Therefore, the asymptotic expansion of $(2/3)R_J(x,az,bz,p)$ for large z follows from eq. (10) in theorem 3. Coefficients $a_k\equiv(-1)^k A_k^J(x,p)$ in eq. (4) are trivially given by formula (55). The Mellin transform $M[f_2^J;k+1]$ in eq. (10) can be obtained from [[15], p. 303, eq. 24] and coefficients $B_k(a,b)$ and $C_k(a,b)$ follow from (11) and (12) respectively.

Function $f_2^J(t)$ satisfies the conditions of proposition 3 with $\mu=3/2$. Therefore, $R_n^J(x,az,bz,p)\le 0$ and, for $n=1,2,3,...$, the bound (29) holds for $(2/3)R_J(x,az,bz,p)$ setting $\rho\equiv\sigma=1/2$, $\mu=3/2$ and $a_n\equiv(-1)^n A_n^J(x,p)$ given in (55).

Introducing the bound $|A_{n+1}^J|\le(3/2)_n s^n/n!$, where $s=\max\{x,p\}$, in (58) we obtain, for $n\ge 1$,

$$|R_n^J(x,az,bz,p)|\le C(c,s,z)\frac{s^n\sqrt{n}}{(cz)^n}, \tag{60}$$

where $C(c,s,z)$ is independent of n. Therefore, expansion (54) is uniformly convergent for $s < cz$. □

Corollary 6. *A uniformly convergent expansion of $R_J(az,bz,cz,p)$ for $0 < p \le az \le bz \le cz$ is given, for $n = 1,2,3,...,$ by*

$$\begin{aligned} R_J(az,bz,cz,p) = &\frac{3}{2\sqrt{abcz}} \sum_{k=0}^{n-1} \frac{p^k}{z^{k+1}} \left[\bar{A}_k^J(a,b,c) \left(\log\left(\frac{z}{p}\right) - \gamma - \psi(k+1) \right) + \right. \\ &\left. B_k(a,b,c) \right] + R_n^J(az,bz,cz,p), \end{aligned} \tag{61}$$

where, for $k = 0,1,2,...,$ the coefficients $\bar{A}_k^J(a,b,c)$ are given by

$$\bar{A}_k^J(a,b,c) = \sum_{j=0}^{k} \sum_{l=0}^{k-j} \frac{(1/2)_l (1/2)_j (1/2)_{k-j-l}}{j!l!(k-j-l)! a^l b^j c^{k-j-l}} \tag{62}$$

and the coefficients $B_k^J(a,b,c)$ are given by

$$B_k^J(a,b,c) = \bar{A}_k^J(a,b,c) \sum_{j=1}^{k} \frac{1}{j} + (-1)^k C_k^J(a,b,c),$$

empty sums being understood as zero and where $C_k^J(a,b,c)$ are given by the recurrence

$$\begin{aligned} C_{k+3} = -\frac{1}{2(k+3)} &\left[(2k+5)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) C_{k+2} + \right. \\ &\left. 2(k+2)\frac{a+b+c}{abc} C_{k+1} + \frac{2k+3}{abc} C_k \right], \end{aligned} \tag{63}$$

with first terms

$$C_0 = 2\log\left(\frac{2\sqrt{abc}}{\Delta}\right) + \frac{4\sqrt{abc}}{3} R_J(a+\Delta, b+\Delta, c+\Delta, \Delta),$$

$$C_1 = \lambda(1 - C_0) - \frac{1}{c} - \sqrt{\frac{c}{ab}} R_F(a,b,c) + \frac{c^2 + ab - (a+b)c}{3\sqrt{abc}} R_D(a,b,c) \tag{64}$$

and

$$\begin{aligned} C_2 = &\frac{\beta}{2} - \frac{a^2+b^2+ab}{2a^2b^2} - \frac{\lambda}{2} C_1 + \frac{1}{4}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) C_0 + \frac{\sqrt{c}(a+b)}{2\sqrt{a^3b^3}} R_F(a,b,c) - \\ &\frac{1}{6}\left[\sqrt{\frac{ab}{c^3}} + \sqrt{\frac{c}{a^3b^3}} \left(cb + ca - a^2 - b^2 - ab\right) \right] R_D(a,b,c), \end{aligned} \tag{65}$$

where

$$\Delta \equiv \sqrt{ab} + \sqrt{ac} + \sqrt{bc},$$

$$\lambda \equiv \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

and

$$\beta \equiv \frac{1}{8}\left(3\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) + 2\frac{a+b+c}{abc}\right).$$

For $n = 1, 2, 3, \ldots$, *the remainder term* $R_n^J(az, bz, cz, p)$ *is positive and a bound for* $(2p\sqrt{abcz}/3)R_n^J(az, bz, cz, p)$ *is given by the right hand side of (24) or (25) putting* $\rho \equiv 1$ *and* $a_n \equiv \bar{A}_n^J(a, b, c)$ *given in (62). In particular, two error bounds are given by*

$$\begin{aligned} 0 \leq & R_n^J(az, bz, cz, p) \leq \frac{3\pi A_n}{2\sqrt{abc}} \frac{p^{n-1/2}}{z^{n+1}}, \\ R_n^J(az, bz, cz, p) \leq & \frac{3}{2\sqrt{abc}} \left[1 + \psi(n+1) + \gamma + \log\left(1 + \frac{nz|\bar{A}_{n-1}^J|}{|p\bar{A}_n^J|}\right)\right] \frac{p^n |\bar{A}_n^J|}{z^{n+3/2}}, \end{aligned} \tag{66}$$

where $A_n \equiv max\{|\bar{A}_n^J|, |\bar{A}_{n-1}^J|\}$.

Proof. After the change of variable $t \to zt^{-1}$, the integral $(2p\sqrt{abcz}/3)R_J(az, bz, cz, p)$ has the form considered in theorem 2 with

$$f(t) \equiv f_3^J(t) = \frac{\sqrt{t}}{\sqrt{(t+a^{-1})(t+b^{-1})(t+c^{-1})}} = \sum_{k=0}^{n-1} \frac{(-1)^k \bar{A}_k^J}{t^{k+1}} + f_n^J(t), \tag{67}$$

where $f_n^J(t) = \mathcal{O}(t^{-n-1})$ as $t \to \infty$, $\rho = 1$ and z replaced by z/p. Therefore, the asymptotic expansion of $(2p\sqrt{abcz}/3)R_J(az, bz, cz, p)$ for large z follows from eq. (8) in theorem 2. Coefficients $a_k \equiv (-1)^k \bar{A}_k^J(a, b, c)$ in eq. (7) are trivially given by formula (62). On the other hand, coefficients $B_k(a, b, c)$ follow from (9), where

$$C_k^J(a, b, c) = \lim_{T\to\infty}\left\{\int_0^T t^k f_3^J(t)dt - \sum_{j=0}^{k-1}(-1)^j \bar{A}_j^J \frac{T^{k-j}}{k-j} - (-1)^k \bar{A}_k^J \log T\right\} \tag{68}$$

We define, for $s > 0$,

$$\alpha_k(a, b, c, s, T) \equiv \sqrt{s}\int_0^T \frac{t^{k+1/2}}{\sqrt{(t+a^{-1})(t+b^{-1})(t+c^{-1})(t+s)}}dt$$

and

$$I_k(a, b, c, T) \equiv \int_0^T \frac{t^{k+1/2}}{\sqrt{(t+a^{-1})(t+b^{-1})(t+c^{-1})}}dt = \lim_{s\to\infty} \alpha_k(a, b, c, s, T).$$

Integrals $\alpha_k(a,b,c,s,T)$ satisfy the recurrence

$$2T^{k+3/2}\sqrt{s(T+a^{-1})(T+b^{-1})(T+c^{-1})(T+s)} = (2k+7)\alpha_{k+4}+$$
$$2(k+3)(a^{-1}+b^{-1}+c^{-1}+s)\alpha_{k+3}+(2k+5)\frac{a+b+c+s(ab+bc+ac)}{abc}\alpha_{k+2}+$$
$$2(k+2)\frac{1+s(a+b+c)}{abc}\alpha_{k+1}+(2k+3)\frac{s}{abc}\alpha_k.$$

Taking the limit $s\to\infty$ we obtain that the integrals $I_k(a,b,c,T)$ satisfy the recurrence

$$\begin{aligned} I_{k+3} =&\frac{1}{2(k+3)}\Big[2T^{k+3/2}\sqrt{(T+a^{-1})(T+b^{-1})(T+c^{-1})}- \\ &(2k+5)(a^{-1}+b^{-1}+c^{-1})I_{k+2}-2(k+2)\frac{a+b+c}{abc}I_{k+1}-\frac{2k+3}{abc}I_k\Big]. \end{aligned} \tag{69}$$

On the other hand, from the differential equation $2(t+a^{-1})(t+b^{-1})(t+c^{-1})(f_3^J/\sqrt{t})'+\big(3t^2+2(a^{-1}+b^{-1}+c^{-1})t+(a+b+c)/abc\big)\, f_3^J/\sqrt{t}=0$, we obtain, for $k=0,1,2,...$,

$$(2k+3)\bar{A}^J_{k+2}-2(k+1)(a^{-1}+b^{-1}+c^{-1})\bar{A}^J_{k+1}+(2k+1)\frac{a+b+c}{abc}\bar{A}^J_k-\frac{2k}{abc}\bar{A}^J_{k-1}=0. \tag{70}$$

If we expand the term $\sqrt{(T+a^{-1})(T+b^{-1})(T+c^{-1})}$ in (69) in inverse powers of T and use the recurrence (70) and the definition (68), we obtain the recurrence (63). Coefficient C_0 can be obtained from [[10], eq. (42)]. Integrating by parts $I_1(a,b,c,T)$, expanding the integrated term in inverse powers of T and after straightforward operations and using the definition (68) we obtain

$$C_1=-\lambda(C_0+1)+\frac{1}{2}\left(\frac{1}{a^2}J(a,b,c)+\frac{1}{b^2}J(b,c,a)+\frac{1}{c^2}J(c,a,b)\right),$$

where

$$J(a,b,c)\equiv\int_0^\infty\sqrt{\frac{t}{(t+a^{-1})^3(t+b^{-1})(t+c^{-1})}}dt.$$

Now, using [[15], p. 73, eqs. $9-11$] and [[16], p. 329, eqs. (12.33)], we obtain (64). Equation (65) follows from $I_2(a,b,c,T)$ after similar steps.

Function $f_3^J(t)$ satisfies the conditions of proposition 2 with $\mu=1$. Therefore, $R_n^J(az,bz,cz,p)\geq 0$ and, for $n=1,2,3,...$, the bounds (24) and (25) hold for $(2p\sqrt{abcz}/3)R_J(az,bz,cz,p)$ setting $\rho=\mu=1$ and $a_n\equiv(-1)^n\bar{A}^J_n(a,b,c)$ given in (62). In particular, the second line of (66) follows after introducing (28) in inequality (25).

Introducing the bound $|\bar{A}^J_n|\leq(3/2)_n/(a^n n!)$ in the first line of (66) we obtain, for $n\geq 1$,

$$R_n^J(az,bz,cz,p)\leq C(a,p,z)\frac{\sqrt{n}p^n}{(az)^n}, \tag{71}$$

where $C(a,p,z)$ is independent of n Therefore, expansion (61) is uniformly convergent for $p < az$. □

z	$R_J(z,2z,3z,1)$	1st order aprox.	Relative error	Relative error bound	2nd order aprox.	Relative error	Relative error bound
10	.0478543705	.0430650136	-.100	.129	.0474540374	-.00837	.0132
20	.0210080906	.0199714360	-.0493	.0617	.0209648222	-.00206	.00306
50	.0067712980	.0066394802	-.0195	.0235	.0067690999	-.000325	.000453
100	.0027989048	.0027718749	-.00966	.0114	.0027986796	-.0000805	.000108

Table 6. *Numerical example of the approximation (61). Second, third and sixth columns represent $R_J(z,2z,3z,1)$, approximation (61) for $n=1$ and approximation (61) for $n=2$ respectively. Fourth and seventh columns represent the respective relative errors $-R_n^J(z,2z,3z,1)/R_J(z,2z,3z,1)$ in (61). Fifth and last columns represent the respective error bounds given by eq. (25).*

Corollary 7. *A uniformly convergent expansion of $R_J(x,y,az,bz)$ for $0 \le x \le y < az$ and $0 < y < bz$ is given, for $n = 1,2,3,\ldots$, by*

$$R_J(x,y,az,bz) = \frac{3}{2}\sum_{k=0}^{n-1}\frac{1}{z^{k+3/2}}\left[-A_{k+1}^F(x^{-1},y^{-1})\left(C_k(a,b)\log(az)+D_k(a,b)\right)+ E_k(a,b)B_k^J(x,y)\right]+R_n^J(x,y,az,bz), \tag{72}$$

where, for $k=0,1,2,\ldots$, the coefficients $A_k^F(x^{-1},y^{-1})$ are given in (34) and the coefficients $B_k^J(x,y)$ are given by the recurrence

$$B_{k+2}^J = -\left[\frac{xyA_{k+1}^F-(x+y)A_{k+2}^F+2A_{k+3}^F}{k+2}+\frac{A_{k+3}^F}{k+1}\right]+ \frac{2k+3}{2k+4}(x+y)\left[\frac{A_{k+2}^F}{k+1}+B_{k+1}^J\right]-\frac{k+1}{k+2}xyB_k^J, \tag{73}$$

where

$$B_0^J = -2\log\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right), \qquad B_1^J = \sqrt{xy}-(x+y)\log\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right). \tag{74}$$

The coefficients C_k, D_k and E_k are given by

$$C_k(a,b) = \sum_{j=0}^{k+1}\frac{(1/2)_j(k+1)}{j!(k+3/2)a^{j-1/2}b^{k-j+2}}F\left(\begin{array}{c}1,k+2-j\\k+5/2\end{array}\middle|1-\frac{a}{b}\right),$$

$$D_k(a,b) = \sum_{j=0}^{k+1}\frac{(1/2)_j(k+1)}{j!(k+3/2)a^{j-1/2}b^{k-j+2}}\left[F'\left(\begin{array}{c}1,k+2-j\\k+5/2\end{array}\middle|1-\frac{a}{b}\right)- \left(\psi\left(k+\frac{3}{2}\right)+\gamma\right)F\left(\begin{array}{c}1,k+2-j\\k+5/2\end{array}\middle|1-\frac{a}{b}\right)\right]$$

and

$$E_k(a,b)=\sum_{j=0}^{k}\frac{(1/2)_j}{j!a^{j+1/2}b^{k+1-j}}. \tag{75}$$

For $n=1,2,3,...,$ *the remainder term* $R_n^J(x,y,az,bz)$ *is positive and a bound for* $(2/3)R_n^J(x,y,az,bz)$ *is given by the right hand side of (31) or (32) putting* $\rho\equiv 1/2$, $\sigma=1$ *and* $a_n\equiv(-1)^{n+1}A_{n+1}^F(x^{-1},y^{-1})$ *given in (34). In particular, two error bounds are given by*

$$\begin{aligned} 0\le & R_n^J(x,y,az,bz)\le\frac{3\sqrt{\pi}n!A_n}{\Gamma(n+1/2)(cz)^{n+1}},\\ R_n^J(x,y,az,bz)\le & \left(\frac{3}{2}\right)_n\left[1+\psi(n+1)+\gamma+\log\left(1+\frac{nz|A_n^F|}{|A_{n+1}^F|}\right)\right]\frac{|A_{n+1}^F|}{n!(cz)^{n+3/2}}, \end{aligned} \tag{76}$$

where $A_n=$*max*$\{|A_n^F|,|A_{n+1}^F|\}$ *and* $c=$*min*$\{a,b\}$.

Proof. The integral $(2/3)R_J(x,y,az,bz)$ has the form considered in theorem 4 with

$$f(t)\equiv f_4^J(t)=\frac{1}{\sqrt{(t+x)(t+y)}}=\sum_{k=0}^{n-1}\frac{(-1)^{k+1}A_{k+1}^F}{t^{k+1}}+f_n^J(t), \tag{77}$$

where $f_n^J(t)=\mathcal{O}(t^{-n-1})$ as $t\to\infty$, $\rho=1/2$ and $\sigma=1$. Therefore, the asymptotic expansion of $(2/3)R_J(x,y,az,bz)$ for large z follows from eq. (14) in theorem 4. Coefficients a_k in eq. (7) are trivially given by $a_k\equiv(-1)^{k+1}A_{k+1}^F(x^{-1},y^{-1})$.

For calculating $B_k\equiv E_k(a,b)B_k^J(x,y)$ we consider the second line in (9). Define, for $k=0,1,2,...,$

$$I_k^J(x,y,T)\equiv\int_0^T t^k f_4^J(t)dt\equiv\int_0^T\frac{t^k}{\sqrt{(t+x)(t+y)}}dt.$$

and

$$\sigma_k^J(x,y)\equiv\lim_{T\to\infty}\left\{I_k^J(x,y,T)+\sum_{j=0}^{k-1}(-1)^jA_{j+1}^F\frac{T^{k-j}}{k-j}+(-1)^kA_{k+1}^F\log(T)\right\}. \tag{78}$$

Integrals $I_k^J(x,y,T)$ satisfy the recurrence

$$I_{k+2}^J=\frac{1}{2(k+2)}\left[2T^{k+1}\sqrt{(T+x)(T+y)}-(2k+3)(x+y)I_{k+1}^J-2(k+1)xyI_k^J\right]. \tag{79}$$

On the other hand, from the differential equation $2(t+x)(t+y)(f_4^J)'+(2t+x+y)f_4^J=0$, we obtain, for $k=0,1,2,...,$

$$2(k+1)A_{k+2}^F-(2k+1)(x+y)A_{k+1}^F+2kxyA_k^F=0. \tag{80}$$

Now we substitute $I^{J}_{k+2}(x,y,T)$ in the definition (78) of $\sigma^{J}_{k+2}(x,y)$ by the right hand side of (79), expand the term $\sqrt{(T+x)(T+y)}$ in inverse powers of T and use recurrence (80). We obtain

$$\begin{aligned}2(k+2)\sigma^{J}_{k+2} =&(-1)^{k+1}\left(2xyA^{F}_{k+1}-2(x+y)A^{F}_{k+2}+2A^{F}_{k+3}\right)-\\&(2k+3)(x+y)\sigma^{J}_{k+1}-2(k+1)xy\sigma^{J}_{k},\end{aligned}$$

from which (73) follows easily by using the second line in (9) and the recurrence (80). Integrals $I^{J}_{0}(x,y,T)$ and $I^{J}_{1}(x,y,T)$ may be calculated by using formula [[15], p. 53, eqs. 3,8]. Then, from the second line in (9) and using $A^{F}_{1}=-1$ and $A^{F}_{2}=-(x+y)/2$ we obtain (74).

Function $f^{J}_{4}(t)$ satisfies the conditions of proposition 4 with $\mu=1$. Therefore, $R^{J}_{n}(x,y,az,bz)\geq 0$ and, for $n=1,2,3,...$, the bounds (31) and (32) hold for $(2/3)R_{J}(x,y,az,bz)$ setting $\rho=1/2$, $\sigma=\mu=1$ and $a_n\equiv(-1)^{n+1}A^{F}_{n+1}(x,y)$ given in (34). In particular, the second line of (76) follows after introducing (28) in inequality (32).

Introducing the bound $|A^{F}_{n}|\leq y^n$ in the first line of (76) we obtain, for $n\geq 1$,

$$R^{J}_{n}(x,y,az,bz)\leq C(y,c,z)\frac{\sqrt{n}y^n}{(cz)^n}, \tag{81}$$

where $C(y,c,z)$ is independent of n Therefore, expansion (72) is uniformly convergent for $y<cz$.

□

p	$R_J(1,2,.9z,z)$	2^{nd}order aprox.	Relative error	Relative error bound	3^{rd}order aprox.	Relative error	Relative error bound
10	.0867349930	.0819312420	-.0554	.1243	.0857670728	-.0112	.0296
20	.0399899363	.0394496318	-.0135	.0276	.0399355289	-.00136	.00325
50	.0135949951	.0135663149	-.00211	.00395	.0135938404	-.0000849	.000184
100	.0058043457	.0058013266	-.000520	.000929	.0058042849	-.0000105	.0000214

Table 7. *Numerical example of the approximation (72). Second, third and sixth columns represent $R_J(1,2,.9z,z)$, approximation (72) for $n=2$ and approximation (72) for $n=3$ respectively. Fourth and seventh columns represent the respective relative errors $-R^{J}_{n}(1,2,.9z,z)/R_J(1,2,.9z,z)$ in (72). Fifth and last columns represent the respective error bounds given by eq. (32).*

4. Conclusions

Following Wong's proposal [[17], example 1], the distributional approach has been used in theorems 1-4 for deriving alternative proofs for the asymptotic expansion of special cases of the integrals considered in theorems 1 and 2 in [17]. Using these results we have derived convergent expansions of $R_F(x,az,bz)$, $R_D(az,bz,x)$,

$R_D(x, az, bz)$, $R_J(x, az, bz, cz)$, $R_J(x, az, bz, p)$, $R_J(az, bz, cz, p)$ and $R_J(x, y, az, bz)$ for $x, y, p < az, bz, cz$ in corollaries 1-7 respectively.

Functions $f(t)$ in the integrand of R_F, R_D and R_J (and, in general, functions $f(t)$ given in (15)) belong to a special kind of functions: the remainder terms in their asymptotic expansions in inverse powers of t satisfy the error test. This fundamental property is used in propositions 1-4 for deriving an accurate error bound for the remainder in the asymptotic expansions given in theorems 1-4 at any order of the approximation. In particular, it has been derived for the expansions of R_F, R_D and R_J in corollaries 1-7. These bounds show that the expansions are convergent when the asymptotic variables are greater than the remaining ones and that the convergence rate increases as this difference between the asymptotic variables and the remaining ones increases.

Expansions given in corollaries 1-7 are generalizations of the corresponding first order approximations given by Carlson and Gustafson [10]. Nevertheless, complete expansions for R_F, R_D and R_J for the asymptotic parameters considered in corollaries 1-7 were also obtained by Carlson [3]. But the calculation of the coefficients given in [3] is not straightforward and the error bounds supplied there are not quite satisfactory. The advantage of the approach presented here is that it supplies a simple algorithm for the calculation of the coefficients of these expansions and more accurate error bounds at any order of the approximation. This algorithm is explicitly given in corollaries 1-7.

The error bound supplied in corollary 5 for $n = 1$ is slightly less accurate than the error bound given in [10] for the first order approximation of $R_J(x, az, bz, p)$. The same comparison holds between corollary 6 for $n = 1$ and the second order approximation of $R_J(az, bz, cz, p)$ given in [10] and between corollary 7 for $n = 1$ and the approximation of $R_J(x, y, az, bz)$ given in [10]. On the other hand, for large z, the error bound supplied in corollary 3 for $n = 1$ is slightly more accurate than the error bound given in [10] for the second order approximation of $R_D(az, bz, x)$ given there. When considering first order approximations for $R_F(x, az, bz)$, $R_D(x, az, bz)$ and $R_J(x, az, bz, cz)$, a comparison between the error bounds given in corollaries 1,2 and 4 and the error bounds given in [10] is more complicated because they are concerned with different approximations.

This work, jointly with [14], present the twelve possible convergent expansions of the symmetric standard elliptic integrals. Coefficients in these expansions are given in terms of elementary functions or in terms of hypergeometric functions or symmetric standard elliptic integrals depending on less parameters. Considerably accurate error bounds are supplied also at any order of the approximation.

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References

[1] M. Abramowitz and I.A. Stegun, Handbook of mathematical functions, *Dover, New York*, 1970.

[2] P.F. Byrd and M. D. Friedman, Handbook of elliptic integrals for engineers and scientists, *Springer-Verlag, New York*, 1971.

[3] B.C. Carlson, The hypergeometric function and the R-function near their branch points, *Rend. Sem. Math. Univ. Politec. Torino*, **Fascicolo speciale** (1985) 63-89.

[4] B.C. Carlson and J.L. Gustafson, Asymptotic expansion of the first elliptic integral, *SIAM J. Math. Anal.*, **16** (1985) 1072-1092.

[5] B.C. Carlson, A table of elliptic integrals of the second kind, *Math. Comp.*, **49** (1987) 595-606.

[6] B.C. Carlson, A table of elliptic integrals of the third kind, *Math. Comp.*, **51** (1988) 267-280.

[7] B.C. Carlson, A table of elliptic integrals: cubic cases, *Math. Comp.*, **53** (1989) 327-333.

[8] B.C. Carlson, A table of elliptic integrals: one quadratic factor, *Math. Comp.*, **56** (1991) 267-280.

[9] B.C. Carlson, A table of elliptic integrals: two quadratic factors, *Math. Comp.*, **59** (1992) 165-180.

[10] B.C. Carlson and J.L. Gustafson, Asymptotic approximations for symmetric elliptic integrals, *SIAM J. Math. Anal.*, **25** (1994) 288-303.

[11] J.L. Gustafson, Asymptotic formulas for elliptic integrals, *Ph. D. Thesis* , Iowa State Univ., Ames, IA, 1982.

[12] E. L. Kaplan, Auxiliary table for the incomplete elliptic integrals, *J. Math. Phys.*, **27** (1948) 11-36.

[13] A.M. Legendre, Traité des fonctions elliptiques, Vol. I. , *Imprimerie de Huzard-Courcier, Paris*, 1825.

[14] J.L. López, Asymptotic expansions of symmetric standard elliptic integrals, *SIAM J. Math. Anal.*, **31**, n^o 4 (2000) 754-775.

[15] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and series, Vol. 1, *Gordon and Breach Science Pub.*, 1990.

[16] N.M. Temme, Special functions: An introduction to the classical functions of mathematical physics, *Wiley and Sons, New York* , 1996.

[17] R. Wong, Explicit error terms for asymptotic expansions of Mellin convolutions, *J. Math. Anal. Appl.*, **72** (1979) 740-756.

[18] R. Wong, Asymptotic approximations of integrals, *Academic Press, New York*, 1989.