Several series containing gamma and polygamma functions

José L. López

Departamento de Matemática Aplicada. Universidad de Zaragoza
50009, Zaragoza, Spain. E-mail: jllopez@posta.unizar.es.

Abstract
The analytic calculation of a generalization of the integral representation of the polylogarithmic functions is presented. As a consequence, several new series, most of them containing gamma and polygamma functions, are calculated.

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I. Introduction

In general, Quantum Field Theories in physics are ill defined because of the divergent ultraviolet behaviour of this kind of theories. The infinities which appear in the calculation of the physical observables make necessary a regularization prescription. Zeta-function regularization is one of the most important regularization schemes used in Quantum Field Theory. It has been mainly utilized to properly define divergent determinants, although in the last few years many applications have been found also in Gravity, Strings and P-Branes Theories. When a Quantum Field Theory is regularized within this scheme, effective quantities like the vacuum energy or the effective vertex and physical observables like the Green Functions become the sum of a certain series. In many cases these series involve, among their coefficients, gamma, polygamma and Riemann zeta functions of the regulator parameter and the physical constants of the theory. For example, in the calculation of the Casimir energy over a Riemann sphere with Dirichlet or Neumann boundary conditions, we find series like [2],

$$\sum_{l=n+1}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(1-s+c)}{k!\Gamma(1-s-k+c)}a^k l^{-2s-k+b+c}, \tag{1}$$

$$\sum_{k=0}^{\infty} \frac{\Gamma(1-s)}{k!\Gamma(1-s-k)}[\zeta(2s+k-1)-1] \tag{2}$$

and some other similar series involving also polygamma functions, where s is the regulator parameter and the remaining coefficients are physical constants of the theory. The problem is that many of these series can not be found in the table books (see for example [3] or [5]). The purpose of this paper is to show a method for calculating a family of series of these kind and some other related to them and give a list of series that may be added to the table books.

The starting point is the calculation of the value of the polylogarithmic functions $\mathrm{Li}_n(x)$ [4] of degree $n > 1$ in the point $x = 1$. These functions are defined by means of the recursive formula [4],

$$\mathrm{Li}_{n+1}(x) \equiv \int_0^x \frac{\mathrm{Li}_n(t)}{t} dt, \qquad |x| \leq 1, \quad n \in \mathbb{N}, \tag{3}$$

where $\mathrm{Li}_1(x) \equiv -\log(1-x)$. With the change of variable $t = 1 - y$ in each one of the n integrals involved in the above definition, the functions $\mathrm{Li}_{n+1}(x)$ may be also defined by means of the formula

$$\mathrm{Li}_{n+1}(x) = (-1)^{n+1} \frac{d}{dz} I_n(z, w, 1-x)\bigg|_{z=0}, \tag{4}$$

where $I_n(z, w, x)$ are the multiple integrals defined by means of the recursive formula

$$I_{n+1}(z, w, x) \equiv \int_x^1 \frac{dy}{y-1} I_n(z, w, y), \qquad 0 \leq x \leq 2, \quad n = 1, 2, 3, ... \tag{5}$$

and

$$I_1(z, w, x) \equiv \int_x^1 \frac{y^z - y^w}{y-1} dy. \tag{6}$$

In particular, $\mathrm{Li}_{n+1}(1) = (-1)^{n+1} dI_n(z, w, 0)/dz|_{z=0}$. In this paper we will deal only with $I_n(z, w, 0)$ and we simplify the notation by writing $I_n(z, w) \equiv I_n(z, w, 0)$.

It is easy to show that the integrals $I_n(z, w)$ are finite for $\mathrm{Re}(z) + n > 0$ and $\mathrm{Re}(w) + n > 0$ $\forall$ $n \geq 1$ and that the convergence dominated theorem can be applied here to show that the derivation of (4) is correct (anyway it will be shown at the end of section 2).

The evaluations, for $x = 0$, of the integrals $I_n(z, w, x)$, its derivatives (16)-(17) and its generalization (22)-(23) are the starting point for calculating series containing gamma, polygamma and Riemann zeta functions (although some other functional and numerical series will be obtained also). Once the values of these multiple integrals are known, the procedure is very simple: take one of these multiple integrals, expand the integrand in power series of z (or w) or in power series of the integration variables after suitable changes of variable, interchange series and integrals when it may be justified and match the resulting series with the value of the integral.

In Section 2, the analytic calculation of the integrals $I_n(z, w, x)$, its derivatives (16)-(17) and its generalization (22)-(23) is performed for $x = 0$. A list of new series that have been obtained by the method explained above is summarized in Section 3.

II. Evaluation of the multiple integral $I_n(z,w)$

In order to calculate $I_n(z,w)$, it will be necessary to solve before a recurrence of difference equations.

Lemma. *The recurrence of difference equations*

$$F_n(z) - F_n(z-1) = -\frac{1}{z} F_{n-1}(z), \qquad n \in \mathbb{N}, \tag{7}$$

with $F_0(z) = 1$, has the following particular solution,

$$F_n(z) \equiv \frac{1}{n!} \Gamma(z+1) \frac{d^n}{dz^n} \left[\frac{1}{\Gamma(z+1)} \right], \qquad z \in \mathbb{C} \setminus \mathbb{Z}^-. \tag{8}$$

Proof. It proceeds by induction over n. It is trivial for $n = 1$ using the recurrence formula of the digamma function [1],

$$\psi(z+1) = \psi(z) + z^{-1}. \tag{9}$$

Now, let us suppose that (8) verifies (7) for a given $n \geq 1$. Then, taking the derivative of (7) with respect to z and using (9), we find it is true for $n+1$. □

Remark. Note that the functions $F_n(z)$ are nothing but algebraic combinations of polygamma functions of order $\leq n$ and the constants $F_k(0)$, which will be used later, are just the coefficients of the Taylor expansion of $\Gamma(z+1)^{-1}$ in powers of z.

Proposition. *For $-z \notin \mathbb{N}$ and $Re(z) + n > 0$, the family of multiple integrals $I_n(z,w)$ defined in (5)-(6) for $x = 0$ is calculated by means of the recurrence*

$$I_n(z,w) = \sum_{k=0}^{n} C_k \left(F_{n-k}(z) - F_{n-k}(w) \right), \tag{10}$$

where the functions $F_n(z)$ are given in (8) and the coefficients C_k may be recurrently obtained from

$$C_k = -\sum_{i=0}^{k-1} C_i F_{k-i}(0), \tag{11}$$

with $C_0 = -1$

Proof. We define $I_0(z,w) = -1$. Then, by direct substitution we may check that, as well as the functions $F_n(z)$ defined in (8), the integrals $-I_n(z,0)$ verify the recurrence (7). Now, we proceed by induction over n to show that (10) is true for $w = 0$. For $n = 1$ we find that both $-I_1(z,0)$ and $F_1(z)$ are particular solutions of (7). But two different particular solutions of (7) may differ only by a constant which is fixed, for example, in $z = 0$,

$$-I_1(z,0) = F_1(z) - F_1(0). \tag{12}$$

Now, we suppose that (10), for $w = 0$, is true for a given $n \geq 1$. Then, the integrals $I_{n+1}(z,0)$ verify the difference equation

$$I_{n+1}(z,0) - I_{n+1}(z-1,0) = -\frac{1}{z} \sum_{k=0}^{n} C_k F_{n-k}(z) \tag{13}$$

(Note that from (11) we have that $\sum_{k=0}^{n} C_k F_{n-k}(0) = 0$). According to Lemma 1, a particular solution of this equation is

$$\sum_{k=0}^{n} C_k F_{n-k+1}(z). \tag{14}$$

Therefore,

$$I_{n+1}(z,0) = \sum_{k=0}^{n} C_k F_{n-k+1}(z) + C_{n+1}, \tag{15}$$

where C_{n+1} is fixed, for example, in $z = 0$ and is given by (11). It can be introduced in the sum of eq. (15) extending it up to $n+1$ because $F_0(z) = 1$. Finally, subtracting $I_n(z,0)$ and $I_n(w,0)$, we find that (10) holds. □

Corollary 1. *For Re$(z)+n > 0$ and $-z \notin \mathbb{N}$, $n, m \in \mathbb{N}$, the family of multiple integrals $I_n^m(z,x)$ defined recursively for $0 \le x \le 2$ by*

$$I_{n+1}^m(z,x) \equiv \int_x^1 \frac{dy}{y-1} I_n^m(z,y), \qquad n \ge 1, \tag{16}$$

where

$$I_1^m(z,x) \equiv \int_x^1 \frac{y^z \log^m y}{y-1} dy, \tag{17}$$

is given, for $x = 0$, by

$$I_n^m(z,0) \equiv I_n^m(z) = \sum_{i=0}^{n} C_i \frac{d^m}{dz^m} F_{n-i}(z). \tag{18}$$

Proof. The integrals $I_n^m(z)$ may be written

$$I_1^m(z) = \int_0^1 \frac{x_1^z \log^m x_1}{x_1 - 1} dx_1 \tag{19}$$

and

$$I_n^m(z) = \int_0^1 \frac{dx_1}{x_1 - 1} \int_{x_1}^1 \frac{dx_2}{x_2 - 1} \cdots \int_{x_{n-1}}^1 \frac{x_n^z \log^m x_n}{x_n - 1} dx_n \qquad \text{for} \qquad n > 1. \tag{20}$$

Consider the function

$$g(x_1, ..., x_n) = \prod_{k=1}^{n} (1 - x_k)^{-1} h(x_n) (-\log(x_n))^m \tag{21}$$

with $h(x_n) = 1$ if $\mathrm{Re}(z) \ge 0$ and $h(x_n) = x_n^{y_0}$ if $\mathrm{Re}(z) < 0$, where $\mathrm{Re}(z) \in (y_0, y_1)$ and $y_0 > -n$. The function $g(x_1, ..., x_n)$ is integrable and dominates the modulus of the integrand in (20) (or (19) for $n = 1$). Therefore, taking the m-th derivative of both members of the equality (10) with respect to z and using the dominated convergence theorem, the proof concludes. □

Corollary 2. *For Re$(z) + n > k - 1$, $m \ge k = 2, 3, 4...$ and $-z \notin \mathbb{N}$, consider the family of multiple integrals $I_{n,k}^m(z,x)$ defined recursively for $0 \le x \le 2$ by*

$$I_{n+1,k}^m(z,x) \equiv \int_x^1 \frac{dy}{y-1} I_{n,k}^m(z,y), \qquad n > 1, \tag{22}$$

where

$$I^m_{1,k}(z,x) \equiv \int_x^1 \frac{y^z \log^m y}{(y-1)^k} dy. \tag{23}$$

Let us define $I^m_{n,k}(z) \equiv I^m_{n,k}(z,0)$. *Then, for* $n = 1,2,3,...$ $I^m_{n,k}(z)$ *is given by the recurrence,*

$$I^m_{n,k}(z) = \frac{1}{k-1}\left[I^m_{n-1,k-1}(z) + zI^m_{n,k-1}(z-1) + mI^{m-1}_{n,k-1}(z-1)\right], \tag{24}$$

where $I^m_{n,1}(z) = I^m_n(z)$ *is given in eq. (18) and* $I^m_{0,k}(z) = 0$.

Proof. It is just an integration by parts in (22) that needs the condition $m \geq k$. In order to assure that the above recurrence is well defined, the condition $\mathrm{Re}(z) + n > k - 1$ is also necessary. □

Corollary 3. For $|z| < 1$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{\infty}\left(\sum_{i=0}^{n} C_i F^{(k)}_{n-i}(0)\right)\frac{z^k}{k!} = \sum_{i=0}^{n} C_i \left(F_{n-i}(z) - F_{n-i}(0)\right), \tag{25}$$

where $F^{(k)}_n(z)$ is the $k-$th derivative of $F_n(z)$. Moreover, this series is absolutely convergent.

Proof This series is nothing but the Taylor expansion of $I_n(z,0)$ on the point $z = 0$. Therefore, the coefficients of this series are just the numbers $I^k_n(0)$ given in (18). The result follows using also eq. (10). On the other hand, it can be proved by induction on n, that $|F^{(k)}_n(0)| \leq 3^n (k+n)!/n!$. Then, the series $\sum_{k=0}^{\infty} F^{(k)}_n(0) z^k/k! = F_n(z)$ is absolutely convergent for $|z| < 1$ and therefore, the series (25) is absolutely convergent. □

Remark. This formula is a generalization of the known power series with zeta function coefficients [1, pg. 259, eq. 6.3.14]

$$\sum_{n=1}^{\infty}(-1)^{n+1}\zeta(n+1)z^n - \gamma = \psi(z+1), \qquad |z| < 1, \tag{26}$$

which is obtained from (25) for $n = 1$ using $F^{(k)}_1(0) \equiv -\psi^{(k)}(1) = (-1)^k k!\zeta(k+1)$ for $k \in \mathbb{N}$ [1, pg. 260, eq. 6.4.2].

Corollary 4. For $|z| < m$, $-z \notin \mathbb{N}$ and $n, m \in \mathbb{N}$ we have

$$\begin{aligned} \int_m^{\infty} \frac{\mathrm{Frac}(x)dx}{(z+x)^{n+2}} &= \frac{(-1)^n}{(n+1)!}\psi^{(n)}(1+z) + \frac{1}{n(n+1)(z+m)^n} \\ &+ \frac{1}{(n+1)}\sum_{k=1}^{m}\frac{1}{(z+k)^{n+1}}. \end{aligned} \tag{27}$$

Proof. We insert the integral representation of the zeta-function [1, pg. 807, eq. 23.2.9]

$$\zeta(k+1) = \sum_{l=1}^{m}\frac{1}{l^{k+1}} + \frac{1}{km^k} - (k+1)\int_m^{\infty}\frac{\mathrm{Frac}(x)dx}{x^{k+2}}, \qquad m = 1,2,3... \tag{28}$$

in $F^{(k)}_1(0) = (-1)^k k!\zeta(k+1)$ in the left hand side of (25). Now, it is trivial to show that

$$\sum_{k=1}^{\infty}|z|^k \int_m^{\infty}\frac{(k+1)\mathrm{Frac}(x)}{x^{k+2}}dx < \infty \qquad \text{for} \quad |z| < m.$$

Therefore, using the dominated convergence theorem we may interchange the sum in k and the integral of (25). After straightforward algebra we obtain

$$\int_m^\infty \frac{\text{Frac}(x)dx}{(z+x)^2} = I(m) + \psi(1+z) - \sum_{k=1}^m \frac{z}{k(z+k)} - \log\left(1+\frac{z}{m}\right), \tag{29}$$

where $I(m) = \psi(m+1) - \ln m + \gamma$. For z in a ball of centre zero and radius $r < m$, the $n-$th derivative with respect to z of the function in the integrand in the left hand side of (29) can be easily bounded by an integrable function. Therefore, applying the dominated convergence theorem we obtain (27). □

Performing changes of variable in the integrals entering the equations (10), (18) and (24) and/or expanding the integrands in power series of z, w or the integration variables, several new (and known) functional and numerical series are obtained. These calculations take use of several versions of the dominated convergence theorem to justify interchanges of series and integrals or derivations inside the integrals, like we have illustrated in corollaries 1 and 4. Some examples of series calculated using these techniques are written in the next section. We will use the following notation: $\zeta(z)$ is the Riemann zeta-function, γ is the Euler's constant, $\psi^{(n)}(z)$ is the $n-$th derivative of the digamma function, $F_n^{(k)}(z)$ is the $k-$th derivative of the functions $F_n(z)$, $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol and γ_k are the generalized γ constants,

$$\gamma_k = \lim_{m\to\infty}\left\{\sum_{n=1}^m \frac{\log^k n}{n} - \frac{\log^{k+1} m}{k+1}\right\}. \tag{30}$$

III. List of new series

1. Series containing binomial coefficients

Expanding the denominator $(y-1)^{-k}$ in eq. (23) for $x=0$ in power series of y and using (18) and (24), we obtain, for $-z \notin \mathbb{N}$, $m \geq k \in \mathbb{N}$,

1.1 $$\sum_{n=0}^\infty \binom{n+k-1}{n} \frac{(-1)^{m+k}}{(z+n+1)^{m+1}} = \frac{1}{m!}\Psi_k^m(z),$$

where the functions $\Psi_k^m(z)$ are defined by means of the recursive formula

$$\Psi_k^m(z) = \frac{1}{k-1}\left[z\Psi_{k-1}^m(z-1) + m\Psi_{k-1}^{m-1}(z-1)\right], \qquad k \geq 2 \tag{31}$$

and $\Psi_1^m(z) = \psi^{(m)}(z+1)$.

If we expand the term $(z+n+1)^{-(m+1)}$ in the above series in power series of z we obtain, for $m \geq k \in \mathbb{N}$ and $|z| < 1$,

1.2 $$\sum_{l=0}^\infty \sum_{n=0}^\infty \binom{n+k-1}{n} \frac{(-1)^l (l+m)!}{l!(n+1)^{m+l+1}} z^l = (-1)^{m+k}\Psi_k^m(z).$$

This formula is a generalization of the $m-$th derivative of the known power series with zeta functions coefficients (26). It is obtained from formula 1.2 for $k=1$.

If we write $\int_m^\infty (z+x)^{-(n+2)}\mathrm{Frac}(x)dx = \sum_{k=m}^\infty \int_0^1 (z+k+x)^{-(n+2)}xdx$, after a straightforward algebra and using (27) we obtain, for $\mathrm{Re}(z)+m>1$, $n,m\in\mathbb{N}$,

1.3
$$\sum_{k=m}^{\infty}\sum_{j=0}^{\infty}\binom{n+j+1}{j}\frac{(-1)^j}{j+2}\frac{1}{(k+z)^{n+j+2}} =$$
$$= \frac{(-1)^n}{(n+1)!}\psi^{(n)}(z+1)+\frac{1}{n(n+1)(m+z)^n}+\frac{1}{n+1}\sum_{k=1}^{m}\frac{1}{(k+z)^{n+1}}.$$

2. Series containing gamma functions

If we set $m=1$ in eqs. (16)-(17) for $x=0$, perform the change of variable $y=1+t$ in eq. (17), expand $(1+t)^z$ and $\log(1+t)$ in power series of t and use eq. (18), we obtain, for $\mathrm{Re}(z)+n>0$, $n\in\mathbb{N}$, $1+z\notin\mathbb{N}$,

2.1
$$\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\frac{\Gamma(k-z)}{m(k+m)^n k!} = (-1)^{n+1}\Gamma(-z)\sum_{k=0}^{n}C_k F_{n-k}^{(1)}(z).$$

If we set $m=k=2$ in eqs. (22)-(23) for $x=0$, perform the change of variable $y=1+t$ in eq. (23), expand $(1+t)^z$ and each $\log(1+t)$ in power series of t and use eqs. (18) and (24), we obtain, for $\mathrm{Re}(z)+n>0$, $n\in\mathbb{N}$, $1+z\notin\mathbb{N}$,

2.2
$$\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(k-z)}{lmk!(l+m+k-1)^n} =$$
$$= -(-1)^n\Gamma(-z)\sum_{k=0}^{n-1}C_k\left(F_{n-k-1}^{(2)}(z)+zF_{n-k}^{(2)}(z-1)+2F_{n-k}^{(1)}(z-1)\right).$$

3. Series containing digamma functions

If we set $m=n=1$ in eqs. (16)-(17) for $x=0$, perform the change of variable $y=1+t$ in eq. (17), expand $\log(1+t)$ in power series of t and use eqs. (10) for $n=1$ and (18), we obtain, for $1-z\notin\mathbb{N}$,

3.1
$$\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\frac{\psi(k+z+1)+\gamma}{n(k+z)} = \psi(z)\psi^{(1)}(z)-\frac{1}{2}\psi^{(2)}(z).$$

Expanding $(y-1)^{-1}$ in eq. (6) for $x=0$ in power series of y and after a straightforward algebra and using [5, pg. 655, eq. 2], we obtain for $1-z$, $1-w\notin\mathbb{N}$,

3.2
$$\sum_{n=0}^{\infty}\frac{(n+w)\psi(n+z+1)-(n+z)\psi(n+w+1)}{(n+z)(n+w)} =$$
$$= \frac{1}{2}\left(\psi^2(w)-\psi^2(z)-\psi^{(1)}(w)+\psi^{(1)}(z)\right).$$

If se set $w = z+1$ in the above formula and use [1, pg. 258, eq. 6.3.5] and [5, pg. 655, eq. 2], we obtain, for $1-z \notin \mathbb{N}$,

3.3 $$\sum_{n=0}^{\infty} \frac{\psi(n+z+1)}{(n+z)(n+z+1)} = \psi^{(1)}(z) + \frac{\psi(z)}{z}.$$

4. Series containing gamma and digamma functions

Taking the derivative of formula 2.1 with respect to z for $n=1$ and using [5, pg. 655, eq. 2], we obtain, for $\mathrm{Re}(z) > -1$, $z \notin \mathbb{N}$,

4.1 $$\sum_{k=1}^{\infty} \frac{(\psi(k+1)+\gamma)\psi(k-z)\Gamma(k-z)}{kk!} = \\ = \Gamma(-z)\left(\psi(-z)(\psi^{(1)}(z+1)-\zeta(2)) - \psi^{(2)}(z+1)\right).$$

5. Series containing polygamma functions

If we perform the change of variable $y = 1+t$ in eq. (6) for $x = w = 0$, expand $(1+t)^z$ in power series of t, take the m derivative with respect to z and use eq. (18), we obtain, for $\mathrm{Re}(z)+n > 0$, $m+1, n \in \mathbb{N}$, $z \notin \mathbb{N}$,

5.1 $$\sum_{k=1}^{\infty} \frac{1}{(-k)^n k!} \frac{d^m \Gamma(k-z)}{d(-z)^m} = \frac{d^m}{dz^m}\left[\Gamma(-z)\sum_{k=0}^{n} C_k \left(F_{n-k}(z) - F_{n-k}(0)\right)\right].$$

If we set $n = m = 1$ in eq. (16) for $x = 0$, expand $(y-1)^{-1}$ in power series of y in the integral (17), integrate by parts in this integral and use eq. (10) for $n = 1$ and eq. (18), we obtain, for $1-z \notin \mathbb{N}$,

5.2 $$\sum_{n=0}^{\infty} \frac{(n+z)\psi^{(1)}(n+z+1) - \psi(n+z+1) - \gamma}{(n+z)^2} = \frac{1}{2}\psi^{(2)}(z) - (\gamma + \psi(z))\,\psi^{(1)}(z).$$

6. Series containing Pochhammer symbols

Setting $m = k = 2$ in eq. (23) for $x = 0$, performing the change of variable $y = 1+t$, expanding each $\log(1+t)$ in power series of t and using [5, pg. 611, eq. 45] and eqs. (18) and (24), we obtain, for $\mathrm{Re}(z) > -1$,

6.1 $$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(n+m-2)!}{nm(z+1)_{n+m-1}} = z\psi^{(2)}(z) + 2\psi^{(1)}(z).$$

Setting $m = 0$ in formula 5.1, subtracting this formula for variable z from itself for variable $-z$ and using [1, pg. 256, eq. 6.1.22], we obtain, for $|z| < n$, $n \in \mathbb{N}$,

6.2 $$\sum_{m=1}^{\infty} \frac{(z)_m - (-z)_m}{m^n m!} = (-1)^n \sum_{k=0}^{n} C_k \left(F_{n-k}(-z) - F_{n-k}(z)\right).$$

If we subtract $I_2^1(z-1)$ from $I_2^1(z)$, integrate by parts $y^{z-1}\log y$, perform the change of variable $y=1+t$, expand $\log(1+t)$ in power series of t and use eqs. (10) and (18), we obtain, for $-z \notin \mathbb{N}$,

6.3 $$\sum_{n=1}^{\infty} \frac{(n-1)!}{n(z+1)_n} = \psi^{(1)}(z) - \frac{1}{z^2}.$$

7. Other series

Inserting the representation [1, pg. 807 eq. 23.2.5] of the zeta-function in the left-hand side of eq. (25) for $n=1$ and using the identity $\log(1-z) = \sum_{k=1}^{\infty} z^k/k$, we obtain, for $|z|<1$,

7.1 $$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\gamma_k n^k z^n = \log(1-z) - \gamma - \psi(1-z).$$

Setting $z=0$ and $m=2$ in eq. (16) for $x=0$, performing the change of variable $y=1+t$ in (17), expanding each $\log(1+t)$ in power series of t and using eq. (10) for $n=1$ and eq. (18), we obtain, for $n \in \mathbb{N}$,

7.2 $$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^n}{km(k+m)^n} = \sum_{k=0}^{n} C_k F_{n-k}^{(2)}(0).$$

Many other series may be calculated by means of similar manipulations of formulas (10), (18) and (24) for $x=0$. All them, as well as the series listed above, will result in algebraic combinations of gamma and polygamma functions.

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