Asymptotic expansions of integrals: the term by term integration method

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Abstract
The classical term by term integration technique used for obtaining asymptotic expansions of integrals requires the integrand to have an uniform asymptotic expansion in the integration variable. A modification of this method is presented in which the uniformity requirement is substituted by a much weaker condition. As we show in some examples, the relaxation of the uniformity condition provides the term by term integration technique a large range of applicability. As a consequence of this generality, Watson's lemma and the integration by parts technique applied to Laplace's and a special family of Fourier's transforms become corollaries of the term by term integration method.

Keywords: Asymptotic expansions, integral representations, term by term integration technique, Watson's lemma, integration by parts method.

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1 Introduction

Asymptotic approximation is an important topic in applied analysis. The solutions to a large kind of applied problems in fluid mechanics, electromagnetism, statistics and many other fields can, by means of integral transforms, be represented by integrals. The analytic calculation of these integrals is usually quite difficult, although they can be approximated in particular limits of interest by means of asymptotic expansions. A lot of techniques and theories have been proposed during the last decades for obtaining asymptotic expansions of functions defined by means of integrals: integration by parts, Watson's lemma, stationary phase, steepest descent, Laplace's method, Mellin transform techniques, etc... (see for example [2], [7], [11] or [12]).

Among all of them, the term by term integration method (TTIM) is one of the most intuitive tech-

niques used for deriving asymptotic approximations of integrals. Let us suppose that an asymptotic expansion of the integrand in some asymptotic sequence of the asymptotic variable is known. Then, if this expansion is uniform in the integration variable, the simple term by term integration of that sequence, if it is well defined, constitutes an asymptotic expansion of the integral [2, pg. 29, Theorem 1.7.5.]. The problem is that the uniformity condition is very uncommon in practise. On the other hand, uniform asymptotic expansions usually present a very complicated form (see for example [10] and references there in for a very complete survey on the subject). Therefore, even if an uniform asymptotic expansion of the integrand is known, the integration of the terms of this expansion may be quite difficult and the asymptotic expansion of the integral has not a computational utility.

Uniformity is a too strong condition to justify the TTIM (integral representations of many special functions for example do not satisfy this requirement). In order to make the TTIM useful for practise, the uniformity condition should be replaced by some weaker (and easier to verify) requirement. In the new version of the TTIM that we introduce in the next section, this requirement is a certain bounding condition over the integrand. A large class of integrals satisfy this requirement, which provides the TTIM a large range of applicability. For example, it is applicable to integral representations of many special functions. In section 3, new asymptotic expansions of the incomplete beta function $B_x(a, b)$ and of certain coefficients related to the Whittaker function $M_{\kappa,\mu}(z)$ are obtained using this method.

On the other hand, from a theoretical point of view, this generalization of the TTIM led us to shed some light upon the idea of unification of asymptotic methods. This idea was already suggested in 1963 by Erdelyi and Wyman [4], [13]. In their work, they show in a lucid and expository way that Darboux's method, Watson's lemma, steepest descents and stationary phase (applied to a certain kind of integrands) can be viewed as particular cases of the method of Laplace. Although from a more modern point of view, following the work of Wong [12], Watson's lemma and integration by parts should be considered as 'fundamental methods': steepest descents, Laplace's, or Perron's method for example are based on Watson's lemma, whereas stationary phase or summability methods for example are based on the integration by parts technique. In section 4 we show that the generalization of the TTIM that we present here is applicable to integrals whose integrand satisfies the conditions required by Watson's lemma. It is also applicable to Laplace transforms when the integrand satisfies the conditions required by the integration by parts method. Besides, it is applicable to a certain family of Fourier transforms that are classically solved by the integration by parts technique. These facts suggest the possibility of considering the TTIM as a 'fundamental' asymptotic technique.

2 A new version of the term by term integration method

We will consider complex functions $I(z)$ defined by means of an integral representation of the form

$$I(z) \equiv \int_{\mathcal{C}} h(w) f(w,z)\, dw, \tag{1}$$

where $z \in \mathbb{C}$, $\mathcal{C}$ is a complex path and $h : \mathcal{C} \longrightarrow \mathbb{C}$ and $f : \mathcal{C} \times (\mathbb{C} \setminus B(0,r_0)) \longrightarrow \mathbb{C}$ (where $B(0,r_0)$ denotes the open disk of center $z=0$ and radius r_0) satisfy the following two hypotheses,
A) The product $h(w)f(w,z)$ is integrable along the path $\mathcal{C}$ for $|z| \geq r_0 > 0$. The path $\mathcal{C}$ may depend on $\mathrm{Arg}(z)$ but not on $|z|$.
B) For $N = 0,1,2,...,N_0$ (N_0 finite or infinite),

$$f(w,z) = \sum_{n=0}^{N} \frac{a_n(w)}{z^{\lambda_n}} + \mathcal{O}_w(z^{-\lambda_{N+1}}), \qquad |z| \to \infty, \tag{2}$$

where $a_n : \mathcal{C} \to \mathbb{C}$, $\{\lambda_n\}$ is a sequence of complex numbers satisfying $\mathrm{Re}(\lambda_{n+1}) > \mathrm{Re}(\lambda_n) > 0 \ \forall\ n \geq 0$ and $n \leq N+1$ and the subscript w in the remainder after N terms, $\mathcal{O}_w(z^{-\lambda_{N+1}})$, means that it may depend also on w.

The classical TTIM states the following [2, pg. 29] (although the method is originally formulated for arbitrary asymptotic sequences, here we need to consider only sequences of the form $\{z^{-\lambda_n}\}$). Suppose that the remainder term in the expansion (2) of $f(w,z)$ can be bounded by a w-independent quantity along all the path $\mathcal{C}$, that is,

$$\mathcal{O}_w(z^{-\lambda_{N+1}}) = \mathcal{O}(z^{-\lambda_{N+1}}) \quad \forall\ w \in \mathcal{C}, \qquad N = 0,1,...,N_0, \tag{3}$$

where $\mathcal{O}(z^{-\lambda_{N+1}})$ is independent of w. Suppose also that

$$\left| \int_{\mathcal{C}} h(w) dw \right| < \infty \quad \text{and} \quad \left| \int_{\mathcal{C}} h(w) a_n(w) dw \right| < \infty \quad \forall\ n \leq N. \tag{4}$$

Then, the asymptotic expansion of the integral $I(z)$ with respect to the sequence $\{z^{-\lambda_n}\}$ is just given by introducing the expansion (2) into (1) and interchanging the sum and the integral,

$$\int_{\mathcal{C}} h(w) f(w,z) dw = \sum_{n=0}^{N} \left[\int_{\mathcal{C}} h(w) a_n(w) dw \right] \frac{1}{z^{\lambda_n}} + \mathcal{O}(z^{-\lambda_{N+1}}), \qquad |z| \to \infty. \tag{5}$$

The evident advantage of this procedure is its great simplicity. But, if $f(w,z)$ and/or $\mathcal{C}$ are not bounded, it may be difficult to prove that the remainder term $\mathcal{O}_w(z^{-\lambda_{N+1}})$ is bounded by a w-independent quantity $\mathcal{O}(z^{-\lambda_{N+1}})$. This happens for example when $f(w,z) = f(w/z)$ and $\mathcal{C}$ is unbounded. In this case, $\mathcal{O}_w(z^{-\lambda_{N+1}}) = \mathcal{O}((w/z)^{\lambda_{N+1}})$ may not be bounded for unbounded values of w. A simple classical example is the following,

Example 1. *the Exponential Integral* [12, p. 14, Eq. 4.1].

$$\mathrm{Ei}(z) = \int_{-\infty}^{z} \frac{e^w}{w} dw, \qquad |\mathrm{Arg}(-z)| < \pi, \qquad z \neq 0, \tag{6}$$

where the integration path is defined by $-\infty <\mathrm{Re}(w) \leq\mathrm{Re}(z)$ and $\mathrm{Im}(w) =\mathrm{Im}(z)$. After the change of variable $w = z - x$ we obtain

$$\mathrm{Ei}(z) = \frac{e^z}{z} \int_0^{\infty} e^{-x} \left(1 - \frac{x}{z}\right)^{-1} dx. \tag{7}$$

Therefore, we can take $h(x) \equiv e^{-x}$,

$$f(x,z) = f\left(\frac{x}{z}\right) \equiv \left(1 - \frac{x}{z}\right)^{-1} = \sum_{n=0}^{N} \left(\frac{x}{z}\right)^n + \mathcal{O}\left(\left(\frac{x}{z}\right)^{N+1}\right) \tag{8}$$

and $a_n(x) \equiv x^n$. This expansion is not uniform in $x \in [0, \infty)$ and the classical term by term integration method of [2] can not be applied.

Integral representations of many other special functions [1] are also examples of failure of the classical TTIM. The argument is similar: they are of (or can be expressed in) the form

$$I(z) \equiv \int_{\mathcal{C}} h(w) f\left(\frac{w}{z}\right) dw, \tag{9}$$

where $\mathcal{C}$ is unbounded and then, the uniformity (3) of the expansion (eq. (2)) does not hold. In the following, we will consider only integrands of the form (9) for which hypothesis B) reads

$$f\left(\frac{w}{z}\right) = \sum_{n=0}^{N} a_n \left(\frac{w}{z}\right)^{\lambda_n} + \mathcal{O}\left(\left(\frac{w}{z}\right)^{\lambda_{N+1}}\right), \qquad |z| \to \infty, \qquad a_n \in \mathbb{C}. \tag{10}$$

In order to show that the term by term integration of the expansion (10) in (9) is an asymptotic expansion of $I(z)$ in the sequence $\{z^{-\lambda_n}\}$, we need to show that the remainder after N terms,

$$\epsilon_N(z) \equiv \int_{\mathcal{C}} h(w) \left[f\left(\frac{w}{z}\right) - \sum_{n=0}^{N} a_n \left(\frac{w}{z}\right)^{\lambda_n} \right] dw, \tag{11}$$

satisfies $\epsilon_N(z) = \mathcal{O}\left(z^{-\lambda_{N+1}}\right)$. Uniformity (3) and the first bound in (4) guarantee this. But condition (3) may be relaxed by using the following observation,

Remark 1. Hypothesis (10) above means that for $N = 0, 1, 2, ..., N_0$,

$$\forall \ \delta > 1 \ \exists \ r_N(\delta) > 0, \ \left|z^{\lambda_{N+1}}\right| \left| f\left(\frac{w}{z}\right) - \sum_{n=0}^{N} a_n \left(\frac{w}{z}\right)^{\lambda_n} \right| \leq \delta \left|a_{N+1} w^{\lambda_{N+1}}\right|$$
$$\forall \ w \in \mathcal{C}, \ |z| \geq r_N |w| \ \text{ and } \ |z| \geq r_0. \tag{12}$$

Therefore, the remainder in (10) is bounded, at least for $|z| \geq r_N|w|$, by the quantity $\delta \left|a_{N+1}(w/z)^{\lambda_{N+1}}\right|$. But, if we can find a function $g : \mathcal{C} \longrightarrow \mathbb{R}^+$, satisfying

$$\left| f\left(\frac{w}{z}\right) \right| \leq g(w) \quad \forall \ w \in \mathcal{C}, \ |z| < r_N|w| \quad \text{and} \quad |z| \geq r_0, \tag{13}$$

then, this remainder is also bounded by a $\mathcal{O}\left(z^{-\lambda_{N+1}}\right)$-quantity for $|z| < r_N|w|$. This is so because for $N = 0, 1, 2, ..., N_0$ and $\forall\ |z| \geq r_0$,

$$\left| z^{\lambda_{N+1}} \left(f\left(\frac{w}{z}\right) - \sum_{n=0}^{N} a_n \left(\frac{w}{z}\right)^{\lambda_n} \right) \right| \leq \left| (r_N w)^{\lambda_{N+1}} \right| \left[g(w) + \sum_{n=0}^{N} \frac{|a_n|}{r_N^{\lambda_n}} \right] \ \ \forall\ \ w \in \mathcal{C}, \ \text{ and } \ |z| < r_N|w|. \quad (14)$$

Joining inequalities (12) and (14) we obtain, for $N = 0, 1, 2, ..., N_0$, a $\mathcal{O}\left(z^{-\lambda_{N+1}}\right)$ bound for the remainder term in (10) uniformly valid $\forall\ |z| \geq r_0$,

$$\left| \left(f\left(\frac{w}{z}\right) - \sum_{n=0}^{N} a_n \left(\frac{w}{z}\right)^{\lambda_n} \right) \right| \leq \left| z^{-\lambda_{N+1}} \right| \left[g(w) + \delta \sum_{n=0}^{N+1} \frac{|a_n|}{r_N^{\lambda_n}} \right] \left| (r_N w)^{\lambda_{N+1}} \right| \ \ \forall\ \ w \in \mathcal{C}. \quad (15)$$

Therefore, the remainder $\epsilon_N(z)$ is bounded by the integral of $h(w)$ times the right hand side of the above inequality. If this integral is finite, then $\epsilon_N(z) = \mathcal{O}\left(z^{-\lambda_{N+1}}\right)$. We can summarize the above discussion in the following

Theorem 1. Consider a complex function $I(z)$ defined by the integral (9) and suppose that the following conditions are satisfied.

i) The product $h(w)f(w/z)$ is integrable along the path $\mathcal{C}$ for $|z| \geq r_0 > 0$. The path $\mathcal{C}$ may depend on $\text{Arg}(z)$ but not on $|z|$.

ii) For $N = 0, 1, 2, ..., N_0$ (N_0 finite or infinite),

$$f\left(\frac{w}{z}\right) = \sum_{n=0}^{N} a_n \left(\frac{w}{z}\right)^{\lambda_n} + \mathcal{O}\left(\left(\frac{w}{z}\right)^{\lambda_{N+1}} \right), \qquad |z| \to \infty, \quad (16)$$

where $a_n \in \mathbb{C}$ and $\{\lambda_n\}$ is a sequence of complex numbers satisfying $\text{Re}(\lambda_{n+1}) > \text{Re}(\lambda_n) > 0\ \forall\ n \geq 0$ and $n \leq N + 1$.

iii) There is a function $g : \mathcal{C} \longrightarrow \mathbb{R}^+$, satisfying

$$\left| f\left(\frac{w}{z}\right) \right| \leq g(w) \quad \forall\ \ w \in \mathcal{C}, \ \ |z| < r_N|w| \quad \text{ and } \quad |z| \geq r_0. \quad (17)$$

iv) For $n = 0, 1, 2, ..., N + 1$,

$$\int_{\mathcal{C}} |h(w) w^{\lambda_n} dw| < \infty \quad \text{ and } \quad \int_{\mathcal{C}} |h(w) g(w) w^{\lambda_n} dw| < \infty. \quad (18)$$

Then, the asymptotic expansion of $I(z)$ with respect to the sequence $\{z^{-\lambda_n}\}$, for $|z| \to \infty$, is given by

$$I(z) = \sum_{n=0}^{N} \left[\int_{\mathcal{C}} h(w) w^{\lambda_n} dw \right] \frac{a_n}{z^{\lambda_n}} + \epsilon_N(z), \quad (19)$$

where the remainder after N terms, $\epsilon_N(z)$, is bounded by

$$|\epsilon_N(z)| \leq \left| \left(\frac{r_N}{z}\right)^{\lambda_{N+1}} \right| \int_{\mathcal{C}} |h(w)| \left[g(w) + \delta \sum_{n=0}^{N+1} \left| \frac{a_n}{r_N^{\lambda_n}} \right| \right] \left| w^{\lambda_{N+1}} \right| dw \quad (20)$$

and δ and $r_N(\delta)$ are defined in (12).

Remark 2. The verification of conditions iii) and iv) of Theorem 1 may depend on $\mathrm{Arg}(z)$. This fact is related with the Stokes lines. If the expansion (16) is valid only in a certain sector of $\mathbb{C}$, then the Stokes rays also depend on this sector.

The main difference between this version and the classical method lies in the substitution of the uniformity requirement (3) by the much weaker condition iii). But, if $r_N(\delta)$ in eq. (12) is unknown, condition iii) may be not easily checked. Nevertheless, in practice, an inequality like $|f(w/z)| \leq g(w)$ usually holds $\forall\ |z| \geq r_0$ and hence, the explicit value of r_N is not needed for checking condition iii). This fact is shown in Example 1.

Example 1 (Continuation). For any $r_0 > 0$ we have that

$$\left|f\left(\frac{x}{z}\right)\right| \equiv \left|1-\frac{x}{z}\right|^{-1} \leq \begin{cases} 1 & \text{if } \ |\mathrm{Arg}(z)| \geq \pi/2 \\ |\sin(\mathrm{Arg}(z))|^{-1} & \text{if } \ 0 < |\mathrm{Arg}(z)| \leq \pi/2 \end{cases} \tag{21}$$

and conditions i)-iv) of Theorem 1 are satisfied with $g(x) \equiv h(\mathrm{Arg}(z))$, where we have denoted $h(\mathrm{Arg}(z))$ the second hand of the above inequality. Applying Theorem 1 and after a straightforward algebra we obtain the well-known result [12, p. 14, Eq. 4.3]

$$\mathrm{Ei}(z) \sim \frac{e^z}{z}\sum_{n=0}^{\infty}\frac{n!}{z^n}, \qquad \mathrm{Arg}(z) \neq 0. \tag{22}$$

Nevertheless, the bound (20) for the remainder term is unpractical unless $r_N(\delta)$ is known. This problem can be solved if the expansion (16) is convergent in some region $|z| \geq r|w| > 0$. Moreover, if (16) is convergent in this region, then the constant r may be used as r_N in (17) to verify condition iii). These points are clarified in the following

Corollary 1. Assume that,

ii') The function $f(t)$ admits a convergent series expansion for $|t| \leq r^{-1}$, $r > 0$,

$$f\,(t) = \sum_{n=0}^{\infty} a_n t^{\lambda_n}, \qquad |t| \leq r^{-1}, \tag{23}$$

for some $r > 0$ and conditions i), iii) and iv) of Theorem 1 hold for $r_N = r$. Then, the asymptotic expansion of $I(z)$ in the sequence $\{z^{-\lambda_n}\}$ is given by (19) and an error bound for the remainder is given by

$$|\epsilon_N(z)| \leq \left|\left(\frac{r}{z}\right)^{\lambda_{N+1}}\right| \int_{\mathcal{C}} |h(w)| \left[g(w) + \sum_{n=0}^{\infty}\left|\frac{a_n}{r^{\lambda_n}}\right|\right] \left|w^{\lambda_{N+1}}\right| |dw| \tag{24}$$

Proof. Hypothesis ii') implies

$$\left|z^{\lambda_{N+1}}\right| \left|f\left(\frac{w}{z}\right) - \sum_{n=0}^{N} a_n \left(\frac{w}{z}\right)^{\lambda_n}\right| \leq \left|(rw)^{\lambda_{N+1}}\right| \sum_{n=N+1}^{\infty}\left|\frac{a_n}{r^{\lambda_n}}\right| \ \ \forall\ w \in \mathcal{C}, \ \ |z| \geq r|w|. \tag{25}$$

And condition iii) with $r_N = r$ implies

$$\left|z^{\lambda_{N+1}}\right| \left| f\left(\frac{w}{z}\right) \sum_{n=0}^{N} a_n \left(\frac{w}{z}\right)^{\lambda_n} \right| \leq \left|(rw)^{\lambda_{N+1}}\right| \left[g(w) + \sum_{n=0}^{N} \left| \frac{a_n}{r^{\lambda_n}} \right| \right] \quad \forall \ w \in \mathcal{C}, \ \ |z| < r|w|. \tag{26}$$

Then, by using inequalities (25) and (26) and the definition of the remainder term (11), the bound (24) follows. This bound is finite by virtue of condition iv). Therefore, $\epsilon_N(z) = \mathcal{O}(z^{-\lambda_N})$ and eq. (19) is the asymptotic expansion of $I(z)$ in the sequence $\{z^{-\lambda_n}\}$. □

Example 1 (Continuation). The expansion (8) is in fact convergent for $|z|/x \geq (N+2)/(N+1)$,

$$f(t) \equiv (1-t)^{-1} = \sum_{n=0}^{\infty} t^n, \qquad |t| \leq \frac{N+1}{N+2}. \tag{27}$$

Therefore, using Corollary 1 with $r \equiv (N+2)/(N+1)$, a bound for the remainder after N terms is given by

$$\epsilon_N(z) \leq (h(\mathrm{Arg}(z)) + N + 2) e^{\mathrm{Re}(z)+1} \frac{(N+1)!}{|z|^{N+2}}. \tag{28}$$

Asymptotic approximations of integral representations of many other special functions [1] have been obtained by using several asymptotic methods. They could also be obtained in a systematic and easy way using this modified TTIM. The procedure would be similar to the employed in Example 1.

3 Nontrivial Examples

The substitution of the uniformity condition (3) by the weaker condition (17) of Theorem 1 provides the TTIM with a large range of applicability. Besides, its easy implementation lets us to obtain asymptotic expansions of more or less sophisticated integrals. In this section we obtain, using Corollary 1, the asymptotic expansion of the incomplete Beta function $B_x(a,b)$ in the sequence $\{a^{-n}\}$. In Example 3, Corollary 1 let us to obtain an asymptotic approximation of the coefficients of the expansion of the Whittaker function $M_{\kappa,\mu}(z)$ in power series of κ from an intricate integral representation.

Example 2. *The incomplete beta function* $B_x(a,b)$. An integral representation of $B_x(a,b)$ is given by [11, p. 128],

$$B_x(a,b) = \int_0^x y^{a-1}(1-y)^{b-1} dy, \qquad a > 0, \ \ b > 0, \ \ 0 \leq x \leq 1. \tag{29}$$

With the change of variable $y = xe^{-t/a}$ we obtain

$$B_x(a,b) = \frac{x^a}{a} \int_0^{\infty} e^{-t}(1 - xe^{-t/a})^{b-1} dt, \tag{30}$$

Therefore, we have $\mathcal{C} \equiv [0,\infty)$, $h(t) \equiv e^{-t}$ and $f(t/a) \equiv (1 - xe^{-t/a})^{b-1}$. The asymptotic expansion of $f(t/a)$ with respect to the sequence $(t/a)^n$ is its Taylor expansion around 0,

$$f\left(\frac{t}{a}\right) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \left(\frac{t}{a}\right)^n, \qquad \left|\frac{t}{a}\right| < -\log x. \tag{31}$$

After some algebra, it may be verified that this expansion is given by

$$f^{(n)}(0) = (1-x)^{b-1} \sum_{k=0}^{n} (-1)^n (1-b)_k \alpha_{n,k} \left(\frac{x}{1-x} \right)^k, \tag{32}$$

where the coefficients $\alpha_{n,k}$ are universal and defined recursively by

$$\begin{aligned} \alpha_{0,0} &= 1, \quad \alpha_{n,k} = 0 \text{ for } k < 0 \text{ or } k > n, \\ \alpha_{n,k} &= \alpha_{n-1,k-1} + k\alpha_{n-1,k}, \text{ for } 0 \le k \le n. \end{aligned} \tag{33}$$

Expansion (31) is convergent for $|t/a| \le (-\log x)/\delta$ for any $\delta > 1$. In order to apply Corollary 1, we can take $r \equiv (-\log x)/\delta$. The function $f(t/a)$ satisfies hypotheses i) and ii') with $a_n \equiv f^{(n)}(0)/n!$ and $\lambda_n \equiv n$. On the other hand, $\forall\ t \in [0, \infty)$, we have that $f(t/a)$ satisfies condition (17) $\forall\ a > 0$ with

$$g(t) \equiv 1 + (1-x)^{b-1}, \tag{34}$$

where $g(t)$ and $h(t)$ satisfy (18). Therefore we can apply Corollary 1 and, after a straightforward algebra, we obtain,

$$B_x(a,b) \sim \frac{x^a (1-x)^{b-1}}{a} \sum_{n=0}^{\infty} (-1)^n \left[\sum_{k=0}^{n} (1-b)_k \alpha_{n,k} \left(\frac{x}{1-x} \right)^k \right] \frac{1}{a^n}. \tag{35}$$

A bound for the remainder after N terms can be obtained by using formula (24). This expansion in the sequence $\{a^{-n}\}$ may be 'added' to the list of expansions of the incomplete beta function (see for example [3], [6], [9] and [11, sec. 11.3] and references there in).

Example 3. The coefficients of the expansion of the Whittaker function $M_{\kappa,\mu}(z)$ in power series of κ,

$$M_{\kappa,\mu}(z) = 2^{2\mu} \Gamma(\mu+1) z^{1/2} \sum_{m=0}^{\infty} F_m^{(\mu)}(z) \frac{(-\kappa)^m}{m!} \tag{36}$$

are given by [5],

$$F_m^{(\mu)}(z) = \frac{e^{z/2} \Gamma(\mu + 1/2)}{\sqrt{\pi z}} \sum_{k=0}^{m} \binom{m}{k} (\ln z)^{m-k} I_k^{(\mu)}(z), \tag{37}$$

where the functions $I_k^{(\mu)}(z)$ are defined by the integral representation

$$I_k^{(\mu)}(z) = 2^{2\mu} \frac{(-1)^k}{2\pi i} \int_{\mathcal{C}} e^{w/2} w^{-(2\mu+1)} \frac{d^k}{d\mu^k} \left(\left(\frac{w}{2} \right)^{\mu+1/2} f\left(\frac{w}{z} \right) \right) dw, \tag{38}$$

$$f\left(\frac{w}{z} \right) \equiv \left(1 + \frac{w}{2z} \right)^{-(\mu+1/2)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\mu + \frac{1}{2} \right)_n \left(\frac{w}{2z} \right)^n, \quad |w| < |2z|, \tag{39}$$

and the contour $\mathcal{C}$ starts at $-\infty$ on the real axes, encircles the points 0 and $-2z$ in the counterclockwise direction and returns to the starting point. We are interested in approximating the coefficients $F_m^{(\mu)}(z)$ for large values of z. Then, we use Corollary 1 for deriving an asymptotic expansion of the integrals

$I_k^{(\mu)}(z)$ for large z. We must take $h(w) \equiv e^{w/2} w^{-(2\mu+1)} \frac{d^k}{d\mu^k} \left(\left(\frac{w}{2}\right)^{\mu+1/2} \cdot \right)$, $a_n \equiv (\mu+1/2)_n/((-2)^n n!)$, $\lambda_n \equiv n$ and $r \equiv 1$. After some algebra we find that the function $g(w)$ may be taken

$$g(w) = e^{\mathrm{Re}(w)/2} \left| \frac{w^N}{(2w)^{\mu+1/2}} \right| \sum_{l=0}^{k} \left(\begin{array}{c} k \\ l \end{array} \right) \left| \ln \frac{w}{2} \right|^{k-l} g_l(w), \tag{40}$$

where

$$g_l(w) = \sum_{n=N+1}^{\infty} \frac{1}{2^n n!} \left| \frac{d^l}{d\mu^l} (\mu + \tfrac{1}{2})_n \right| + e^{2|\mathrm{Im}(\mu)|\pi} |2w|^{\mathrm{Re}(\mu)+l+1} + \sum_{n=0}^{N} \frac{|w|^n}{n!} \left| \frac{d^l}{d\mu^l} (\mu + \tfrac{1}{2})_n \right|, \tag{41}$$

provided the contour $\mathcal{C}$ is such that all its points w satisfy $|w| > 1$ and $|w + 2z| \geq 1$.

Then, using Corollary 1 and after straightforward operations we obtain that the functions $I_k^{(\mu)}(z)$ admit asymptotic expansions

$$I_k^{(\mu)}(z) \sim \sum_{n=0}^{\infty} \left[\sum_{j=0}^{k} \binom{k}{j} \left(\frac{d^j (\mu+\frac{1}{2})_n}{d(-\mu)^j} \right) \frac{d^{k-j}}{d\mu^{k-j}} \left(\frac{1}{\Gamma(\mu-n+\frac{1}{2})} \right) \right] \frac{(-1)^n}{n! z^n}. \tag{42}$$

Introducing this expansion into (37) we obtain an asymptotic expansion of $F_m^{(\mu)}(z)$. This result may be checked by substituting now (37) into (36), grouping equal powers of $\log(z)$ and reordering sums. One obtains in this way the familiar asymptotic expansion of the Whittaker function in the sequence $\{z^{-n}\}$.

4 Corollaries of the term by term integration method

As we see in Theorem 1 or Corollary 1, the TTIM does not require a too special form for the integrand, but only the bounding and integrability conditions iii)-iv). On the other hand, other classical techniques (such as those mentioned in the introduction) are adapted to particular forms of the integrand. Therefore, if those kind of integrands satisfy conditions i)-iv) of Theorem 1, the TTIM becomes a more general technique. In this section we show that, at least, Watson's lemma and the integration by parts technique applied to Laplace transforms and a special family of Fourier transforms are corollaries of this version of the TTIM.

Corollary 2. (*Watson's lemma* [7 p. 113]). Suppose that $I(z)$ is defined by the integral

$$I(z) = \int_0^{\infty} e^{-zt} q(t) dt, \tag{43}$$

where

a) $q : \mathbb{C} \longrightarrow \mathbb{C}$ has a finite number of discontinuities and infinities in the positive real axis.

b) As $t \to 0^+$,

$$q(t) \sim \sum_{n=0}^{\infty} a_n t^{(n+a-b)/b}, \tag{44}$$

where $a_n \in \mathbb{C}$, $b > 0$ and $\mathrm{Re}(a) > 0$. And

c) The integral (43) has abscissa of convergence $\xi \equiv r_0 \cos(\mathrm{Arg}(z)) > 0$ (it is integrable for $|z| \geq r_0 > 0$ and $|\mathrm{Arg}(z)| < \pi/2$).

Then, for $|z| \to \infty$ and $|\mathrm{Arg}(z)| < \pi/2$,

$$I(z) \sim \sum_{n=0}^{\infty} \Gamma\left(\frac{n+a}{b}\right) \frac{a_n}{z^{(n+a)/b}}. \tag{45}$$

And this is actually a strong asymptotic expansion of $I(z)$,

$$|\epsilon_N(z)| \leq \left| r_N^{(N+a+1)/b} \right| \frac{e^{-\mathrm{Im}(a)\mathrm{Arg}(z)/b}}{|z^{(N+a+1)/b}|} \left[\frac{K}{r_0} + \delta \sum_{n=0}^{N+1} \left| \frac{a_n}{r_N^{(n+a)/b}} \right| \right] \frac{\Gamma((N+\mathrm{Re}(a)+1)/b+1)}{(\cos(\mathrm{Arg}(z))^{(N+\mathrm{Re}(a)+1)/b+1}}, \tag{46}$$

where $K > 0$ is a bound of $|q(t)|$ in a certain interval given bellow.

Proof. For any $T > 0$ we write

$$I(z) = \int_0^T e^{-zt} q(t) dt + \int_T^{\infty} e^{-zt} q(t) dt = \frac{1}{z} \int_0^{zT} e^{-w} q\left(\frac{w}{z}\right) dw + e^{-zT} \int_0^{\infty} e^{-zt} q\left(t+T\right) dt. \tag{47}$$

Using condition c), the second integral in the last line satisfies

$$\left| \int_0^{\infty} e^{-zt} q\left(t+T\right) dt \right| \leq \int_{-T}^{\infty} \left| e^{-\xi t} q\left(t+T\right) \right| dt \leq e^{\xi T} \int_0^{\infty} \left| e^{-\xi t} q\left(t\right) \right| dt < \infty \tag{48}$$

and therefore,

$$I(z) = \frac{1}{z} \int_0^{zT} e^{-w} q\left(\frac{w}{z}\right) dw + \mathcal{O}(e^{-zT}). \tag{49}$$

Condition b) above means that

$$\frac{w}{z} q\left(\frac{w}{z}\right) \sim \sum_{n=0}^{\infty} a_n \left(\frac{w}{z}\right)^{(n+a)/b} \tag{50}$$

and then, hypothesis iii) of Theorem 1 is satisfied for $\lambda_n \equiv (n+a)/b$ and $f(w/z) \equiv (w/z) q(w/z)$. Using condition a) above, we can choose T such that $0 < T <$ first singularity of $q(t)$ apart from an eventual singularity at $t = 0$. Therefore,

$$\forall \ r_N^{-1} > 0, \quad \exists \ K > 0, \quad \left| q\left(\frac{w}{z}\right) \right| \leq K \quad \text{for} \quad r_N^{-1} \leq \frac{w}{z} \leq T \tag{51}$$

and condition iii) of Theorem 1 is satisfied with $g(w) \equiv K|w|/r_0$ for any $r_0 > 0$. Condition iv) of Theorem 1 is trivially satisfied for the integral in eq. (49) with $h(w) \equiv w^{-1} e^{-w}$ and $\mathcal{C} \equiv [0, zT)$. On the other hand, using condition c) above, the first requirement of hypothesis i) holds, but not the second one, because $\mathcal{C}$ depends on $|z|$. The independence of $|z|$ is used in the proof of Theorem 1 to obtain that the coefficients of the expansion (19) and the bound given in eq. (20) for $|\epsilon_N(z) z^{\lambda_{N+1}}|$ are independent of $|z|$. Nevertheless, for the case $\mathcal{C} \equiv [0, zT)$, the dependence of this bound on $|z|$ can be trivially eliminated,

$$|\epsilon_N(z)| \leq \left| \left(\frac{r_N}{z}\right)^{(N+a+1)/b} \right| \int_0^{\infty \exp(i\mathrm{Arg}(z))} |e^{-w}| \left[\frac{K}{r_0} + \frac{\delta}{w} \sum_{n=0}^{N+1} \left| \frac{a_n}{r_N^{(n+a)/b}} \right| \right] \left| w^{(N+a+1)/b} \right| |dw|, \tag{52}$$

and after straightforward operations, we obtain (46). On the other hand, we are going to prove in what follows that the coefficients of the expansion (19) are also independent of $|z|$, but for a quantity $\mathcal{O}(e^{-zT})$. This coefficients are given by

$$\int_0^{zT} e^{-w} w^{(n+a)/b-1} dw = \int_0^{\infty e^{i\mathrm{Arg}(zT)}} e^{-w} w^{(n+a)/b-1} dw - \int_{zT}^{\infty e^{i\mathrm{Arg}(zT)}} e^{-w} w^{(n+a)/b-1} dw. \tag{53}$$

The second integral on the right hand side above can be written

$$\int_{zT}^{\infty e^{i\mathrm{Arg}(zT)}} e^{-w} w^{(n+a)/b-1} dw = M_n(z) e^{-zT}, \tag{54}$$

where

$$M_n(z) \equiv \int_0^{\infty e^{i\mathrm{Arg}(zT)}} e^{-w} (w+zT)^{(n+a)/b-1} dw. \tag{55}$$

A bound for the modulus of the integrand in the above integral is given by

$$\left|(w+zT)^{(n+a)/b-1}\right| \leq e^{2\pi|\mathrm{Im}(a/b)|}(|w|+T|z|)^{I_n}, \qquad |z| \geq r_0, \tag{56}$$

where we have abbreviated $I_n \equiv \mathrm{Int}(\mathrm{Re}((n+a)/b))$ and chosen $r_0 \geq T^{-1}$. Using (56) and after straightforward operations, we find that $|M_n(z)|$ is bounded by a $\mathcal{O}(z^{I_n})$ quantity. Therefore, the left hand side of eq. (54) is a $\mathcal{O}(e^{-zT})$ quantity. We can apply Theorem 1 and we obtain (19), where the integrals inside the brackets equals $\Gamma((n+a)/b)$, but for a $\mathcal{O}(e^{-zT})$ quantity and (45) holds. □

Corollary 3. (*Integration by parts technique for Laplace transforms* [2 p. 78]). Suppose that $I(z)$ is defined by the integral

$$I(z) = \int_a^b e^{-zt} q(t) dt, \tag{57}$$

where $0 \leq a < b \leq \infty$, $\mathrm{Re}(z) \geq r_0 > 0$ and $q : [a,b] \longrightarrow \mathbb{C}$ has N continuous derivatives in $[a,b]$, where $[a,b]$ must be set $[a,\infty)$ if $b = \infty$.

Then, for $|z| \to \infty$,

$$I(z) = \frac{e^{-az}}{z}\left[\sum_{n=0}^{N} \frac{q^{(n)}(a)}{z^n} + \mathcal{O}(z^{-(N+1)})\right] - \frac{e^{-bz}}{z}\left[\sum_{n=0}^{N} \frac{q^{(n)}(b)}{z^n} + \mathcal{O}(z^{-(N+1)})\right] \quad \text{if} \quad b < \infty, \tag{58}$$

$$I(z) = \frac{e^{-az}}{z}\left[\sum_{n=0}^{N} \frac{q^{(n)}(a)}{z^n} + \mathcal{O}(z^{-(N+1)})\right] \quad \text{if} \quad b = \infty \tag{59}$$

and the remainder term $\epsilon_N(z)$ is bounded by

$$|\epsilon_N(z)| \leq \left|\left(\frac{r_N}{z}\right)^{N+1}\right| \left[\left(\epsilon_1 K + \delta \sum_{n=0}^{N+1} \frac{|q^{(n)}(a)|}{n! r_N^n}\right)\left|e^{-az}\right| + \epsilon_2\left(K + \delta \sum_{n=0}^{N+1} \frac{|q^{(n)}(b)|}{n! r_N^n}\right)\left|e^{-bz}\right|\right] \frac{(N+1)!}{(\cos(\mathrm{Arg}(z)))^{N+2}}, \tag{60}$$

where $\epsilon_1 = \epsilon_2 = 1$ and $K \geq 1$ is a bound for $|q(t)|$ in $[a, b]$ if $b < \infty$ and $\epsilon_1 = 2^{N+2}$, $\epsilon_2 = 0$ and $K \geq 1$ is a bound for $|e^{-zt}q(t)|$ in $[a, \infty)$ if $b = \infty$.

Proof. We consider first the case $b < \infty$. We extend the function $q(t)$ to a $C^{(N)}$-function in $[a, \infty)$ with $q(t) \equiv 0$ in a neighborhood of ∞. The explicit extension of $q(t)$ to $[b, \infty)$ is not required. Nevertheless, a construction of this extension is indicated for example in [8, p. 418, Exs. 11 and 12]. Then, we can write

$$I(z) = \int_a^\infty e^{-zt}q(t)dt - \int_b^\infty e^{-zt}q(t)dt \tag{61}$$

and perform the changes of variable $t = a + w/z$ and $t = b + w/z$ in each one of the respective integrals on the right hand side of the above equation,

$$I(z) = \frac{e^{-az}}{z}\int_0^{\infty\exp(i\mathrm{Arg}(z))} e^{-w}q\left(\frac{w}{z} + a\right)dw - \frac{e^{-bz}}{z}\int_0^{\infty\exp(i\mathrm{Arg}(z))} e^{-w}q\left(\frac{w}{z} + b\right)dw. \tag{62}$$

But, using that $q(t)$ has N continuous derivatives in the interval $[a, b]$,

$$q\left(\frac{w}{z} + t\right) = \sum_{n=0}^{N} \frac{q^{(n)}(t)}{n!}\left(\frac{w}{z}\right)^n + \mathcal{O}\left(\left(\frac{w}{z}\right)^{N+1}\right) \quad \forall\ t \in [a, b] \text{ and } \forall\ w \in [0, \infty e^{i\mathrm{Arg}(z)}). \tag{63}$$

Therefore, the integrands in the right hand side of equation (62) satisfy hypothesis ii) of Theorem 1 up to N terms with $f(w/z) \equiv q(w/z + a)$, $a_n \equiv q^{(n)}(a)/n!$ and $f(w/z) \equiv q(w/z + b)$, $a_n \equiv q^{(n)}(b)/n!$ respectively and $\lambda_n \equiv n$.

On the other hand, using that $q(t)$ is a continuous function in $[a, \infty)$ and $q(t) \equiv 0$ around ∞, we have that $\exists\ K \geq 1$ and $K < \infty$ satisfying $|q(t)| \leq K\ \forall\ t \in [a, \infty)$. Therefore, $f(w/z) \leq K\ \forall\ w \in [0, \infty e^{i\mathrm{Arg}(z)})$ and conditions i), iii) and iv) of Theorem 1 are satisfied with $\mathcal{C} \equiv [0, \infty e^{i\mathrm{Arg}(z)})$, $g(w) \equiv K$ for any $r_0 > 0$ and $h(w) \equiv e^{-w}$ for each one of the integrals of equation (62). Then, using (63) and applying this theorem we obtain (58) from (62). The bound (60) follows trivially from (20).

The proof is similar for the case $b = \infty$ setting the second integral in eq. (61) equal to zero. But in this case, $|q(t)|$ may not be bounded in $[0, \infty)$. Nevertheless, $|e^{-zt}q(t)|$ must be bounded $\forall\ t \in [a, \infty)$ by some constant $K \geq 1\ \forall\ |z| \geq r_0$. Therefore, $|e^{-r_0 w/z}q\left(w/z + a\right)| \leq K\ \forall\ |z| \geq r_0$, $\forall\ w \in [0, \infty e^{i\mathrm{Arg}(z)})$ and then, $|q\left(w/z + a\right)| \leq Ke^{\mathrm{Re}(w)/2}\ \forall\ |z| \geq 2r_0$, $\forall\ w \in [0, \infty e^{i\mathrm{Arg}(z)})$ and conditions of Theorem 1 are satisfied. The bound (60) follows from eq. (20) for $g(w) \equiv Ke^{\mathrm{Re}(w)/2}$. □

Corollary 4. (*Integration by parts technique for Fourier transforms*, [2, p. 78], [12 p. 15]). Suppose that $I(x)$ is defined by the integral

$$I(x) = \int_a^b e^{ixt}q(t)dt, \tag{64}$$

where $0 \leq a < \infty$, $0 < b < \infty$, $x \in \mathbb{R}$, $|x| \geq x_0$ and $q : [a, b] \longrightarrow \mathbb{C}$ has N continuous derivatives in the interval $[a, b]$.

Then, for $|x| \to \infty$,

$$I(x) = \frac{ie^{iax}}{x}\left[\sum_{n=0}^{N} \frac{q^{(n)}(a)}{(-ix)^n} + \mathcal{O}(x^{-(N+1)})\right] - \frac{ie^{ibx}}{x}\left[\sum_{n=0}^{N} \frac{q^{(n)}(b)}{(-ix)^n} + \mathcal{O}(x^{-(N+1)})\right] \tag{65}$$

and the remainder term $\epsilon_N(z)$ is bounded by

$$|\epsilon_N(z)| \leq \left(\frac{r_N}{|x|}\right)^{N+1} \left[2K + \delta \sum_{n=0}^{N+1} \frac{|q^{(n)}(a)| + |q^{(n)}(b)|}{n! r_N^n}\right] (N+1)! \tag{66}$$

Proof. Similar to that of the above corollary setting $z \equiv -ix$. □

We have excluded the case $b = \infty$ in this corollary because in this case, the bound $|f(iw/x)| \leq K$ is not enough to guarantee the fulfillment of condition iv) of Theorem 1. Nevertheless, eqs. (65)-(66) remain valid for $b = \infty$ if the conditions of the following corollary are satisfied.

Corollary 5. Suppose that $I(x)$ is defined by eq. (64) with $b \equiv \infty$ and

a) $\exists\ \alpha \in (0, \pi/2)$ if $x > 0$ or $\alpha \in (-\pi/2, 0)$ if $x < 0$ satisfying

$$\int_0^{\infty e^{i\alpha}} \left|q(w+a)e^{ixw}dw\right| < \infty \quad \forall\ |x| \geq x_0. \tag{67}$$

b) The function $q(w+a)$ is analytic in the sector $S_\alpha \equiv \{w \in \mathbb{C},\ 0 \leq |\mathrm{Arg}(w)| \leq \alpha\}$.

Then, for $|x| \to \infty$, eq. (65) and the bound (66) hold setting $q^{(n)}(b) = 0\ \forall\ n \geq 0$.

Proof. We perform changes of variable $t = a + s$ in the integral (64) for $b \equiv \infty$. Then, using condition b) and the residue theorem,

$$I(x) = e^{iax} \int_0^{\infty \exp(i\alpha)} e^{ixs} q\,(s+a)\,ds. \tag{68}$$

With the change of variable $s = iw/x$, the above equation reads

$$I(x) = \frac{ie^{iax}}{x} \int_0^{\infty \exp i(\alpha \mp \pi/2)} e^{-w} q\left(\frac{iw}{x} + a\right) dw, \tag{69}$$

where $\mp$ stands for $x > 0$ and $x < 0$ respectively. Using condition b) again, the integrand of the above equation satisfies hypothesis ii) of Theorem 1 with $a_n \equiv i^n q^{(n)}(a)/n!$, $\lambda_n \equiv n$ and $f(w/x) \equiv q(iw/x + a)$,

$$q\left(\frac{iw}{x} + a\right) = \sum_{n=0}^{N} \frac{q^{(n)}(a)}{n!} \left(\frac{iw}{x}\right)^n + \mathcal{O}\left(\left(\frac{w}{x}\right)^{N+1}\right). \tag{70}$$

On the other hand, using conditions a) and b), $\exists\ K \geq 1$ and $K < \infty$ satisfying $|q(s+a)e^{ixs}| \leq K\ \forall\ s \in [0, \infty e^{i\alpha})$ and $|x| \geq x_0$. Therefore, $|q(s+a)| \leq Ke^{|x_0 s \sin\alpha|}\ \forall\ s \in [0, \infty e^{i\alpha})$. Then, $|f(w/x)| \leq Ke^{|w \sin\alpha|/2}$ $\forall\ w \in [0, \infty e^{i(\alpha \mp \pi/2)})$ and $|x| \geq 2x_0$ and the integral of equation (69) satisfies also conditions i), iii) and iv) of Theorem 1 with $g(w) \equiv Ke^{|w \sin\alpha|/2}$, $h(w) \equiv e^{-w}$, $r_0 \equiv 2x_0$ and $\mathcal{C} \equiv [0, \infty e^{i(\alpha \mp \pi/2)})$. Then, using (70) and applying this theorem, we obtain (65) for $q^{(n)}(b) \equiv 0\ \forall\ n \geq 0$. □

5 Final comments and summary.

A modification of the TTIM [2, pg. 29, Theorem 1.7.5.] for obtaining asymptotic expansions of integrals in some complex parameter z (for large $|z|$), has been introduced in Theorem 1 and Corollary 1. It

basically demands the integrand to satisfy two conditions. The first one is the knowledge of the expansion (convergent or asymptotic) of the integrand in power series of w/z, w being the integration variable. The second requirement is the fulfillment of the bound (17) and the integrability conditions (18). These properties are much more frequent in practise than the uniformity condition (3) required by the classical TTIM. Therefore, this version happens to be more useful for solving practical problems, as it has been shown in Example 1.

As it does not demand a too special form for the integrand, one of the advantages of this version of the TTIM is its wide range of applicability. The other advantage is that it maintains the simplicity of the classical method: once the bounding condition (17) is found and (18) is checked, the procedure to derive the asymptotic expansion is trivial, just expand the integrand in the sequence $\{(w/z)^{\lambda_n}\}$ and interchange the integral and sum operations. This has been shown in Examples 2-3.

We have shown in Corollaries 2-5 that Watson's lemma and the integration by parts technique applied to certain families of integrals are corollaries of the TTIM. On the other hand, several classical asymptotic techniques such as steepest descent, stationary phase, summability, Laplace's or Perron's methods are based on Watson's lemma or integration by parts [12]. This suggests the possibility of obtaining these asymptotic methods as corollaries of the TTIM. This point is subject of present investigations.

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