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Analytic expansions of thermonuclear reaction rates

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Abstract
The evaluation of thermonuclear reaction rates requires the calculation of several thermonuclear functions. These functions can be written as the Laplace transform of locally integrable functions which have an asymptotic expansion in negative rational powers of their variable. In this paper we obtain asymptotic expansions of the Laplace transform of these kinds of functions for small values of the parameter of the transformation. Error bounds are obtained at any order of the approximation for a large family of Laplace transforms which include thermonuclear functions. Then we apply this asymptotic theory to the calculation of convergent expansions of four thermonuclear functions in powers of the dimensionless Sommerfeld parameter. Some of these expansions also involve logarithmic terms in the dimensionless Sommerfeld parameter. Accurate error bounds are given at any order of the approximation.

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1. Introduction

The energy released by stars is due to nuclear reactions that occur near their centres where the motion of nuclei is in thermal equilibrium. The state of stellar matter is such that only the lightest elements contribute to these reactions because of the Coulomb repulsion between nuclei (see [5] and references therein for a full explanation of this phenomenon). Charged particle reaction rates for high-temperature and low-density thermonuclear plasma in the cosmological and stellar nucleosynthesis depend strongly on the penetrability of the Coulomb barrier and the velocity distribution of the reacting particles [4]. In fact, the reaction rate r_{ij} of reacting particles i and j, in the case of nonrelativistic nuclear reactions (low energy) taking place in a nondegenerate environment, is usually expressed as $r_{ij} = n_i n_j \langle \sigma v \rangle$, where n_i and

n_j are the particle number densities of the reacting particles i and j respectively [2, 6–9]. The symbol $\langle \sigma v \rangle$ represents the reaction probability integral [9, 11]:

$$\langle \sigma v \rangle \equiv \sqrt{\frac{8}{\pi \mu \kappa^3 T^3}} \int_0^\infty E f(E) \sigma(E) \, \mathrm{d}E \tag{1}$$

where μ is the reduced mass of the reacting particles, T is the temperature, κ is the Boltzmann constant, $\sigma(E)$ is the reaction cross section and $f(E)$ is the distribution of energy of the reacting particles.

Theoretical considerations and experimental data suggest several possible forms for the cross section $\sigma(E)$ and the energy distribution $f(E)$ in the above formula. For example, for nonresonant nuclear reactions, the cross section has the form [7, 9]

$$\sigma(E) = \frac{S_0 + S_1 E + S_2 E^2}{E} \exp\left\{ -2\pi \sqrt{\frac{\mu}{2}} \frac{z_i z_j e^2}{h\sqrt{E}} \right\}$$

where S_0, S_1 and S_2 are experimental constants, h is Planck's constant, e is the electron charge and $z_i e$ and $z_j e$ are the charges of the reacting particles. For isotropic Maxwell–Boltzmann kinetic-energy distributions, the distribution of energy $f(E)$ has the form

$$f(E) = \mathrm{e}^{-E/kT}.$$

After introducing the cross section and energy distribution in (1), we have

$$\langle \sigma v \rangle = \sqrt{\frac{8}{\pi \mu}} \sum_{j=0}^{2} \frac{S_j}{(\kappa T)^{1/2-j}} \int_0^\infty y^j \, \mathrm{e}^{-y} \, \mathrm{e}^{-\tilde{z}/\sqrt{y}} \, \mathrm{d}y$$

where $y = E/(\kappa T)$ is a dimensionless integration variable and $\tilde{z} \equiv 2\pi \sqrt{\frac{\mu}{2\kappa T}} \frac{z_i z_j e^2}{h}$ is the dimensionless Sommerfeld parameter. For light reacting particles (typically ^{3}He or ^{4}He) this parameter is of order $\tilde{z} \sim 10^{-4} T^{-1/2}$, that is, $0 < \tilde{z} \ll 1$ unless T is near the absolute zero. Therefore, the evaluation of reaction rates, in the standard nonresonant Maxell–Boltzmann case, requires the calculation (or approximation) of the integrals $\int_0^\infty y^j \, \mathrm{e}^{-y} \, \mathrm{e}^{-\tilde{z}/\sqrt{y}} \, \mathrm{d}y$, $j = 1, 2, 3$, where $\tilde{z}$ is a small parameter. Deviations from the ideal physical situation described above require a more general integral of the following form [9].

Nonresonant case:

$$I_1(\tilde{z}) \equiv \int_0^\infty y^\nu \, \mathrm{e}^{-ay} \, \mathrm{e}^{-\tilde{z} y^{-1/\rho}} \, \mathrm{d}y \qquad \nu \in \mathbb{R} \quad a > 0 \quad \tilde{z} > 0 \quad \rho \in \mathbb{Q}^+.$$

Physical situations different from the ideal nonresonant Maxell–Boltzmann case translate into modifications of the cross section $\sigma(E)$ for the reacting particles and/or their energy distribution $f(E)$. Then, the computation of $\langle \sigma v \rangle$ requires the calculation of integrals different from $I_1(\tilde{z})$. The integrals to be evaluated in these nonstandard physical situations are listed below.

If the thermonuclear fusion plasma is not in thermodynamic equilibrium and there is a cut-off of the high-energy tail of the Maxwell–Boltzmann distribution, the thermonuclear function to be evaluated is [2, 9] as follows.

Nonresonant case with high-energy cut-off:

$$I_2(\tilde{z}) \equiv \int_0^d y^\nu \, \mathrm{e}^{-ay} \, \mathrm{e}^{-\tilde{z} y^{-\rho}} \, \mathrm{d}y \qquad \nu \in \mathbb{R} \quad a > 0 \quad \tilde{z} > 0 \quad \rho \in \mathbb{Q}^+ \quad d \in \mathbb{R}^+.$$

If due to plasma effects, a depletion of the Maxwell–Boltzmann distribution has to be taken into account, the thermonuclear function is [2, 9, 10]

Nonresonant case with depleted tail:

$$I_3(\tilde{z}) \equiv \int_0^\infty y^\nu \, e^{-ay} \, e^{-by^\delta} \, e^{-\tilde{z}y^{-\rho}} \, dy \qquad \nu \in \mathbb{R} \quad a > 0 \quad b > 0 \quad \tilde{z} > 0 \quad \rho, \delta \in \mathbb{Q}^+.$$

The Coulomb barrier seen by a reacting particle in dense ionized matter may be modified by the surrounding cloud of electrons. The electron screening effects for the reacting particles modify the cross section and then the thermonuclear function is [2, 7, 9, 11] as follows.

Screened nonresonant case:

$$I_4(\tilde{z}) \equiv \int_0^\infty y^\nu \, e^{-ay} \, e^{-\tilde{z}(y+b)^{-\rho}} \, dy \qquad \nu \in \mathbb{R} \quad a > 0 \quad \tilde{z} > 0 \quad \rho \in \mathbb{Q}^+ \quad b > 0.$$

If the cross section has a broad single resonance, it can be calculated using the Breit–Wigner formula. Then, the thermonuclear function to be evaluated is [8, 9] as follows.

Resonant case:

$$I_5(\tilde{z}) \equiv \int_0^\infty \frac{y^\nu \, e^{-ay} \, e^{-\tilde{z}y^{-\rho}}}{(b-y)^2 + g^2} \, dy \qquad \nu, b, g \in \mathbb{R} \quad a > 0 \quad \tilde{z} > 0 \quad \rho \in \mathbb{Q}^+ \quad g \neq 0.$$

A depletion of the tail of the Maxwell–Boltzmann distribution in the presence of a resonance leads to the thermonuclear function [9] as follows.

Resonant case with depleted tail:

$$I_6(\tilde{z}) \equiv \int_0^\infty \frac{y^\nu \, e^{-ay-by^\delta} \, e^{-\tilde{z}y^{-\rho}}}{(b-y)^2 + g^2} \, dy \qquad \nu, b, g \in \mathbb{R} \quad a > 0 \quad \tilde{z} > 0 \quad \rho, \delta \in \mathbb{Q}^+ \quad g \neq 0.$$

Some analytical approximations to $I_1(\tilde{z})$ can be found in [3, 4] and [11]. But the more exhaustive investigation in the calculation of the integrals $I_1(\tilde{z}), \ldots, I_6(\tilde{z})$ has been developed by Haubold, Mathai and Anderson [2, 6–9]: they write these integrals in terms of G or H functions of some positive power of $\tilde{z}$. Then, from the known expansions of these two functions in powers of their argument [13, 14], these authors obtain convergent expansions of these thermonuclear functions for small $\tilde{z}$. From the asymptotic approximations of G and H for large values of their argument, they also obtain asymptotic approximations of $I_1(\tilde{z}), \ldots, I_6(\tilde{z})$ for large values of $\tilde{z}$. The thermonuclear function $I_1(\tilde{z})$ may be written as a G function with variable $\tilde{z}^2$ and then it can be expanded as a power series of $\tilde{z}^2$ [8]. But each of the remaining integrals $I_2(\tilde{z}), \ldots, I_6(\tilde{z})$ is written as an infinite series of G functions, which in turn becomes a double or triple series in powers of $\tilde{z}$.

The first purpose of this work is to obtain asymptotic (in fact convergent) expansions of the six thermonuclear functions in the form of a simple series of powers of $\tilde{z}$. This more simple analytical expression clarifies the analytic properties of the thermonuclear functions as functions of $\tilde{z}$ and simplifies their numerical evaluation. Of course, the expansion of $I_1(\tilde{z})$ obtained here agrees with that obtained by Haubold, Mathai and Anderson [2, 7–9]. Moreover, we will obtain accurate error bounds for the remainder at any order of the approximation. To face these challenges we require the distributional method to approximate Laplace transforms near the origin ([17], chapter 6). Then, the first step is to write every thermonuclear function $I_1(\tilde{z}), \ldots, I_6(\tilde{z})$ in the form of a Laplace transform:

$$\mathcal{L}_f(z) \equiv \int_0^\infty f(t) \, e^{-zt} \, dt \tag{2}$$

with a parameter z proportional to $\tilde{z}$ or some positive power of $\tilde{z}$.

With the change of variable $y = \tilde{z}^\rho t$, $I_1(\tilde{z})$ reads

$$I_1(\tilde{z}) = \tilde{z}^{\rho(\nu+1)} \mathcal{L}_{f_1}(a\tilde{z}^\rho) \qquad f_1(t) \equiv t^\nu \, e^{-t^{-1/\rho}}. \tag{3}$$

With the change of variable $y = d(t+1)^{-1/\rho}$, $I_2(\tilde{z})$ reads

$$I_2(\tilde{z}) = \frac{\mathrm{d}^{\nu+1}}{\rho}\,\mathrm{e}^{-\tilde{z}d^{-\rho}}\mathcal{L}_{f_2}(\mathrm{d}^{-\rho}\tilde{z}) \qquad f_2(t) \equiv (1+t)^{-(\nu+1)/\rho-1}\,\mathrm{e}^{-ad(1+t)^{-1/\rho}}. \tag{4}$$

With the change of variable $y = t^{-1/\rho}$, $I_3(\tilde{z})$ reads

$$I_3(\tilde{z}) = \frac{1}{\rho}\mathcal{L}_{f_3}(\tilde{z}) \qquad f_3(t) \equiv t^{-(\nu+1)/\rho-1}\,\mathrm{e}^{-at^{-1/\rho}-bt^{-\delta/\rho}}. \tag{5}$$

With the change of variable $y = u^{-1/\rho} - b$, $I_4(\tilde{z})$ reads

$$I_4(\tilde{z}) = \frac{\mathrm{e}^{ab}}{\rho}\int_0^{b^{-\rho}} u^{-1/\rho-1}(u^{-1/\rho}-b)^{\nu}\,\mathrm{e}^{-\tilde{z}u}\,\mathrm{e}^{-au^{-1/\rho}}\,\mathrm{d}u.$$

Writing this integral as the difference $\int_0^{b^{-\rho}} = \int_0^{\infty} - \int_{b^{-\rho}}^{\infty}$ and performing the change of variable $u = t + b^{-\rho}$ in the second integral, we have

$$I_4(\tilde{z}) = \frac{\mathrm{e}^{ab}}{\rho}\left[\mathcal{L}_{f_4^{(1)}}(\tilde{z}) - \mathrm{e}^{-b^{-\rho}\tilde{z}}\mathcal{L}_{f_4^{(2)}}(\tilde{z})\right]$$
$$f_4^{(1)}(t) \equiv t^{-1/\rho-1}(t^{-1/\rho}-b)^{\nu}\,\mathrm{e}^{-at^{-1/\rho}}$$
$$f_4^{(2)}(t) \equiv f_4^{(1)}(t+b^{-\rho}).$$

With the change of variable $y = t^{-1/\rho}$, $I_5(\tilde{z})$ reads

$$I_5(\tilde{z}) = \frac{1}{\rho}\mathcal{L}_{f_5}(\tilde{z}) \qquad f_5(t) \equiv \frac{t^{-(\nu+1)/\rho-1}\,\mathrm{e}^{-at^{-1/\rho}}}{(b-t^{-1/\rho})^2+g^2}. \tag{6}$$

With the change of variable $y = t^{-1/\rho}$, $I_6(\tilde{z})$ reads

$$I_6(\tilde{z}) = \frac{1}{\rho}\mathcal{L}_{f_6}(\tilde{z}) \qquad f_6(t) \equiv \frac{t^{-(\nu+1)/\rho-1}\,\mathrm{e}^{-at^{-1/\rho}-bt^{-\delta/\rho}}}{(b-t^{-1/\rho})^2+g^2}.$$

Asymptotic expansions of Laplace transform (2) near the origin (small z) of functions $f(t)$ that (i) are locally integrable on $[0,\infty)$ and (ii) have an asymptotic expansion in integer powers of t^{-1} have been fully investigated by Wong ([17], chapter 6). But for $\rho^{-1}\notin\mathbb{N}$, the functions $f_1(t),\ldots,f_6(t)$ above do not admit an asymptotic expansion in integer powers of t^{-1}, but in integer powers of $t^{-1/\rho}$. Then, Wong's method is not directly applicable.

The second purpose of this paper is then: (i) the generalization of Wong's distributional method to obtain formal asymptotic expansions of Laplace transform (2) near the origin of locally integrable functions $f(t)$ on $[0,\infty)$ which have an asymptotic expansion in negative rational powers of t, (ii) show the asymptotic character of these expansions and (iii) obtain error bounds at any order of the approximation for a large family of Laplace transforms which include the thermonuclear functions.

The paper is organized as follows. In section 2 we introduce the above mentioned generalization of Wong's method. As an illustration of the power of this method, we obtain in section 3 convergent (and asymptotic) expansions in powers of the dimensionless Sommerfeld parameter $\tilde{z}$ of $I_1(\tilde{z})$, $I_2(\tilde{z})$, $I_3(\tilde{z})$ and $I_5(\tilde{z})$ (for I_2, I_3 and I_5 only in the case $\rho\in\mathbb{N}$). The computations of the expansions for I_2, I_3, I_4, I_5 and I_6 in the general case $\rho\in\mathbb{Q}^+$ are more involved and we relegate them to a forthcoming paper. We also obtain error bounds for the expansions mentioned. Several numerical examples are shown as illustrations in section 4.

2. Distributional approach for Laplace transforms

In the following, $f(t)$ denotes a locally integrable function on $[0, \infty)$ which satisfies

$$f(t) = \sum_{k=K}^{n-1} \frac{a_k}{t^{k/s+\beta}} + f_n(t) \qquad s \in \mathbb{N} \quad K \in \mathbb{Z} \quad 0 < \operatorname{Re}\beta \leqslant 1/s \tag{7}$$

$\{a_k, k = K, K+1, K+2, \ldots\}$ is a sequence of complex numbers and $f_n(t) = \mathcal{O}(t^{-n/s-\beta})$ when $t \to \infty$. (In chapter 6 of [17], only the case $s = 1$ and $\beta \in \mathbb{R}$, $0 < \beta \leqslant 1$, is considered.) In the following we use the notation introduced in [17]. Empty sums must be understood as zero.

2.1. *Asymptotic expansion of $L_f(z)$ for small z*

We denote by $\mathcal{S}$ the space of rapidly decreasing functions on $[0, \infty)$ and by $\langle \mathbf{\Lambda}, \varphi \rangle$ the image of a tempered distribution $\mathbf{\Lambda}$ acting over a function $\varphi \in \mathcal{S}$. Since $f(t)$ in (7) is a locally integrable function on $[0, \infty)$, it defines a distribution $\mathbf{f}$:

$$\langle \mathbf{f}, \varphi \rangle \equiv \int_0^\infty f(t)\varphi(t)\, \mathrm{d}t.$$

The distributions associated with $t^{-k-\beta}$, $k = 0, 1, 2, \ldots, n-1$, are given by [17, chapter 5]

$$\langle \mathbf{t^{-k-}}^{\beta}, \varphi \rangle \equiv \frac{1}{(\beta)_k} \int_0^\infty t^{-\beta} \varphi^{(k)}(t)\, \mathrm{d}t \qquad \text{if} \quad 0 < \operatorname{Re}\beta < 1$$

$$\langle \mathbf{t^{-k-}}^{\beta}, \varphi \rangle \equiv \frac{1}{(i \operatorname{Im}\beta)_{k+1}} \int_0^\infty t^{-i \operatorname{Im}\beta} \varphi^{(k+1)}(t)\, \mathrm{d}t \qquad \text{if} \quad 1 \neq \beta = 1 + i \operatorname{Im}\beta$$

where $(\beta)_k$ denotes the Pochhammer's symbol of β, and

$$\langle \mathbf{t^{-k-1}}, \varphi \rangle \equiv -\frac{1}{k!} \int_0^\infty \log(t) \varphi^{(k+1)}(t)\, \mathrm{d}t.$$

To assign a distribution to the function $f_n(t)$ introduced in (7), we first define recursively the kth integral $f_{n,k}(t)$ of $f_n(t)$ by $f_{n,0}(t) \equiv f_n(t)$ and, for $k = 0, 1, 2, \ldots, n/s - 1$ (with n being a multiple of s),

$$f_{n,k+1}(t) \equiv -\int_t^\infty f_{n,k}(u)\, \mathrm{d}u = \frac{(-1)^{k+1}}{k!} \int_t^\infty (u-t)^k f_n(u)\, \mathrm{d}u. \tag{8}$$

For $\beta \neq 1/s$, it is trivial to show that $f_{n,n/s}(t)$ is bounded on $[0, T]$ for any $T > 0$ and is $\mathcal{O}(t^{-\beta})$ as $t \to \infty$. For $\beta = 1/s$ we have $f_{n,n/s}(t) = \mathcal{O}(t^{-1/s})$ as $t \to \infty$ and $f_{n,n/s}(t) = \mathcal{O}(\log(t))$ as $t \to 0^+$. Therefore, for $0 < \operatorname{Re}\beta \leqslant 1/s$ we can define the distribution associated with $f_n(t)$ by

$$\langle \mathbf{f_n}, \varphi \rangle \equiv (-1)^{n/s} \langle \mathbf{f_{n,n/s}}, \varphi^{(n/s)} \rangle \equiv (-1)^{n/s} \int_0^\infty f_{n,n/s}(t) \varphi^{(n/s)}(t)\, \mathrm{d}t.$$

Once we have assigned a distribution to each function involved in the identity (7), we are interested in finding an identity between these distributions. In fact, this relation is established in the following two lemmas.

Lemma 1. *For $0 < \operatorname{Re}\beta < 1/s$, $s \in \mathbb{N}$, $n \geqslant K + 1$, and $n = s, 2s, 3s, \ldots$, the identity*

$$\mathbf{f} = \sum_{k=K}^{n-1} a_k \mathbf{t^{-k/s-}}^{\beta} + \sum_{k=0}^{n/s-1} \frac{(-1)^k}{k!} M[f; k+1] \delta^{(\mathbf{k})} + \mathbf{f_n}$$

holds for any function $\varphi \in \mathcal{S}$*, where* δ *is the delta distribution in the origin and* $M[f; k+1]$ *denotes de Mellin transform of* $f(t)$*:* $\int_0^\infty t^k f(t)\,\mathrm{d}t$*, or its analytic continuation.*

Proof. It is a trivial generalization of lemma 1 in [12] from real to complex values of β. □

Lemma 2. *For* $\mathrm{Re}\,\beta = 1/s$, $s \in \mathbb{N}$, $n \geqslant K+1$ *and* $n = s, 2s, 3s, \ldots,$ *the identity*

$$\mathbf{f} = \sum_{k=K}^{n-1} a_k \mathbf{t}^{-\mathbf{k/s}-\beta} + \sum_{k=0}^{n/s-1} b_{(k+1)s}\delta^{(\mathbf{k})} + \mathbf{f_n}$$

holds for any rapidly decreasing function $\varphi \in \mathcal{S}$*, where, for* $n = 0, s, 2s, \ldots$

$$b_{n+s} = \frac{(-1)^{n/s}}{(n/s)!}\left[\int_0^1 t^{n/s} f_n(t)\,\mathrm{d}t + \int_1^\infty t^{n/s} f_{n+s}(t)\,\mathrm{d}t + \sum_{k=0}^{s-2} \frac{(n/s)!\,a_{n+k}}{(k/s+\beta-1)_{n/s+1}} + \sum_{k=1}^{n/s+1}\sum_{j=n}^{n+s-1} \frac{(n/s-k+2)_{k-1} a_j}{(j/s+\beta-k)_k}\right] \tag{9}$$

$$= \frac{(-1)^{n/s}}{(n/s)!}\left\{M[f; n/s+1] + \frac{a_{n+s-1}}{1/s-\beta} + \sum_{k=0}^{s-2}\left[\frac{(n/s)!}{(k/s+\beta-1)_{n/s+1}} - \frac{1}{k/s+\beta-1}\right]a_{n+k} + \sum_{k=1}^{n/s+1}\sum_{j=n}^{n+s-1} \frac{(n/s-k+2)_{k-1} a_j}{(j/s+\beta-k)_k}\right\} \tag{10}$$

if $\mathrm{Im}\,\beta \neq 0$*, or*

$$b_{n+s} = \frac{(-1)^{n/s}}{(n/s)!}\left[\int_0^1 t^{n/s} f_n(t)\,\mathrm{d}t + \int_1^\infty t^{n/s} f_{n+s}(t)\,\mathrm{d}t + \sum_{k=0}^{s-2} \frac{(n/s)!\,a_{n+k}}{((k+1)/s-1)_{n/s+1}} + \sum_{k=1}^{n/s}\sum_{j=n}^{n+s-1} \frac{(n/s-k+2)_{k-1} a_j}{((j+1)/s-k)_k}\right] \tag{11}$$

$$= \frac{(-1)^{n/s}}{(n/s)!}\left\{\lim_{z\to n/s}\left[M[f; z+1] + \frac{a_{n+s-1}}{z-n/s}\right] + \sum_{k=0}^{s-2}\left[\frac{(n/s)!}{((k+1)/s-1)_{n/s+1}} - \frac{1}{(k+1)/s-1}\right]a_{n+k} + \sum_{k=1}^{n/s}\sum_{j=n}^{n+s-1} \frac{(n/s-k+2)_{k-1} a_j}{((j+1)/s-k)_k}\right\} \tag{12}$$

if $\mathrm{Im}\,\beta = 0$.

Proof. Let $f_0(t) \equiv f(t) - \sum_{k=K}^{-1} a_k t^{-k/s-\beta}$. Then, for $n = 0, s, 2s, \ldots$

$$f_{n+s}(t) = f_n(t) - \sum_{k=n}^{n+s-1} \frac{a_k}{t^{k/s+\beta}}$$

and

$$f_{n+s,n/s}(t) = f_{n,n/s}(t) - (-1)^{n/s} \sum_{k=0}^{s-1} \frac{a_{n+k}}{(k/s+\beta)_{n/s}} \frac{1}{t^{k/s+\beta}}.$$

From this it follows, by integration, that

$$\int_0^t f_{n,n/s}(u)\,\mathrm{d}u = f_{n+s,n/s+1}(t) + (-1)^{n/s} a_{n+s-1} g_{n/s}(\beta, t) - (-1)^{n/s} \sum_{k=0}^{s-2} \frac{a_{n+k} t^{1-(k/s+\beta)}}{(k/s+\beta-1)_{n/s+1}} + b_{n+s}$$

where

$$g_n(\beta, t) \equiv \begin{cases} \log t/n! & \text{if} \quad \mathrm{Im}\,\beta = 0 \\ -t^{-i\,\mathrm{Im}\,\beta}/(i\,\mathrm{Im}\,\beta)_{n+1} & \text{if} \quad \mathrm{Im}\,\beta \neq 0 \end{cases}$$

and we have defined the integration constant

$$b_{n+s} \equiv -\lim_{t\to 0}[f_{n+s,n/s+1}(t) + (-1)^{n/s} a_{n+s-1} g_{n/s}(\beta, t)].$$

From here, the proof is the same as the proof of lemma 2 in [12], replacing $\log t$ by $(n/s)!g_{n/s}(\beta, t)$ and $(k+1)/s$ by $k/s + \beta$ in that proof. □

To apply lemmas 1 and 2 to the integral (2) we choose $\varphi(t) = \mathrm{e}^{-zt}$, which belong to $\mathcal{S}$ for $\mathrm{Re}\,z \geqslant 0$. We will also need the following lemma.

Lemma 3. *Let $f(t)$ verify (7). Then, for $0 < \mathrm{Re}\,\beta \leqslant 1$, $k = 0, 1, 2, \ldots$ and $n = s, 2s, 3s, \ldots$ the following identities hold:*

$$\langle \mathbf{f}, \varphi\rangle = \int_0^\infty f(t)\,\mathrm{e}^{-zt}\,\mathrm{d}t$$

$$\langle \delta^{(k)}, \varphi\rangle = z^k$$

$$\langle \mathbf{t}^{-\mathbf{k}/\mathbf{s}-\beta}, \varphi\rangle = \Gamma(1 - k/s - s) z^{k/s+\beta-1}$$

$$\langle \mathbf{t}^{-\mathbf{k}-\mathbf{1}}, \varphi\rangle = \frac{(-1)^{k+1}}{k!}(\gamma + \log z) z^k$$

$$\langle \mathbf{f}_{\mathbf{n},\mathbf{n}/\mathbf{s}}, \varphi^{(n/s)}\rangle = (-1)^{n/s} z^{n/s} \int_0^\infty f_{n,n/s}(t)\,\mathrm{e}^{-zt}\,\mathrm{d}t.$$

Proof. It is a straightforward generalization of the analogue equations given in [17, chapter 6, section 5] from real to complex values of β. □

With these preparations, we are now able to obtain asymptotic expansions of the integrals (2) for small z in the following two theorems.

Theorem 1. *Let $f(t)$ be a locally integrable function on $[0, \infty)$ which satisfies (7) with $\beta \neq 1/s$. Then, for $\mathrm{Re}\,z > 0$, and $n = s, 2s, 3s, \ldots$*

$$\int_0^\infty f(t)\,\mathrm{e}^{-zt}\,\mathrm{d}t = \sum_{k=K}^{-1} a_k \Gamma(1 - k/s - \beta) z^{k/s+\beta-1} + \sum_{k=0}^{n/s-1} z^k \left[M_k + \sum_{j=0}^{s-1} \Gamma(1 - k - j/s - \beta) a_{sk+j} z^{\beta+j/s-1} \right] + R_{n,s}(z) \qquad (13)$$

where

$$M_k \equiv \begin{cases} (-1)^k M[f; k+1]/k! & \text{if} \quad \mathrm{Re}\,\beta \neq 1/s \\ b_{(k+1)s} & \text{if} \quad \mathrm{Re}\,\beta = 1/s \end{cases}$$

and, for $k = 0, 1, 2, \ldots$, the coefficients $b_{(k+1)s}$ are given by (9) or (10).

The remainder term is defined by

$$R_{n,s}(z) \equiv z^{n/s} \int_0^\infty f_{n,n/s}(t)\, \mathrm{e}^{-zt}\, \mathrm{d}t \tag{14}$$

where $f_{n,n/s}(t)$ *is defined in (8).*

Proof. For $\operatorname{Re}\beta \neq 1/s$ it follows from lemmas 1 and 3. For $\operatorname{Re}\beta = 1/s$ it follows from lemmas 2 and 3, and using formula

$$\langle t^{-k/s-\beta}, \varphi\rangle = \frac{1}{(\nu)_{[k/s]}} \langle t^{-\nu}, \varphi^{([k/s])}\rangle \qquad \text{if} \quad k/s \notin \mathbb{N} \tag{15}$$

with $\nu \equiv k/s + \beta - [k/s]$. □

Theorem 2. *Let* $f(t)$ *be a locally integrable function on* $[0, \infty)$ *which satisfies (7) with* $\beta = 1/s$. *Then, for* $\operatorname{Re} z > 0$ *and* $n = s, 2s, 3s, \ldots$

$$\begin{aligned}\int_0^\infty f(t)\, \mathrm{e}^{-zt}\, \mathrm{d}t = {} & \sum_{k=K}^{-1} a_k \Gamma(1-(k+1)/s) z^{(k+1)/s-1} \\ & + \sum_{k=0}^{n/s-1} z^k \left[a_{s(k+1)-1} \frac{(-1)^{k+1}}{k!} (\log z + \gamma) + b_{(k+1)s} \right. \\ & \left. \times \sum_{j=0}^{s-2} a_{sk+j} \Gamma(1-k-(j+1)/s) z^{(j+1)/s-1} \right] + R_{n,s}(z)\end{aligned} \tag{16}$$

where γ *is the Euler constant and, for* $k = 0, 1, 2, \ldots,$ *the coefficients* $b_{(k+1)s}$ *are given by (11) or (12).*

The remainder term $R_{n,s}(z)$ *is given in (14).*

Proof. From lemmas 2 and 3 and formula

$$\langle t^{-(k+1)/s}, \varphi_\eta\rangle = \frac{-1}{((k+1)/s-1)!} \langle \log t, \varphi_\eta^{((k+1)/s)}\rangle \qquad \text{if} \quad (k+1)/s \in \mathbb{N}.$$

or formula (15) with $\beta = 1/s$ if $(k+1)/s \notin \mathbb{N}$, we immediately obtain formula (16) with $R_{n,s}(z)$ given in (14). □

2.2. Error bounds

In the following theorem we show that expansions (13) and (16) are not only formal, but also true asymptotic expansions for small z.

Theorem 3. *In the region of validity of expansions (13) and (16), the remainder term* $R_{n,s}(z)$ *verifies*

$$|R_{n,s}(z)| \leqslant C_n |z|^{n/s+\operatorname{Re}\beta-1} \tag{17}$$

if $s > 1$ *or* $0 < \operatorname{Re}\beta < 1$ *and*

$$|R_{n,s}(z)| \leqslant C_n |z|^n |\log z| \tag{18}$$

if $s = \operatorname{Re}\beta = 1$, *where the constants* C_n *are independent of* $|z|$ *(it may depend on the remaining parameters of the problem).*

Proof. On one hand, $f_n(t) = \mathcal{O}(t^{-n/s-\beta})$ for $t \to \infty$ (with $0 < \operatorname{Re}\beta \leqslant 1/s$) then, there is a certain $t_0 \in (0, \infty)$ and a constant $C_{1,n}$ such that $|f_n(t)| \leqslant C_{1,n} t^{-n/s-\operatorname{Re}\beta}\ \forall\ t \in [t_0, \infty)$.

Introducing this bound in definition (8) of $f_{n,n/s}(t)$ we obtain the bound $|f_{n,n/s}(t)| \leqslant C_{2,n}t^{-\mathrm{Re}\,\beta}\ \forall\ t \in [t_0, \infty)$, where $C_{2,n}$ is a certain positive constant and $0 < \mathrm{Re}\,\beta \leqslant 1/s$. On the other hand, $f_{n,n/s}(t)$ is bounded on any compact interval in $[0, \infty)$ for $\beta \neq 1/s$ and $f_{n,n/s}(t)$ is bounded on any compact interval in $(0, \infty)$ and $\mathcal{O}(\log t)$ as $t \to 0^+$ for $\mathrm{Re}\,\beta = s = 1$. Then, $\forall\ t \in [0, t_0]$, $|f_{n,n/s}(t)| \leqslant C_{3,n}t^{-\mathrm{Re}\,\beta}$ for $0 < \mathrm{Re}\,\beta < 1$ and $|f_{n,n}(t)| \leqslant C_{3,n}(|\log t| + 1)$ for $\mathrm{Re}\,\beta = s = 1$, where $C_{3,n}$ is a certain positive constant.

If we divide the integration interval $[0, \infty)$ in the definition (14) of $R_{n,s}(z)$ at the point t_0 and introduce these bounds in each of the intervals $[0, t_0]$ and $[t_0, \infty)$, we obtain bounds (17) and (18). □

The bounds given in theorem 3 are not useful for numerical computations unless we are able to calculate the constants C_n in terms of the dates of the problem. The property $f_n(t) = \mathcal{O}(t^{-n/s-\beta})$ when $t \to \infty$ implies that $\exists\ t_0 > 0$ and $c_n > 0, |f_n(t)| \leqslant c_nt^{-n/s-\mathrm{Re}\,\beta}\ \forall\ t \in [t_0, \infty)$. The following two propositions show that, if the bound $|f_n(t)| \leqslant c_nt^{-n/s-\mathrm{Re}\,\beta}$ holds $\forall\, t \in [0, \infty)$ then, the constants C_n in theorem 3 can be calculated in terms of the constant c_n.

Proposition 1. *If, for $s > 1$ or $0 < \mathrm{Re}\,\beta < 1$, the remainder $f_n(t)$ in expansion (7) of the function $f(t)$ satisfies the bound $|f_n(t)| \leqslant c_nt^{-n/s-\mathrm{Re}\,\beta}\ \forall\ t \in [0, \infty)$ for some positive constant c_n, then the remainder $R_{n,s}(z)$ in expansion (13) satisfies*

$$|R_{n,s}(z)| \leqslant \frac{c_n\pi}{\sin(\pi\mathrm{Re}\,\beta)\Gamma(n/s + \mathrm{Re}\,\beta)}|z|^{n/s+\mathrm{Re}\,\beta-1}. \tag{19}$$

Proof. Introducing the bound $|f_n(t)| \leqslant c_nt^{-n/s-\mathrm{Re}\,\beta}$ in the definition (8) of $f_{n,n/s}(t)$ we obtain

$$|f_{n,n/s}(t)| \leqslant \frac{c_n\Gamma(\mathrm{Re}\,\beta)}{\Gamma(n/s + \mathrm{Re}\,\beta)t^{\mathrm{Re}\,\beta}} \qquad \forall\, t \in [0, \infty).$$

Introducing this bound in definition (14) of $R_{n,s}(z)$ we obtain (19). □

Proposition 2. *If, for $s = \mathrm{Re}\,\beta = 1$, each remainder $f_n(t)$ in expansion (7) of the function $f(t)$ satisfies the bound $|f_n(t)| \leqslant c_nt^{-n-1}, \forall\ t \in [0, \infty)$ for some positive constant c_n, then the remainder $R_{n,s}(z)$ in expansions (13) satisfies*

$$|R_{n,1}(z)| \leqslant \frac{\bar{c}_n\pi}{\Gamma(n + 1/2)}|z|^{n-1/2} \tag{20}$$

where $\bar{c}_n \equiv Max\{c_n, c_{n-1} + |a_{n-1}|\}$ and

$$|R_{n,1}(z)| \leqslant \left\{\frac{1}{(n-1)!}[(c_{n-1} + |a_{n-1}|)\varepsilon + c_n] + \frac{c_n}{n!}\Theta(z, \varepsilon)\right\}|z|^n \tag{21}$$

where ε is an arbitrary positive number, and

$$\Theta(z, \varepsilon) \equiv \begin{cases} \mathrm{e}^{-1} - \log(\varepsilon\mathrm{Re}\,z) & \text{if} \quad \varepsilon\mathrm{Re}\,z < 1 \\ \dfrac{\mathrm{e}^{-\varepsilon\mathrm{Re}\,z}}{\varepsilon\mathrm{Re}\,z} & \text{if} \quad \varepsilon\mathrm{Re}\,z \geqslant 1. \end{cases} \tag{22}$$

For small enough z and fixed n, the optimum value for ε is given approximately by

$$\varepsilon = \frac{c_n}{n(c_{n-1} + |a_{n-1}|)}. \tag{23}$$

Proof. From $|f_{n-1}(t)| \leqslant c_{n-1}t^{-n}\ \forall\ t \in [0, \infty)$ and $f_n(t) = f_{n-1}(t) - a_{n-1}t^{-n}$ we obtain $|f_n(t)| \leqslant (c_{n-1}+|a_{n-1}|)t^{-n}\ \forall\, t \in [0, \infty)$. To obtain bound (21) we divide the integral defining $f_{n,n}(t)$ in (8) by a fixed point $u = \varepsilon \geqslant t$ and use the bound $|f_n(t)| \leqslant (c_{n-1} + |a_{n-1}|)t^{-n}$ in

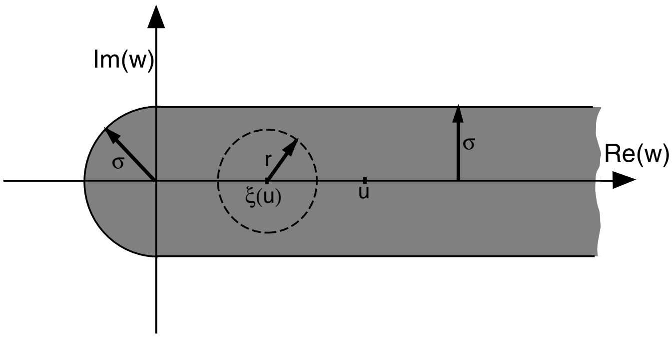

Figure 1. Analyticity region W for the function $g(w)$ considered in lemma 4. The integration variable u in (8) is real and unbounded and therefore, the analyticity region for $g(w)$ in (26) must contain the positive real axis. The circle of radius r centred at $\xi(u)$, with $0 < \xi(u) < u$, used in the proof of lemma 4 must be contained in this region and therefore, $r < \sigma$.

the integral over $[t, \varepsilon]$ and the bound $|f_n(t)| \leqslant c_n t^{-n-1}$ in the integral over $[\varepsilon, \infty)$. Using $u - t \leqslant u$ in the integral over $[t, \varepsilon]$ we obtain

$$|f_{n,n}(t)| \leqslant \frac{1}{(n-1)!}\left[(c_{n-1} + |a_{n-1}|)\log\left(\frac{\varepsilon}{t}\right) + \frac{c_n}{\varepsilon}\right] \qquad \forall\, t \in [0, \varepsilon] \quad \varepsilon > 0. \tag{24}$$

On the other hand, $\forall\, t \in [0, \infty)$ we introduce the bound $|f_n(t)| \leqslant c_n t^{-n-1}$ in the integral definition of $f_{n,n}(t)$ and perform the change of variable $u = tv$. We obtain

$$|f_{n,n}(t)| \leqslant \frac{c_n}{n!}\frac{1}{t} \qquad \forall\, t \in [0, \infty). \tag{25}$$

If we divide the integral on the right-hand side of (14) at the point $t = \varepsilon$ and use bound (25) in the integral over $[\varepsilon, \infty)$ and bound (24) in the integral over $[0, \varepsilon]$, we obtain (21) and (22). For small z and fixed n, this bound takes its optimum value, approximately, for ϵ given in (23).

Now we derive (20). From $|f_{n-1}(t)| \leqslant c_{n-1}t^{-n}$ and $f_n(t) = f_{n-1}(t) - a_{n-1}t^{-n}\,\forall\, t \in [0, \infty), n \in \mathbb{N}$, we obtain $|f_n(t)| \leqslant (c_{n-1} + |a_{n-1}|)t^{-n}\,\forall\, t \in [0, \infty)$. Then, we have both $|f_n(t)| \leqslant c_n t^{-n-1/2}$ if $t \geqslant 1$ and $|f_n(t)| \leqslant (c_{n-1} + |a_{n-1}|)t^{-n-1/2}$ if $t \leqslant 1$. Therefore, $|f_n(t)| \leqslant \bar{c}_n t^{-n-1/2}\,\forall\, t \in [0, \infty)$. Then, $f_n(t)$ satisfies the bound required in proposition 1 with $s = 1$, $\operatorname{Re}\beta = 1/2$ and c_n replaced by $\bar{c}_n$. Repeating now the calculations of the proof of proposition 1, we obtain (20). □

The following lemma introduces a family of functions $f(t)$ which verify the bound $|f_n(t)| \leqslant c_n t^{-n/s-\operatorname{Re}\beta}\,\forall\, t \in [0, \infty)$. Moreover, for these functions $f(t)$, the constants c_n can be easily obtained from $f(t)$.

Lemma 4. *Suppose that $f(t)$ verifies (7) and consider the function $g(u) \equiv u^{-\beta s} f(u^{-s}) - \sum_{k=K}^{-1} a_k u^k$. If $g(w)$ is a bounded analytic function in the region W of the complex w-plane comprised by the points situated at a distance $<\sigma$ from the positive real axis (see figure 1), then,*

$$|f_n(t)| \leqslant C r^{-n} t^{-n/s-\operatorname{Re}\beta}$$

where C is a bound of $|g(w)|$ in W and $0 < r < \sigma$.

Proof. From asymptotic expansion (7) and the Lagrange formula for the remainder in the Taylor expansion of $g(u)$ at $u = 0$, we have

$$g(u) = \sum_{k=0}^{n-1} a_k u^k + R_n(u)$$

where

$$R_n(u) = \frac{1}{n!} \left. \frac{\mathrm{d}^n g(u)}{\mathrm{d}u^n} \right|_{u=\xi} u^n \qquad \xi \in (0, u).$$

Using the Cauchy formula for the derivative of an analytic function,

$$\frac{\mathrm{d}^n g(u)}{\mathrm{d}u^n} = \frac{n!}{2\pi \mathrm{i}} \int_{\mathcal{C}} \frac{g(w)}{(w - \xi)^{n+1}} \, \mathrm{d}w \tag{26}$$

where $\mathcal{C}$ is a circle of radius r around ξ contained in the region W. Then, for fixed ξ and r, performing the change of variable $w = \xi + r\,\mathrm{e}^{\mathrm{i}\theta}$, and using $|g(\xi + r\,\mathrm{e}^{\mathrm{i}\theta})| \leqslant C$ for $\theta \in [0, 2\pi)$ with C independent of θ, r and ξ, we obtain the desired result. □

Lemma 5. *If expansion (7) verifies the error test, then*

$$|f_n(t)| \leqslant |a_n| t^{-n/s - \mathrm{Re}\,\beta} \qquad \text{and} \qquad |f_n(t)| \leqslant |a_{n-1}| t^{-(n-1)/s - \mathrm{Re}\,\beta}.$$

Proof. A proof of the first inequality can be found in [15, p 68]. The second inequality follows from the first one, from $\mathrm{sign}(f_n(t)) \neq \mathrm{sign}(f_{n-1}(t))$ and

$$f_n(t) = f_{n-1}(t) - \frac{a_{n-1}}{t^{(n-1)/s+\beta}}.$$

□

Corollary 1. *If $f(t)$ verifies the hypotheses of lemma 4, then $R_{n,s}(z)$ satisfies the bounds given in proposition 1 or 2 with $c_n = Cr^{-n}$. Moreover, the expansions given in theorems 1 and 2 are convergent.*

Corollary 2. *If expansion (7) of $f(t)$ verifies the error test, then $R_{n,s}(z)$ satisfies the bounds given in proposition 1 or 2 replacing c_n by $|a_n|$ and c_{n-1} by 0. Moreover, the expansions given in theorems 1 and 2 are convergent when* $\lim_{n\to\infty} a_n (ez)^{n/s} (n/s)^{1/2 - n/s - \mathrm{Re}\,\beta} = 0$.

3. Convergent expansions of thermonuclear functions

Asymptotic expansions in powers of $\tilde{z}$ of $I_1(\tilde{z}), \ldots, I_6(\tilde{z})$ may be obtained by applying theorem 1 or 2 to $\mathcal{L}_{f_1}(\tilde{z}), \ldots, \mathcal{L}_{f_6}(\tilde{z})$ respectively. Error bounds for these expansions and their convergence follow from proposition 1 or 2 or corollary 1 or 2.

We consider in this section the integrals $\mathcal{L}_{f_1}(\tilde{z})$, $\mathcal{L}_{f_2}(\tilde{z})$, $\mathcal{L}_{f_3}(\tilde{z})$ and $\mathcal{L}_{f_5}(\tilde{z})$ ($\mathcal{L}_{f_2}(\tilde{z})$, $\mathcal{L}_{f_3}(\tilde{z})$ and $\mathcal{L}_{f_5}(\tilde{z})$ only in the case $\rho \in \mathbb{N}$). Integrals $\mathcal{L}_{f_2}(\tilde{z})$, $\mathcal{L}_{f_3}(\tilde{z})$, $\mathcal{L}_{f_4}(\tilde{z})$, $\mathcal{L}_{f_5}(\tilde{z})$ and $\mathcal{L}_{f_6}(\tilde{z})$ in the general case $\rho \in \mathbb{Q}^+$ are relegated to a forthcoming paper.

3.1. Nonresonant case

We rewrite the rational number ρ defining the function $f_1(t)$ in (3) as $\rho = \bar{\rho}/\delta$, with $\bar{\rho}, \delta \in \mathbb{N}$, δ and $\bar{\rho}$ being relative primes:

$$f_1(t) = t^{\nu}\,\mathrm{e}^{-t^{-\delta/\bar{\rho}}}. \tag{27}$$

Then, this function satisfies

$$f_1(t) = \sum_{k=K}^{n-1} \frac{A_{k-K}}{t^{k/\bar{\rho}+\beta}} + f_n(t) \qquad A_k \equiv \begin{cases} \dfrac{(-1)^{(k/\delta)}}{(k/\delta)!} & \text{if} \quad k = \dot{\delta} \\ 0 & \text{if} \quad k \neq \dot{\delta} \end{cases} \tag{28}$$

where $K \equiv Int(-\bar{\rho}\nu)$, $\beta \equiv Fr(-\bar{\rho}\nu)/\bar{\rho}$ and $f_n(t) = \mathcal{O}(t^{-n/\bar{\rho}-\beta})$ when $t \to \infty$.

Corollary 3. *For $\bar{\rho}\nu \notin \mathbb{Z}$, $z > 0$ and $n, \bar{\rho}, \delta \in \mathbb{N}$ with n being a multiple of $\bar{\rho}$ and $n - K$ being a multiple of δ,*

$$\mathcal{L}_{f_1}(z) = \sum_{k=0}^{-K-1} A_k \Gamma\left(\nu + 1 - \frac{k}{\bar{\rho}}\right) z^{k/\bar{\rho}-\nu-1} + \sum_{k=0}^{n/\bar{\rho}-1} z^k \left[\frac{(-1)^k \bar{\rho}}{\delta k!} \Gamma\left(-\frac{\bar{\rho}(\nu+k+1)}{\delta}\right) \right.$$
$$\left. + \sum_{j=0}^{\bar{\rho}-1} A_{\bar{\rho}k+j-K} \Gamma\left(1 - k - \frac{j}{\bar{\rho}} - \beta\right) z^{\beta+j/\bar{\rho}-1} \right] + R_n(z). \tag{29}$$

A bound for the remainder is given by

$$|R_n(z)| \leqslant \frac{|A_{n-K}|\pi}{\sin(\pi\beta)\Gamma(n/\bar{\rho}+\beta)} z^{n/\bar{\rho}+\beta-1}. \tag{30}$$

Expansion (29) is convergent.

Proof. Apply theorem 1 to the integral (2) with $s = \bar{\rho}$, $f(t) = f_1(t)$ given in (27), $a_k = A_{k-K}$ given in (28) and β and K given above. After the change of variable $t = u^{-\bar{\rho}/\delta}$, the Mellin transform of $f_1(t)$ reads

$$M[f_1; k+1] = \frac{\bar{\rho}}{\delta} \int_0^\infty u^{-\bar{\rho}(\nu+k+1)/\delta-1} \, e^{-u} \, du = \frac{\bar{\rho}}{\delta} \Gamma\left(-\frac{\bar{\rho}(\nu+k+1)}{\delta}\right). \tag{31}$$

Expansion (29) follows after introducing (31) in theorem 1.

Now we derive the bound (30). We write $f_n(t)$ in (28) as

$$f_n(t) = t^\nu r_{n-K}(t) \tag{32}$$

where $r_n(t)$ represents the nth remainder of the Taylor expansion of $e^{-t^{-\delta/\bar{\rho}}}$ in powers of $t^{-\delta/\bar{\rho}}$:

$$e^{-t^{-\delta/\bar{\rho}}} = 1 - \frac{|A_\delta|}{t^{\delta/\bar{\rho}}} + \frac{|A_{2\delta}|}{t^{2\delta/\bar{\rho}}} - \cdots + (-1)^{[(n-1)/\delta]} \frac{|A_{[(n-1)/\delta]\delta}|}{t^{[(n-1)/\delta]\delta/\bar{\rho}}} + r_n(t) \tag{33}$$

for $n = 1, 2, 3, 4, \ldots$ and $[\cdot]$ is the integer part function. Using the Lagrange formula for the remainder $r_n(t)$, we find that two consecutive error terms, $r_n(t)$ and $r_{n+\delta}(t)$, have opposite sign: $\text{sign}(r_n(t)) = (-1)^{[(n-1)/\delta]+1}$. Then, expansion (33) verifies the error test and we have (lemma 5):

$$|r_n(t)| \leqslant \frac{|A_{[(n-1)/\delta]\delta+\delta}|}{t^{([(n-1)/\delta]\delta+\delta)/\bar{\rho}}}. \tag{34}$$

Using that $n - K$ is a multiple of δ, from (32) and (34) we obtain $|f_n(t)| \leqslant |A_{n-K}| t^{-n/\bar{\rho}-\text{Re}\,\beta}$. Then, from corollary 2, bound (30) follows from proposition 1 with $c_n = |A_{n-K}|$. From the second formula in (28) and (30) we have $\lim_{n\to\infty} R_n(z) = 0$. □

Corollary 4. *For $\nu\bar{\rho} \in \mathbb{Z}$, $z > 0$, $\bar{\rho}, \delta \in \mathbb{N}$ and $n = \bar{\rho}\delta, 2\bar{\rho}\delta, 3\bar{\rho}\delta, \ldots$*

$$\mathcal{L}_{f_1}(z) = \sum_{k=-\bar{\rho}\nu-1}^{-1} A_{k-\alpha+3}\Gamma(1-(k+1)/\bar{\rho})z^{(k+1)/\bar{\rho}-1}$$
$$+ \sum_{k=0}^{n/\bar{\rho}-1} z^k \left[A_{\bar{\rho}(\nu+k+1)} \frac{(-1)^{k+1}}{k!}(\log z + \gamma) + \frac{(-1)^k}{k!} B_k \right.$$
$$\left. + \sum_{j=0}^{\bar{\rho}-2} A_{\bar{\rho}k+j+\bar{\rho}\nu+1}\Gamma(1-k-(j+1)/\bar{\rho})z^{(j+1)/\bar{\rho}-1} \right] + R_n(z) \tag{35}$$

where the coefficients A_k are given in (28) and the coefficients B_k are given by

$$B_k \equiv \frac{\bar{\rho}(-1)^{\bar{\rho}(\nu+k+1)/\delta}}{\delta\,(\bar{\rho}(\nu+k+1)/\delta)!}\psi\left(\frac{\bar{\rho}(\nu+k+1)}{\delta}+1\right)$$
$$+ \sum_{j=0}^{\bar{\rho}-2}\left[\frac{k!}{((j+1)/\bar{\rho}-1)_{k+1}} - \frac{1}{(j+1)/\bar{\rho}-1}\right] A_{\bar{\rho}k+j+\bar{\rho}\nu+1}$$
$$+ \sum_{i=1}^{k}\sum_{j=k\bar{\rho}}^{\bar{\rho}(k+1)-1} \frac{(k-i+2)_{i-1}A_{j+\bar{\rho}\nu+1}}{((j+1)/\bar{\rho}-i)_i} \tag{36}$$

where ψ is the digamma function ([1], equation (6.3.1)).

For $\bar{\rho} > 1$, a bound for the remainder is given in (30), whereas for $\bar{\rho} = \beta = 1$, two bounds are given by

$$|R_n(z)| \leqslant \frac{\bar{c}_n\pi}{\Gamma(n+1/2)} z^{n-1/2} \tag{37}$$

and

$$|R_n(z)| \leqslant \left\{ \frac{1}{(n-1)!}[|A_{n+\nu}|\varepsilon + |A_{n+\nu+1}|] + \frac{|A_{n+\nu+1}|}{n!}\Theta(z,\varepsilon) \right\} z^n \tag{38}$$

where $\bar{c}_n = Max\{|A_{n+\nu+1}|, |A_{n+\nu}|\}$, $\Theta(z,\epsilon)$ is defined in (22) and ε is an arbitrary positive number whose optimum value is given by $|A_{n+\nu+1}|/(n|A_{n+\nu}|)$. Expansion (35) is convergent.

Proof. To obtain expansion (35) we apply theorem 2 to the integral (2) with $s = \bar{\rho}$, $f(t) = f_1(t)$ given in (27), $\beta = 1/\bar{\rho}$, $K = -\bar{\rho}\nu - 1$ and $a_k = A_{k+\bar{\rho}\nu+1}$. The coefficient B_k in (35) is $b_{(k+1)\bar{\rho}}$ given by (12) with $a_k = A_{k+\bar{\rho}\nu+1}$. The Mellin transform in formula (12) is given by (31) with k replaced by z. When $z \to n$, there are two singular terms in the limit of formula (12): $A_{\bar{\rho}(n+1+\nu)}/(z-n)$ and $\Gamma(-\bar{\rho}(\nu+z+1)/\delta)$. Setting $z = n + \eta$, expanding these terms at $\eta = 0$ and using (28)(b) we obtain (36).

For $\bar{\rho} > 1$, bound (30) is obtained as in corollary 3. For $\bar{\rho} = \beta = 1$, the bounds (37) and (38) are obtained from proposition 2 and corollary 2. From (37) we have $\lim_{n\to\infty} R_n(z) = 0$. □

3.2. Nonresonant case with depleted tail

The function $f_3(t)$ in (5) satisfies

$$f_3(t) = \sum_{k=K}^{n-1} \frac{A_{k-K}}{t^{k/\rho+\beta}} + f_n(t) \tag{39}$$

where $K \equiv Int(\nu+\rho+1)$, $\beta \equiv Fr(\nu+\rho+1)/\rho$ and the coefficients A_k are

$$A_k \equiv \sum_{j=0}^{k} \frac{(-a)^{k-j}}{(k-j)!} B_j \qquad B_j \equiv \begin{cases} \dfrac{(-b)^{(j/\delta)}}{(j/\delta)!} & \text{if} \quad j = \dot{\delta} \\ 0 & \text{if} \quad j \neq \dot{\delta} \end{cases} \tag{40}$$

and $f_n(t) = \mathcal{O}(t^{-n/\rho-\beta})$ when $t \to \infty$.

Corollary 5. *For $\nu \notin \mathbb{Z}$, $z > 0$ and $n, \rho, \delta \in \mathbb{N}$ with n being a multiple of ρ and $n - K$ being a multiple of δ,*

$$\mathcal{L}_{f_3}(z) = \sum_{k=0}^{-K-1} A_k \Gamma\left(-\frac{(k+\nu+1)}{\rho}\right) z^{(k+\nu+1)/\rho} + \sum_{k=0}^{n/\rho-1} z^k \left[\frac{(-1)^k}{k!}\rho M_k + \sum_{j=0}^{\rho-1} A_{\rho k+j-K} \Gamma\left(1-k-\frac{j}{\rho}-\beta\right) z^{\beta+j/\rho-1}\right] + R_n(z) \tag{41}$$

where the coefficients M_k are given by

$$M_k \equiv \begin{cases} \Gamma(\nu-\rho k+1)(a+b)^{\rho k-\nu-1} & \text{if} \quad \delta = 1 \\ \displaystyle\sum_{j=0}^{\infty} \frac{(-a)^j}{\delta j!} \Gamma\left(\frac{j-k\rho+\nu+1}{\delta}\right) b^{(k\rho-j-\nu-1)/\delta} & \text{if} \quad \delta > 1. \end{cases} \tag{42}$$

A bound for the remainder is given by

$$|R_n(z)| \leqslant \frac{\pi c_n}{\sin(\pi\beta)\Gamma(n/\rho+\beta)} z^{n/\rho+\beta-1} \tag{43}$$

where we can take $c_n = |A_{n-K}|$ if δ is odd. In any case, we can take $c_n = \mathrm{e}^{a+b}$. Expansion (41) is convergent.

Proof. To obtain expansion (41), we apply theorem 1 to the integral (2) with $s = \rho$, $f(t) = f_3(t)$ given in (5), $a_k = A_{k-K}$ given in (40) and β and K as given above. After the change of variable $t = u^{-\rho}$, the Mellin transform of $f_3(t)$ reads

$$M[f_3; k+1] = \rho \int_0^\infty u^{\nu-\rho k}\, \mathrm{e}^{-bu^\delta}\, \mathrm{e}^{-au}\, \mathrm{d}u.$$

For $\delta = 1$, this is an elementary integral given by the first line in the right hand side of (42). If $\delta > 1$ we expand e^{-au} in powers of u and interchange sum and integral to obtain the second line in the right hand side of (42).

Now we obtain bound (43). We write $f_n(t)$ in (39) as

$$f_n(t) = t^{-(\nu+1)/\rho-1} r_{n-K}(t)$$

where $r_n(t)$ is the remainder of the Taylor expansion of $\mathrm{e}^{-at^{-1/\rho}}\, \mathrm{e}^{-bt^{-\delta/\rho}}$ in powers of $t^{-1/\rho}$:

$$r_{n-K}(t) = \sum_{j=0}^{[(n-K-1)/\delta]} \sum_{l=0}^{\delta-1} \frac{(-a)^{j\delta+l}}{(j\delta+l)!} t^{-(j\delta+l)/\rho} r^2_{n-K-j\delta-l}(t) + r^1_{n-K}(t) r^2_0(t)$$

where $r_n^1(t)$ and $r_n^2(t)$ are, respectively, the remainders of the expansions of $\mathrm{e}^{-at^{-1/\rho}}$ and $\mathrm{e}^{-bt^{-\delta/\rho}}$ in powers of $t^{-1/\rho}$:

$$\mathrm{e}^{-at^{-1/\rho}} = \sum_{k=0}^{n-1} \frac{(-a)^k}{k!} t^{-k/\rho} + r_n^1(t) \tag{44}$$

$$\mathrm{e}^{-bt^{-\delta/\rho}} = \sum_{k=0}^{n-1} b_k t^{-k/\rho} + r_n^2(t) \qquad b_k \equiv \begin{cases} \dfrac{(-b)^{(k/\delta)}}{(k/\delta)!} & \text{if} \quad k = \dot{\delta} \\ 0 & \text{if} \quad k \neq \dot{\delta}. \end{cases}$$

We write $n - K = m\delta$ for some $m \in \mathbb{N}$. Therefore, $r^2_{n-K-j\delta-l}(t) = r^2_{(m-j)\delta}(t)$ for $l = 0, 1, 2, \ldots, \delta - 1$. Using the fact that $\mathrm{sign}\big(r_n^2(t)\big) = (-1)^{\lfloor (n-1)/\delta \rfloor + 1}$ (see paragraph following equation (33)), we conclude that $\mathrm{sign}\big(r^2_{n-K-j\delta-l}(t)\big) = (-1)^{m-j}$ for $l = 0, 1, 2, \ldots, \delta - 1$. Defining

$$\widetilde{a_j} \equiv (-a)^{j\delta} t^{-j\delta/\rho} \sum_{l=0}^{\delta-1} \frac{(-at^{-1/\rho})^l}{(j\delta + l)!}$$

and using the fact that $\sum_{l=0}^{\delta-1} (-at^{-1/\rho})^l/(j\delta + l)! > 0$, we have $\mathrm{sign}(\widetilde{a_j}) = (-1)^{j\delta} = (-1)^j$ for odd δ. Then, taking into account that $r_0(t) > 0$ and $\mathrm{sign}\big(r^1_{n-K}(t)\big) = (-1)^{m\delta} = (-1)^m$, we have, for $m = 1, 2, 3, \ldots,$

$$\mathrm{sign}(r_{n-K}(t)) = \mathrm{sign}\left(\sum_{j=0}^{[(n-K-1)/\delta]} \widetilde{a_j} r^2_{n-K-j\delta}(t) + r^1_{n-K}(t) r^2_0(t) \right) = (-1)^m.$$

We conclude that the function $f_3(t)$ verifies the error test for odd δ and, from corollary 2, we obtain (43) with $c_n = |A_{n-K}|$.

For any δ, by corollary 1, the remainder in expansion (41) verifies the bounds given in propositions 1 and 2 with $c_n = Cr^{-n}$, where C is a bound of $g(w) = \mathrm{e}^{-aw}\,\mathrm{e}^{-bw^\delta}$ in the region W considered in lemma 4 and $0 < r < \infty$. We take $r = 1$ and then $C = \mathrm{e}^{a+b}$. Therefore, bound (43) holds with $c_n = \mathrm{e}^{a+b}$ and, from this bound, we have $\lim_{n\to\infty} R_n(z) = 0$. □

Corollary 6. *For $\nu \in \mathbb{Z}$, $z > 0$, $\rho, \delta \in \mathbb{N}$ and $n = \rho\delta, 2\rho\delta, 3\rho\delta, \ldots$*

$$\begin{aligned} \mathcal{L}_{f_3}(z) = {} & \sum_{k=\nu+\rho}^{-1} A_{k-\nu-\rho} \Gamma(1 - (k+1)/\rho) z^{(k+1)/\rho - 1} \\ & + \sum_{k=0}^{n/\rho - 1} z^k \left[A_{\rho k - \nu - 1} \frac{(-1)^{k+1}}{k!} (\log z + \gamma) + \frac{(-1)^k}{k!} C_k \right. \\ & \left. + \sum_{j=0}^{\rho-2} A_{\rho k + j - \nu - \rho} \Gamma(1 - k - (j+1)/\rho) z^{(j+1)/\rho - 1} \right] + R_n(z) \end{aligned} \tag{45}$$

where the coefficients A_k are given in (40) and the coefficients C_k are given by

$$\begin{aligned} C_k \equiv {} & \rho C_k' + \sum_{j=0}^{\rho-2} \left[\frac{k!}{((j+1)/\rho - 1)_{k+1}} - \frac{1}{(j+1)/\rho - 1} \right] A_{\rho k + j - \nu - \rho} \\ & + \sum_{i=1}^{k} \sum_{j=k\rho}^{\rho(k+1)-1} \frac{(k - i + 2)_{i-1} A_{j-\nu-\rho}}{((j+1)/\rho - i)_i} \end{aligned} \tag{46}$$

where

$$C'_k \equiv \sum_{j=0;\, j-\rho k+\nu+1\neq\dot{\delta}}^{\rho k-\nu-1} \frac{(-a)^j}{j!\delta} b^{(\rho k-j-\nu-1)/\delta}\Gamma\left(\frac{j-\rho k+\nu+1}{\delta}\right)$$
$$+\sum_{j=0;\, j-\rho k+\nu+1=\dot{\delta}}^{\rho k-\nu-1} \frac{(-a)^j}{j!\delta} B_{\rho k-\nu-1-j}\psi\left(\frac{\rho k-\nu-1-j}{\delta}+1\right)$$
$$+\sum_{j=0}^{\infty}\frac{(-1)^{\rho k-\nu+j}a^{\rho k-\nu+j}}{(\rho k-\nu+j)!\delta}\Gamma\left(\frac{j+1}{\delta}\right)b^{-(j+1)/\delta} \tag{47}$$

and B_j *are defined in (40).*

For $\rho > 1$*, a bound for the remainder is given by (43) with* $c_n = \mathrm{e}^{a+b}$ *for any* δ *or* $c_n = |A_{n-\nu-\rho}|$ *for odd* δ*. For* $\rho = \beta = 1$*, two bounds are given by*

$$|R_n(z)| \leqslant \frac{\bar{c}_n\pi}{\Gamma(n+1/2)}z^{n-1/2} \tag{48}$$

and, for any $\epsilon > 0$*,*

$$|R_n(z)| \leqslant \left\{\frac{1}{(n-1)!}[(c_{n-1}+|A_{n-\nu-\rho-1}|)\varepsilon + c_n] + \frac{c_n}{n!}\Theta(z,\varepsilon)\right\} z^n \tag{49}$$

where $\Theta(z,\varepsilon)$ *is defined in (22). In these formulae we can take, for odd* δ*,* $c_n = |A_{n-\nu-\rho}|$*,* $c_{n-1} = 0$ *and* $\bar{c}_n = Max\{|A_{n-\nu-\rho}|, |A_{n-\nu-\rho-1}|\}$*. For any* δ *we can take* $\bar{c}_n = Max\{\mathrm{e}^{a+b}, \mathrm{e}^{a+b}+|A_{n-\nu-\rho-1}|\}$ *and* $c_n = c_{n-1} = \mathrm{e}^{a+b}$*. Expansion (45) is convergent.*

Proof. To obtain expansion (45) we apply theorem 2 to integral (2) with $s = \rho$, $f(t) = f_3(t)$ given in (5), $\beta = 1/\rho$, $K = \nu+\rho$ and $a_k = A_{k-\nu-\rho}$. The coefficient C_k in (45) is $b_{(k+1)\rho}$ given by (12) with $a_n = A_{n-\nu-\rho}$. The Mellin transform in formula (12) is given by (42) replacing k by z. When $z \to n$, there are two singular terms in the limit in (12): $A_{\rho n-\nu-1}/(z-n)$ and $\Gamma((j-\rho z+\nu+1)/\delta)$ when $(j-n\rho+\nu+1)/\delta - 1 \in \mathbb{Z}^-$. Setting $z = n+\eta$, expanding these terms at $\eta = 0$ and using (40) we obtain (46) and (47).

For $\rho > 1$, the error bound (43) is obtained as in corollary 5. For $\rho = \beta = 1$, bounds (37) and (38) are obtained as in corollary 4: using corollary 2 for odd δ and corollary 1 for any δ. Using any of these bounds we have $\lim_{n\to\infty} R_n(z) = 0$. □

3.3. Resonant case

The function $f_5(t)$ given in (6) satisfies

$$f_5(t) = \sum_{k=K}^{n-1}\frac{A_{k-K}}{t^{k/\rho+\beta}} + f_n(t) \qquad A_k \equiv \frac{(-1)^k}{\sin\theta}\sum_{j=0}^{k}\frac{a^{k-j}}{(k-j)!}\frac{\sin[(j+1)\theta]}{(b^2+g^2)^{j/2+1}} \tag{50}$$

where $K \equiv Int(\nu+\rho+1)$, $\beta \equiv Fr(\nu+\rho+1)/\rho$, $\theta \equiv \arctan(g/b)$ and $f_n(t) = \mathcal{O}(t^{-n/\rho-\beta})$ when $t \to \infty$.

Corollary 7. *For* $\nu \notin \mathbb{Z}$*,* $z > 0$*,* $\rho \in \mathbb{N}$ *and* $n = \rho, 2\rho, 3\rho, \ldots$

$$\mathcal{L}_{f_5}(z) = \sum_{k=0}^{-K-1} A_k\Gamma(-(k+\nu+1)/\rho)z^{(k+\nu+1)/\rho} + \sum_{k=0}^{n/\rho-1} z^k\left[\frac{(-1)^k\rho}{gk!}M_k\right.$$
$$\left.+\sum_{j=0}^{\rho-1}A_{\rho k+j-K}\Gamma(1-k-j/\rho-\beta)z^{\beta+j/\rho-1}\right] + R_n(z) \tag{51}$$

where the coefficients M_k are given by

$$M_k \equiv \Gamma(\nu+1-\rho k)\,\mathrm{Im}[(b-\mathrm{i}g)^{\nu-\rho k}\,\mathrm{e}^{a(b-\mathrm{i}g)}\Gamma(\rho k-\nu, a(b-\mathrm{i}g))] \tag{52}$$

and $\Gamma(z,x)$ denotes the incomplete gamma function ([1], equation (653)). A bound for the remainder is given by

$$|R_n(z)| \leqslant \frac{\pi c_n}{\sin(\pi\beta)\Gamma(n/\rho+\beta)} z^{n/\rho+\beta-1} \tag{53}$$

where

$$c_n = \frac{1}{\sin\theta}\sum_{k=0}^{n-1}\frac{a^{n-k}}{(n-k)!}\frac{|\sin[(k+1)\theta]|}{(b^2+g^2)^{k/2+1}} + \frac{2}{g^2(b^2+g^2)^{n/2}}. \tag{54}$$

Expansion (51) is convergent.

Proof. We apply theorem 1 to integral (2) with $s=\rho$, $f(t)=f_5(t)$ given in (6), $a_k=A_{k-K}$ given in (50) and β and K as given above. The Mellin transform of $f_5(t)$ reads

$$M[f_5;k+1] = \rho\int_0^\infty \frac{u^{\nu-\rho k}}{g^2+(b-u)^2}\,\mathrm{e}^{-au}\,\mathrm{d}u = \frac{\rho}{w_2-w_1}[I(w_1)-I(w_2)]$$

where $w_1\equiv b+\mathrm{i}g$, $w_2\equiv b-\mathrm{i}g$ and

$$I(w) \equiv \int_0^\infty \frac{u^{\nu-\rho k}}{u+w}\,\mathrm{e}^{-au}\,\mathrm{d}u = \Gamma(\nu+1-\rho k)w^{\nu-\rho k}\,\mathrm{e}^{aw}\Gamma(\rho k-\nu, aw)$$

where we have used [16, p 325, equation (13)]. Then (51) follows from theorem 1 after straightforward computations. To obtain the error bound (53) we apply proposition 1 to the function $f_5(t)$. We write

$$f_n(t) = t^{-(\nu+1)/\rho-1}r_{n-K}(t) \qquad r_n(t) \equiv \sum_{k=0}^{n-1} c_k^2 t^{-k/\rho} r_{n-k}^1(t) + r_n^2(t) r_0^1(t) \tag{55}$$

where $r_n^1(t)$ is the remainder in expansion (44) of $\mathrm{e}^{-at^{-1/\rho}}$ in powers of $t^{-1/\rho}$ and $r_n^2(t)$ is the remainder in the expansion of $[(b-t^{-1/\rho})^2+g^2]^{-1}$ in powers of $t^{-1/\rho}$:

$$\frac{1}{(b-t^{-1/\rho})^2+g^2} = \sum_{k=0}^{n-1} c_k^2 t^{-k/\rho} + r_n^2(t) \qquad c_k^2 \equiv \frac{(-1)^k\sin[(k+1)\theta]}{(b^2+g^2)^{k/2+1}\sin\theta}.$$

Expansion (44), as well as expansion (33), satisfies the error test and then, from lemma 5, $\left|r_n^1(t)\right| \leqslant a^n t^{-n/\rho}/n!$. On the other hand

$$\frac{1}{(b-u)^2+g^2} = \frac{1}{u-w_1}\frac{1}{u-w_2} \qquad \text{and} \qquad \frac{1}{u-w} = -\frac{1}{w}\sum_{k=0}^{n-1}\left(\frac{u}{w}\right)^k + \frac{(u/w)^n}{u-w}.$$

Therefore,

$$\begin{aligned} r_n^2(t) &= -\frac{1}{w_1}\sum_{k=0}^{n-1}\left(\frac{u}{w_1}\right)^k \frac{(u/w_2)^{n-k}}{u-w_2} + \frac{(u/w_1)^n}{u-w_1}\frac{1}{u-w_2} \\ &= \frac{u^n}{u-w_2}\left[\frac{1-(w_2/w_1)^n}{w_2-w_1}\frac{1}{w_2^n} + \frac{1}{u-w_1}\frac{1}{w_1^n}\right]. \end{aligned}$$

After trivial manipulations we obtain $\left|r_n^2(t)\right| \leqslant 2g^{-2}t^{-n/\rho}(b^2+g^2)^{-n/2}$. Collecting these bounds for $r_n^1(t)$ and $r_n^2(t)$ in (55) we find

$$|r_n(t)| \leqslant \left[\sum_{k=0}^{n-1}\left|c_k^2\right|\frac{a^{n-k}}{(n-k)!} + \frac{2}{g^2(b^2+g^2)^{n/2}}\right]t^{-n/\rho}.$$

From proposition 1 we obtain (53) and (54) and $\lim_{n\to\infty} R_n(z) = 0$. □

Corollary 8. *For* $\nu \in \mathbb{Z}$, $z > 0$, $\rho \in \mathbb{N}$ *and* $n = \rho, 2\rho, 3\rho, \ldots$

$$\mathcal{L}_{f_5}(z) = \sum_{k=\nu+\rho}^{-1} A_{k-\nu-\rho}\Gamma(1-(k+1)/\rho)z^{(k+1)/\rho-1} + \sum_{k=0}^{n/\rho-1} z^k \left[A_{\rho k-\nu-1}\frac{(-1)^{k+1}}{k!}(\log z+\gamma) + \frac{(-1)^k}{k!}B_k + \sum_{j=0}^{\rho-2} A_{\rho k+j-\nu-\rho}\Gamma(1-k-(j+1)/\rho)z^{(j+1)/\rho-1} \right] + R_n(z). \tag{56}$$

The coefficients A_k *are given in (50) and the coefficients* B_k *are*

$$B_k \equiv \rho B'_k + \sum_{j=0}^{\rho-2}\left[\frac{k!}{((j+1)/\rho-1)_{k+1}} - \frac{1}{(j+1)/\rho-1}\right]A_{\rho k+j-\nu-\rho} + \sum_{i=1}^{k}\sum_{j=k\rho}^{\rho(k+1)-1}\frac{(k-i+2)_{i-1}A_{j-\nu-\rho}}{((j+1)/\rho-i)_i} \tag{57}$$

where

$$B'_k \equiv \frac{(-1)^{\rho k-\nu-1}}{(\rho k-\nu-1)!}\frac{\psi(\rho k-\nu)}{w_2-w_1}[m_k(w_1)-m_k(w_2)] \tag{58}$$

with

$$m_k(w) \equiv w^{\nu-\rho k}\left[\mathrm{e}^{aw}\Gamma(\rho k-\nu) + \frac{(aw)^{\rho k-\nu}}{\nu-\rho k}\,{}_1F_1(1,\rho k+1-\nu,aw)\right] \tag{59}$$

where ${}_1F_1(a,b,z)$ *denotes Kummer's confluent hypergeometric function ([1], p 504, equation (13.1.2)).*

For $\rho > 1$, *a bound for the remainder* $R_n(z)$ *is given in (53) and (54). For* $\rho = \beta = 1$, *two bounds are given by (48) and (49) with* $\bar{c}_n = Max\{c_n, c_{n-1} + |A_{n-\nu-\rho-1}|\}$ *and* c_n *given in (54) for* $\Theta(z,\epsilon)$ *defined in (22). Expansion (56) is convergent.*

Proof. To obtain expansion (56) we apply theorem 2 to integral (2) with $s = \rho$, $f(t) = f_5(t)$ given in (6), $\beta = 1/\rho$, $K = \nu + \rho$ and $a_k = A_{k-\nu-\rho}$. The coefficients B_k in (56) are $b_{(k+1)\rho}$ given by (12) with $a_k = A_{k-\nu+\rho}$. For $k = 0, 1, 2, \ldots$, the Mellin transform in formula (12) is given by (52) replacing k by z. When $z \to n$, there are two singular terms in the limit in (12): $A_{\rho n-\nu-1}/(z-n)$ and $\Gamma(\nu+1-\rho z)$. Setting $z = n + \eta$, expanding these terms at $\eta = 0$ and using (50) we obtain (57)–(59). The error bounds (48), (49) and (53) and (54) are obtained as in corollary 7. From (53) and (54) we see that $\lim_{n\to\infty} R_n(z) = 0$. □

3.4. Nonresonant case with high-energy cut-off

The function $f_2(t)$ given in (4) satisfies

$$f_2(t) = \sum_{k=K}^{n-1}\frac{A_{k-K}}{t^{k/\rho+\beta}} + f_n(t)$$

where $K \equiv Int(\nu+\rho+1)$, $\beta = Fr(\nu+\rho+1)/\rho$ and the coefficients A_k are defined by

$$A_k \equiv \sum_{j=0}^{k} B_{k-j} E_j \qquad B_j \equiv \begin{cases} \begin{pmatrix} -(\nu+1)/\rho - 1 \\ j/\rho \end{pmatrix} & \text{if} \quad j = \dot{\rho} \\ 0 & \text{if} \quad j \neq \dot{\rho} \end{cases} \tag{60}$$

$$E_j = \sum_{i=0}^{j} \frac{(-ad)^i}{i!} b_{i,j-i} \qquad b_{i,j} \equiv \begin{cases} \begin{pmatrix} -i/\rho \\ j/\rho \end{pmatrix} & \text{if} \quad j = \dot{\rho} \\ 0 & \text{if} \quad j \neq \dot{\rho} \end{cases} \tag{61}$$

and $f_n(t) = \mathcal{O}(t^{-n/\rho-\beta})$ when $t \to \infty$.

Corollary 9. *For* $\nu \notin \mathbb{Z}$, $z > 0$, $\rho \in \mathbb{N}$ *and* $n = \rho, 2\rho, 3\rho, \ldots$

$$\mathcal{L}_{f_2}(z) = \sum_{k=0}^{-K-1} A_k \Gamma\left(-\frac{k+\nu+1}{\rho}\right) z^{(k+\nu+1)/\rho} + \sum_{k=0}^{n/\rho-1} z^k \left[(-1)^k M_k \right.$$
$$\left. + \sum_{j=0}^{\rho-1} A_{\rho k+j-K} \Gamma(1-k-j/\rho-\beta) z^{\beta+j/\rho-1} \right] + R_n(z). \tag{62}$$

Coefficients M_k *are given by*

$$M_k \equiv \sum_{j=0}^{\infty} \frac{\Gamma((j+\nu+1)/\rho - k)}{\Gamma((j+\nu+1)/\rho+1)} \frac{(-ad)^j}{j!}. \tag{63}$$

A bound for the remainder is given by

$$|R_n(z)| \leqslant \frac{C\pi}{r^n \sin(\pi\beta)\Gamma(n/\rho+\beta)} z^{n/\rho+\beta-1} \tag{64}$$

where C *is a bound of* $g(w) = (1+w^\rho)^{-(\nu+1)/\rho-1}\, \mathrm{e}^{-adw(1+w^\rho)^{-1/\rho}}$ *in the region* W *considered in lemma 4 with*

$$0 < r < |\sin(\pi/\rho)| \quad \text{if} \quad \rho \geqslant 3 \quad \text{and} \quad 0 < r < 1 \quad \text{if} \quad \rho = 1, 2. \tag{65}$$

Expansion (62) is convergent.

Proof. To obtain expansion (62) we apply theorem 1 to integral (2) with $s = \rho$, $f(t) = f_2(t)$ given in (4), $a_k = A_{k-K}$ given in (60) and (61) and β and K given above.

The Mellin transform of $f_2(t)$ reads

$$M[f_2; k+1] = \int_0^\infty t^k (1+t)^{-(\nu+1)/\rho-1}\, \mathrm{e}^{-ad(1+t)^{-1/\rho}}\, \mathrm{d}t.$$

Expanding $\mathrm{e}^{-ad(1+t)^{-1/\rho}}$ in powers of $(1+t)^{-1/\rho}$ and interchanging sum and integral we obtain

$$M[f_2; k+1] = \sum_{j=0}^{\infty} \frac{(-ad)^j}{j!} \int_0^\infty t^k (1+t)^{-(j+\nu+1)/\rho-1}\, \mathrm{d}t.$$

From here, (63) follows after straightforward computations.

On the other hand, by corollary 1, the remainder in expansion (62) verifies the bounds given in propositions 1 and 2 with $c_n = Cr^{-n}$, where C is a bound of $g(w) = w^{-\nu-\rho-1} f(w^{-\rho}) = (1+w^\rho)^{-(\nu+1)/\rho-1}\, \mathrm{e}^{-adw(1+w^\rho)^{-1/\rho}}$ in the region W considered in lemma 4. In that lemma we must take $0 < r < \sigma =$ distance of the nearest ρ-root of -1 to the positive real axis. Therefore, bounds (64) and (65) hold. From corollary 1 we have $\lim_{n\to\infty} R_n(z) = 0$. □

Corollary 10. *For $\nu \in \mathbb{Z}$, $z > 0$, $\rho \in \mathbb{N}$ and $n = \rho, 2\rho, 3\rho, \ldots$*

$$\mathcal{L}_{f_2}(z) = \sum_{k=\nu+\rho}^{-1} A_{k-\nu-\rho}\Gamma(1-(k+1)/\rho)z^{(k+1)/\rho-1} + \sum_{k=0}^{n/\rho-1} z^k \left[A_{\rho k-\nu-1}\frac{(-1)^{k+1}}{k!}(\log z+\gamma) + \frac{(-1)^k}{k!}B_k + \sum_{j=0}^{\rho-2} A_{\rho k+j-\nu-\rho}\Gamma(1-k-(j+1)/\rho)z^{(j+1)/\rho-1} \right] + R_n(z) \tag{66}$$

where the coefficients A_k are given in (60) and the coefficients B_k are

$$B_k \equiv k!C_k + \sum_{j=0}^{\rho-2}\left[\frac{k!}{((j+1)/\rho-1)_{k+1}} - \frac{1}{(j+1)/\rho-1}\right]A_{\rho k+j-\nu-\rho} + \sum_{i=1}^{k}\sum_{j=k\rho}^{\rho(k+1)-1}\frac{(k-i+2)_{i-1}A_{j-\nu-\rho}}{((j+1)/\rho-i)_i} \tag{67}$$

where

$$C_k \equiv \sum_{j=0;\, j+\nu+1\neq\dot{\rho}}^{\rho k-\nu-1} \frac{(-ad)^j}{j!}\frac{\Gamma((j+\nu+1)/\rho-k)}{\Gamma((j+\nu+\rho+1)/\rho)} + \sum_{j=0;\, j+\nu+1=\dot{\rho}}^{\rho k-\nu-1}\frac{(-1)^{j+k-(j+\nu+1)/\rho}(ad)^j}{j!(k-(j+\nu+1)/\rho)!\Gamma((j+\nu+1)/\rho+1)} \times[\psi(k+1)-\psi(k+1-(j+\nu+1)/\rho)] + \sum_{j=\rho k-\nu}^{\infty}\frac{(-ad)^j}{j!}\frac{\Gamma((j+\nu+1)/\rho-k)}{\Gamma((j+\nu+1)/\rho+1)}.$$

For $\rho > 1$, a bound for the remainder $R_n(z)$ is given by (64) with $\beta = 1/\rho$ and C and r given there. For $\rho = \beta = 1$, two bounds are given by (48) and (49) with $c_n = Cr^{-n}$, $\bar{c}_n = Max\{Cr^{-n}, Cr^{-n+1} + |A_{n-\nu-\rho-1}|\}$ and C and r given in corollary 9. Expansion (66) is convergent.

Proof. To obtain expansion (66) we apply theorem 2 to integral (2) with $s = \rho$, $f(t) = f_2(t)$ given in (4), $\beta = 1/\rho$, $K = \nu + \rho$ and $a_k = A_{k-\nu-\rho}$. The coefficient B_k in (66) is $b_{(k+1)\rho}$ given by (12) with $a_k = A_{k-\nu-\rho}$. The Mellin transform in formula (12) is given by (63) with k replaced by z. When $z \to n$, there are two singular terms in the limit of (12): $A_{\rho n-\nu-1}/(z-n)$ and $\Gamma((j+\nu+1)/\rho - z)$ when $(j+\nu+1)/\rho - n - 1 \in \mathbb{Z}^-$. Setting $z = n+\eta$, expanding these terms at $\eta = 0$ and using (60) and (61) we obtain (67).

The error bounds are obtained as in corollary 9. From corollary 1 we have $\lim_{n\to\infty} R_n(z) = 0$. □

4. Numerical experiments

The following tables (tables 1–8) show numerical experiments about the approximation and the accuracy of the error bounds supplied by corollaries 3–10. In these tables, the second

Table 1. Approximation supplied by (29) and error bounds given by (30).

Parameter values: $\nu = \frac{1}{2}, \delta = 1, \bar{\rho} = 3, n = 3, 6$

z	$\mathcal{L}_{f_1}(z)$	First order approximation	Relative error	Relative error bound	Second order approximation	Relative error	Relative error bound
0.1	17.5163009	18.073008	0.031	0.053	17.5176537	7.72×10^{-5}	8.89×10^{-5}
0.01	709.943936	712.372622	0.0034	0.0042	709.9444	6.52×10^{-7}	6.94×10^{-7}
0.001	25260.95627	25269.5468	3.4×10^{-4}	3.7×10^{-4}	25260.95642	5.996×10^{-9}	6.16×10^{-9}
0.0001	844353.3251	844381.777	3.37×10^{-5}	3.5×10^{-5}	844353.3251	5.76×10^{-11}	5.83×10^{-11}
0.00001	2.74011004×10^7	2.74011922×10^7	3.35×10^{-6}	3.4×10^{-6}	2.74011004×10^7	5.65×10^{-13}	5.68×10^{-13}

Table 2. Approximation supplied by (35) and error bounds given by Min{(37), (38)}.

Parameter values: $\nu = 1, \delta = 1, \bar{\rho} = 1, n = 1, 2$

z	$\mathcal{L}_{f_1}(z)$	First order approximation	Relative error	Relative error bound	Second order approximation	Relative error	Relative error bound
0.5	2.7339389	2.5193579	0.08	0.15	2.717029	0.006	0.013
0.1	91.391428	91.324077	7.4×10^{-4}	0.002	91.390435	1.0×10^{-5}	3.4×10^{-5}
0.05	381.709889	381.67065	1.0×10^{-4}	3.5×10^{-4}	381.709606	7.4×10^{-7}	2.9×10^{-6}
0.01	9902.485857	9902.475369	1.0×10^{-6}	6.0×10^{-4}	9902.485843	1.5×10^{-9}	9.9×10^{-9}
0.005	39802.82776	39802.82194	1.5×10^{-7}	1.0×10^{-6}	39802.827757	1.0×10^{-10}	8.7×10^{-10}
0.001	999003.628	999003.627	1.4×10^{-9}	1.9×10^{-8}	999003.628	1.9×10^{-13}	3.0×10^{-12}

Table 3. Approximation supplied by (41) and error bounds given by (43).

Parameter values: $\nu = -4.9, \delta = 3, \rho = 2, a = 0.2, b = 1.7, n = 2, 8$

z	$\mathcal{L}_{f_3}(z)$	First order approximation	Relative error	Relative error bound	Second order approximation	Relative error	Relative error bound
0.1	77.6482713	93.66645	0.2	0.48	77.645714	3.29×10^{-5}	2.2×10^{-4}
0.05	316.159579	335.287252	0.06	0.16	316.159166	1.3×10^{-6}	9.0×10^{-6}
0.015	3436.34558	3464.26795	0.008	0.025	3436.34556	5.0×10^{-9}	3.9×10^{-8}
0.005	29661.49798	29703.24225	0.001	0.005	29661.49798	3.56×10^{-11}	2.78×10^{-10}
0.002	178048.65	178108.68	0.0003	0.001	178048.65	1.5×10^{-12}	4.46×10^{-12}

Table 4. Approximation supplied by (45) an error bounds given by (43).

Parameter values: $\nu = 0, \delta = 3, \rho = 2, a = 1, b = 1, n = 6, 12$

z	$\mathcal{L}_{f_3}(z)$	First order approximation	Relative error	Relative error bound	Second order approximation	Relative error	Relative error bound
0.2	0.3255718	0.3404663	0.046	0.054	0.325571	7.8×10^{-7}	9.0×10^{-7}
0.1	0.4667515	0.469481	0.0058	0.0067	0.4667514	1.24×10^{-8}	1.39×10^{-8}
0.08	0.511643	0.513221	0.003	0.0034	0.511643	3.3×10^{-9}	3.7×10^{-9}
0.05	0.602377	0.6028736	0.0008	0.0009	0.602377	2.0×10^{-10}	2.4×10^{-10}
0.02	0.756695	0.756747	6.8×10^{-5}	7.4×10^{-5}	0.756695	1.15×10^{-12}	1.22×10^{-12}
0.005	0.923849	0.923851	1.8×10^{-6}	1.88×10^{-6}	0.923849	4.8×10^{-16}	4.9×10^{-16}

Table 5. Approximation supplied by (51) and error bounds given by (53).

Parameter values: $\nu=-1.4, a=0.2, \rho=2, b=1, g=2, n=4,6$							
z	$\mathcal{L}_{f_5}(z)$	Second order approximation	Relative error	Relative error bound	Third order approximation	Relative error	Relative error bound
0.002	2.7176126	2.7176126	6.34×10^{-5}	1.96×10^{-4}	2.7177846	1.0×10^{-7}	1.2×10^{-7}
0.001	3.1652507	3.1652046	1.4×10^{-5}	6.85×10^{-5}	3.165206	1.0×10^{-8}	2.17×10^{-8}
0.0005	3.691259	3.691246	3.4×10^{-6}	2.38×10^{-5}	3.691259	1.16×10^{-9}	3.77×10^{-9}
0.0001	5.2611778	5.261177	1.48×10^{-7}	2.0×10^{-6}	5.2611778	6.5×10^{-12}	6.5×10^{-11}
0.00005	6.11595839	6.11595813	4.19×10^{-8}	7.2×10^{-7}	6.115958387	7.0×10^{-13}	1.0×10^{-11}
0.00001	8.63234549	8.63234546	2.75×10^{-9}	6.3×10^{-8}	8.63234549	3.9×10^{-15}	1.997×10^{-13}

Table 6. Approximation supplied by (56) and error bounds given by (48) and (49).

Parameter values: $\nu=-2, a=1.2, \rho=3, b=1, g=2, n=6,9$							
z	$\mathcal{L}_{f_5}(z)$	Second order approximation	Relative error	Relative error bound	Third order approximation	Relative error	Relative error bound
0.1	−0.16898599	−0.17026623	0.007	0.01	−0.16897995	3.6×10^{-5}	1.0×10^{-4}
0.01	0.60470192	0.6046355	1.0×10^{-4}	1.3×10^{-4}	0.60470195	5.0×10^{-8}	1.4×10^{-7}
0.001	2.83539549	2.83539227	1.1×10^{-6}	1.3×10^{-6}	2.8353955	5.1×10^{-11}	1.4×10^{-10}
0.0001	8.3317184	8.3317182	1.8×10^{-8}	2.0×10^{-8}	8.3317184	8.2×10^{-14}	2.2×10^{-13}
0.00001	20.9436177	20.94361772	3.4×10^{-10}	3.8×10^{-10}	20.9436177	1.7×10^{-16}	4.0×10^{-16}

Table 7. Approximation supplied by (62) and error bound given by (64).

Parameter values: $\nu=-2.7, a=0.2, \rho=3, d=1, n=6,9$							
z	$\mathcal{L}_{f_2}(z)$	Second order approximation	Relative error	Relative error bound	Third order approximation	Relative error	Relative error bound
0.01	18.413698	18.4140914	2.1×10^{-5}	3.3×10^{-3}	18.413701	1.9×10^{-7}	1.6×10^{-5}
0.002	49.1073814	49.107511	2.6×10^{-6}	2.0×10^{-4}	49.107382	3.3×10^{-9}	2.0×10^{-7}
0.001	74.081196	74.081266	9.0×10^{-7}	6.6×10^{-5}	74.081196	5.5×10^{-10}	3.0×10^{-8}
0.0005	111.261787	111.261823	3.2×10^{-7}	2.0×10^{-5}	111.261787	8.6×10^{-11}	4.86×10^{-9}
0.0001	282.828257	282.828264	2.5×10^{-8}	1.4×10^{-6}	282.828257	1.0×10^{-12}	6.5×10^{-11}

Table 8. Approximation supplied by (66) and error bounds given by (48) and (49).

Parameter values: $\nu=-2, a=0.5, \rho=2, d=1, n=4,6$							
z	$\mathcal{L}_{f_2}(z)$	Second order approximation	Relative error	Relative error bound	Third order approximation	Relative error	Relative error bound
0.1	3.1817197	3.17711956	0.0014	0.004	3.18175944	1.24×10^{-5}	9.4×10^{-4}
0.05	5.12288486	5.12064893	4.0×10^{-4}	5.0×10^{-3}	5.1228827	4.12×10^{-7}	1.0×10^{-4}
0.01	14.0419098	14.0416086	2.14×10^{-5}	1.6×10^{-4}	14.04190942	2.45×10^{-8}	6.7×10^{-7}
0.005	21.01263467	21.0125168	5.6×10^{-6}	4.0×10^{-5}	21.0126346	3.75×10^{-9}	7.9×10^{-8}
0.001	51.1531027	51.15309049	2.38×10^{-7}	1.46×10^{-6}	51.15310268	1.4×10^{-11}	5.8×10^{-10}
0.0005	74.01286165	74.0128571	6.0×10^{-8}	3.6×10^{-7}	74.01286165	4.9×10^{-12}	7.0×10^{-11}

column represents $\mathcal{L}_{f_i}(z)$. The third and sixth columns represent the approximation for the two given values of n. Fourth and seventh columns represent the respective relative errors, and fifth and last columns are the respective relative error bounds.

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