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The Stokes phenomenon as a boundary-value problem

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Abstract
We show that the Stokes phenomenon is related to a boundary-value problem in two dimensions: for a large class of functions and near the Stokes lines, the subdominant multiplier satisfies a two-dimensional boundary-value problem of convection–diffusion type with discontinuous Dirichlet conditions at the boundary. The solution of this problem is approximated by an error function of a certain combination of the polar variables of the plane which measures the distance to the Stokes line. Then, we offer a different and very simple explanation of the smoothing of the Stokes phenomenon showing the universality of the error function as smoothing factor.

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1. Introduction and development

Consider the following asymptotic approximation of the Bessel function $K_\mu(z)$ as $z \to \infty$ [[14], chapter 6, pp 236, 237]:

$$k_\mu(z) \sim z^{-1/2}\,\mathrm{e}^{-z}\sum_{n=0}^{\infty}\frac{A_n(\mu)}{z^n}, \qquad -\frac{5\pi}{2} < \arg z - \pi < \delta,$$

$$k_\mu(z) \sim z^{-1/2}\,\mathrm{e}^{-z}\sum_{n=0}^{\infty}\frac{A_n(\mu)}{z^n} + z^{-1/2}\,\mathrm{e}^{z}\sum_{n=0}^{\infty}\frac{B_n(\mu)}{z^n}, \qquad -\delta < \arg z - \pi < \frac{3\pi}{2},$$

with $0 < \delta < \pi/2$, $A_0(\mu) = 1$, $B_0(\mu) = 2\mathrm{i}\cos(\pi\mu)$ and, for $n = 1, 2, 3, \ldots$,

$$A_n(\mu) = \frac{\cos(\pi\mu)}{\sqrt{2\pi}}\frac{\Gamma(n+\mu+1/2)\Gamma(n-\mu+1/2)}{(-2)^n n!}, \qquad B_n(\mu) = 2\mathrm{i}\cos(\pi\mu)(-1)^n A_n(\mu).$$

These two asymptotic expansions have a common region of validity $-\delta < \arg z - \pi < \delta$, where they differ by the inclusion of the second series in the second expansion. This apparent

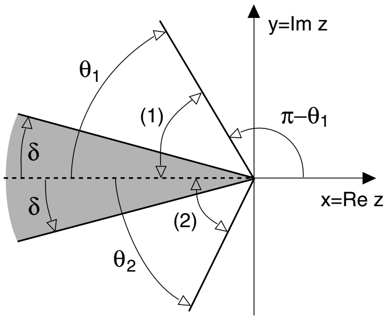

Figure 1. The case $m = 1$, $S_1 = \{z, -\theta_1 < \arg z - \pi < \theta_2\}$. The first approximation in (1) is valid in the sector $\pi - \theta_1 < \arg z < \pi + \delta$, whereas the second approximation is valid in the sector $\pi - \delta < \arg z < \pi + \theta_2$. The sector D_1 is the shaded region.

contradiction is the origin of the *Stokes phenomenon* that, in general, may be stated as follows. Consider a complex function $w(z)$ analytic in a sector of the complex z plane defined by $S_m = \left\{z, -\theta_1 < \arg z - \frac{\pi}{m} < \theta_2\right\}$, where $\theta_1, \theta_2 > 0$ and m is a positive integer number (see figure 1 for the case $m = 1$). Suppose that, when $|z| \to \infty$,

$$w(z) \sim z^a\, \mathrm{e}^{-\alpha z^m} \sum_{n=0}^{\infty} \frac{a_n}{z^n} \qquad \text{and} \qquad w(z) \sim z^a\, \mathrm{e}^{-\alpha z^m} \sum_{n=0}^{\infty} \frac{a_n}{z^n} + z^b\, \mathrm{e}^{\beta z^m} \sum_{n=0}^{\infty} \frac{b_n}{z^n}.$$

Or, equivalently,

$$z^{-b}\, \mathrm{e}^{\alpha z^m} w(z) \sim z^{a-b} \sum_{n=0}^{\infty} \frac{a_n}{z^n} \qquad \text{and} \qquad z^{-b}\, \mathrm{e}^{\alpha z^m} w(z) \sim z^{a-b} \sum_{n=0}^{\infty} \frac{a_n}{z^n} + \mathrm{e}^{(\alpha+\beta) z^m} \sum_{n=0}^{\infty} \frac{b_n}{z^n}. \tag{1}$$

The first expansion in (1) is uniformly valid in $-\theta_1 < \arg z - \frac{\pi}{m} < \delta$ and the second one in $-\delta < \arg z - \frac{\pi}{m} < \theta_2$. In these formulae a and b are rational numbers, $\alpha, \beta \geqslant 0$ with $\alpha\beta \neq 0$, $\delta > 0$ with $\delta < \theta_1$, $\delta < \theta_2$ and a_n and b_n are sequences of complex numbers. The two asymptotic expansions in (1) have a common region of validity $D_m = \left\{z, \left|\arg z - \frac{\pi}{m}\right| < \delta\right\}$ (see figure 1), where they differ by the inclusion of the second series in the second expansion; but in the Poincaré sense, the additional term in the second expansion is negligible. It may become 'visible' when, for z in the middle of the sector D_m ($\arg z \simeq \pi/m$ and $z^m \simeq -|z|^m$), we cut the dominant series at an integer $n = N$ in such a way that the last term of that dominant series, which is of the order $\mathcal{O}(z^{a-b-N})$, is of the same size as the dominant term of the subdominant series, which is of the order $\mathcal{O}\big(\mathrm{e}^{(\alpha+\beta)z^m}\big)$. This happens when, for $\arg z \simeq \pi/m$, we have that $|z|^{a-b-N} \simeq \mathrm{e}^{-(\alpha+\beta)|z|^m}$, that is, when $N = \lfloor a - b + (\alpha+\beta)|z|^m / \log|z| \rfloor$. Then, the Stokes multiplier U is defined by the coefficient that appears in front of the subdominant term when $z^{-b}\, \mathrm{e}^{\alpha z^m} w(z)$ is approximated up to the order $\mathcal{O}\big(\mathrm{e}^{(\alpha+\beta)z^m}\big)$ in D_m by choosing $N = \lfloor a - b + (\alpha+\beta)|z|^m/\log|z| \rfloor$ [2, 13, 14 (equation (6.1.10)), 15]:

$$z^{-b}\, \mathrm{e}^{\alpha z^m} w(z) = z^{a-b} \sum_{n=0}^{N} \frac{a_n}{z^n} + \mathrm{i}U\, \mathrm{e}^{(\alpha+\beta)z^m}. \tag{2}$$

In the above example $w(z) = K_\mu(z)$ we have $a = b = -1/2, m = 1, \alpha = \beta = 1$ and $N = \lfloor 2|z|/\log|z| \rfloor$. Observe that the first term on the right-hand side above is of the order $\mathcal{O}(z^{a-b})$ uniformly in $\arg z$. The non-uniform behaviour in $\arg z$ is concentrated in the second term containing the factor U.

Since Stokes discovered this phenomenon [16–18], the traditional view has been that a discontinuous change in the constant multiplier U associated with subdominant asymptotic expansion takes place when we cross a Stokes line: $U = 0$ for $\frac{\pi}{m} - \delta < \arg z < \frac{\pi}{m}$ and $U = 1$ for $\frac{\pi}{m} < \arg z < \frac{\pi}{m} + \delta$. This discontinuous nature of the multiplier U together with the inherent vagueness associated with the precise location of its jumps (Stokes lines) has enveloped the Stokes phenomenon with a certain air of mystery.

In 1989, Berry removed some of that mysterious air. He showed (formally) that the change in the multiplier U is not abrupt, but smooth, and is universally described by an error function [1, 2]. For a wide class of functions, the functional form of this rapid but smooth transition is found to possess a universal structure approximated by an error function. Since Berry's work, several mathematicians have offered different rigorous proofs of Berry's theory. The first one was proposed by Olver in [10, 11], who used uniform exponentially improved asymptotic expansions for functions defined by Laplace integrals. A different proof was proposed by Boyd [4] using the previous results of Jones [5]. He developed an exponentially improved asymptotic theory for functions defined by a Stieltjes transform. A different proof was introduced by Paris [12] considering functions defined by Mellin–Barnes integrals and constructing uniform exponentially improved asymptotic expansions from these integrals. Chapter 6 of [14] contains a good introduction to the Stokes phenomenon as well as several illustrative examples. The survey paper [13] contains a more detailed explanation of the history of the Stokes phenomenon and its explanation.

There are many other interesting papers related to the Stokes phenomenon where several philosophic-mathematical explanations of the phenomenon are argued. For example, in [3], asymptotic superfactorial series are considered which, as well as factorial series, exhibit the Stokes phenomenon. The argument based on Borel summation works for factorial series [1], but not for superfactorial series. A modification of that argument is given in [3] which also shows the persistence of the Stokes phenomenon in superfactorial series. In [9], the Stokes phenomenon is interpreted via the method of matched asymptotic expansions: a genuine method of singularly perturbed partial differential equations. That interpretation, as well as the one given in this paper, relates the Stokes phenomenon with a partial differential equation. In [8], a new representation of a function, different from the standard expansion in asymptotic series, is introduced. It is argued there that the Stokes phenomenon comes up naturally from that representation. In [7, 12], it is argued that the origin of the phenomenon lies on the fact that we are approximating an analytic function $w(z)$ by non-analytic multivalued functions $z^a\, e^{z^m}$ and $z^b\, e^{-z^m}$.

From definition (2) we have that the Stokes multiplier $U(z)$ depends on N and N depends on $|z|$. At certain values of $r = |z|$, when $a - b + (\alpha + \beta)r^m/\log r \in \mathbb{N}$, it happens that $N \to N + 1$ and then $U(z)$ changes abruptly. Let us denote those special values of r by r_k, $k = 0, 1, 2, \ldots$. If $w(z)$ is an analytic function of z in D_m then, from this definition, the Stokes multiplier $U(z)$ is also an analytic function of z in every subdomain $D_{m,k} = D_m \bigcap \{r_k < |z| < r_{k+1}\}$.

The following argument offers a new and simple derivation of the smoothing of the Stokes phenomenon. In the following, we denote $\arg z = \theta$. For the sake of clearness, consider at this moment $m = 1$ in formula (2):

$$z^{-b}\, e^{\alpha z} w(z) = z^{a-b} \sum_{n=0}^{N} \frac{a_n}{z^n} + \mathrm{i} U\, e^{(\alpha+\beta)z}. \tag{3}$$

We want to approximate U by an alternative function (that we also denote by U) such that (i) it is real in the sector D_1 and (ii) far away from the origin (large r), $U = 0$ at one side of that sector (at $\theta = \pi - \delta$) and $U = 1$ at the other side (at $\theta = \pi + \delta$). Applying the Laplace

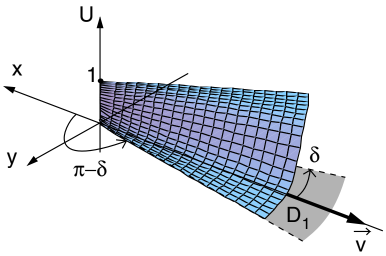

Figure 2. Approximate solution (6) of the boundary-value problem (5) in the sector D_1, that is, approximate form of the Stokes' multiplier U in that sector.

(This figure is in colour only in the electronic version)

operator to the imaginary part of the equality (3) and using (i) and $\Delta\,\mathrm{Re}\,w = \Delta\,\mathrm{Im}\,w = 0$ (w is analytic) we find that U satisfies a convection–diffusion equation for large $r = |z|$:

$$-\frac{1}{2(\alpha+\beta)}\Delta U + \vec{v}\cdot\vec{\nabla}U = 0, \qquad r\to\infty, \qquad -\delta < \theta-\pi < \delta, \quad (4)$$

with a convection vector $\vec{v} \equiv (-1, \tan[(\alpha+\beta)y])$. On the other hand, from (ii), U must change rapidly from the value $U = 0$ to $U = 1$ when crossing the Stokes line $\theta = \pi$ in the positive sense. This has two consequences. First, we can consider a small δ and approximate $\tan[(\alpha+\beta)y] \simeq 0$, that is, $v \simeq (-1, 0)$. Second, equation (4) must be supplemented with the boundary conditions $U(r, \pi-\delta) = 0$ and $U(r, \pi+\delta) = 1$. Then, we are left with the boundary-value problem:

$$\begin{cases} \Delta U + 2(\alpha+\beta)U_x = 0 & \text{in} \quad D_1, \\ U(r,\pi-\delta) = 0, & U(r,\pi+\delta) = 1. \end{cases} \quad (5)$$

This is a convection–diffusion boundary-value problem defined in a sector-shaped domain of angular width 2δ with discontinuous Dirichlet datum at the corner of the sector: $r = 0$. This problem has been studied in [6], where it is shown that, when supplemented with the radiation condition: $U(r,\theta) = o(r^{-1/2}\,\mathrm{e}^{r[1-\cos\theta]})$ as $r \to \infty$ and $\pi-\delta < \theta < \pi+\delta$, the problem has a unique solution. That unique solution is approximated for large r (large $|z|$) by an error function [6]:

$$U(r,\theta) \sim \frac{1}{2} + \frac{1}{2}\left[\sqrt{2(\alpha+\beta)r}\,\sin\left(\frac{\theta-\pi}{2}\right)\right], \quad (6)$$

as $r \to \infty$ in the sector D_1. In fact, this function satisfies exactly the equation $U + 2(\alpha+\beta)U_x = 0$, and for large r it satisfies approximately the boundary condition: $U(r,\pi-\delta) = 0, U(r,\pi+\delta) = 1$.

The solution of a convection boundary-value problem of the form $\{\vec{v}\cdot\vec{\nabla}U = 0$ in a sector D_1 and $U|_{\partial D_1} = U_0\}$ is constant along straight lines parallel to the convection vector $\vec{v} = (v_1, v_2)$. More precisely, $U(x,y) = U_0(x_0, y_0)$, where (x_0, y_0) is a point of ∂D_1 verifying $\frac{x-x_0}{v_1} = \frac{y-y_0}{v_2}$. That is, the convection vector $\vec{v}$ pulls the boundary-value U_0 inside the domain D_1. When we add a term ΔU to the equation (and supplement it with a radiation condition to assure a unique solution), the problem becomes an elliptic problem of convection–diffusion type: $\{-\Delta U + \vec{v}\cdot\vec{\nabla}U = 0$ in D_1 and $U|_{\partial D_1} = U_0\}$, like problem (5). In problem (5), U_0 is discontinuous at $r = 0$ and then the convection vector $\vec{v}$ pulls that discontinuity inside D_1. But the solution of the elliptic problem (5) is continuous in D_1; the diffusion term

ΔU smoothes that discontinuity, transforming the discontinuity of the convection problem in a fast (but smooth) transition between the value $U = 0$ at $\theta = \pi - \delta$ to the value $U = 1$ at $\theta = \pi + \delta$.

The analysis of the case $m \neq 1$ is similar. Replace z by $z^{1/m}$ (and then $U(r, \theta)$ by $U(r^{1/m}, \theta/m)$) in (2). Instead of (3) we have

$$z^{-b/m}\, e^{\alpha z} w(z^{1/m}) = z^{(a-b)/m} \sum_{n=0}^{N} \frac{a_n}{z^{n/m}} + \mathrm{i}U(r^{1/m}, \theta/m)\, e^{(\alpha+\beta)z}.$$

From here, the discussion for $U(r^{1/m}, \theta/m)$ is the same as the preceding discussion elaborated for $U(r, \theta)$ from (3). Therefore, for large r, $U(r^{1/m}, \theta/m)$ is approximated by the right-hand side of (6) and then, for $z = r\, e^{\mathrm{i}\theta} \in D_m$,

$$U(r, \theta) \sim \frac{1}{2} + \frac{1}{2}\left[\sqrt{2(\alpha+\beta) r^m}\, \sin\left(\frac{m\theta - \pi}{2}\right)\right] \qquad \text{as} \quad r \to \infty. \tag{7}$$

This is the approximation for the Stokes multiplier U found first time by Berry and subsequently proved more rigorously by Olver, Boyd or Paris using different techniques and applied to several examples.

2. Final remarks

From the definition of the Stokes multiplier U in equation (2) we see that if $w(z)$ is an analytic function of z in D_m, then $U(z)$ is also an analytic function of z in $D_{m,k}$. Then both $\mathrm{Re}\, U$ and $\mathrm{Im}\, U$ must satisfy a Poisson equation: $\Delta\, \mathrm{Re}\, U = \Delta\, \mathrm{Im}\, U = 0$ in $D_{m,k}$. When we are trying to approximate $U(z)$ for large $|z|$, we set $\mathrm{Im}\, U \sim 0$ and $U \sim \mathrm{Re}\, U$. With this approximation we destroy the analyticity of U because the imaginary part of any analytic function of a complex variable z is not the null function unless it is a constant function. Then, it is no longer true that $\Delta U = \Delta\, \mathrm{Re}\, U = 0$, but we have shown that the Stokes multiplier U satisfies approximately a convection–diffusion boundary-value problem with discontinuous Dirichlet datum (5). The solution of this problem is approximated by an error function with an argument that measures the distance to the Stokes line and produces a rapid variation of that error function from 0 to 1 when crossing that Stokes line (6). This is a well-known result shown by other authors using different techniques. In this paper, we have related the Stokes phenomenon, which is a genuine problem of analytic functions and asymptotic expansions to a boundary-value problem. Under this view, (i) the Poisson equation satisfied by the real and imaginary parts of the function w and its asymptotic series in (2) translates into a convection–diffusion equation satisfied by the Stokes multiplier U (see (4)), (ii) the Stokes line is just the line defined by the convection vector $\vec{v}$ of (4) and (iii) the limit values $U = 0$ and $U = 1$ are the boundary conditions (see (5)). Then, both problems have the same approximate solution given by an error function with the appropriate argument, an argument that measures the distance to the Stokes line or the distance to the line defined by the convection vector $\vec{v}$.

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Endnotes

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