Approximations of Orthogonal Polynomials in Terms of Hermite Polynomials

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ABSTRACT

Several orthogonal polynomials have limit forms in which Hermite polynomials show up. Examples are limits with respect to certain parameters of the Jacobi and Laguerre polynomials. In this paper we are interested in more details of these limits and we give asymptotic representations of several orthogonal polynomials in terms of Hermite polynomials. In fact we give finite exact representations that have an asymptotic character. From these representations the well-known limits can be derived easily. Approximations of the zeros of the Gegenbauer polynomials $C_n^\gamma(x)$ and Laguerre polynomials $L_n^\alpha(x)$ are derived (for large values of γ and α, respectively) in terms of zeros of the Hermite polynomials and compared with numerical values. We also consider the Jacobi polynomials and the so-called Tricomi-Carlitz polynomials.

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1. Introduction

It is well known that the Hermite polynomials

$$H_n(x) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!\,(n-2k)!}(2x)^{n-2k} \tag{1.1}$$

play a crucial role in certain limits of the classical orthogonal polynomials. For example, the Gegenbauer polynomials $C_n^\gamma(x)$, which are defined by the generating function

$$(1-2xw+w^2)^{-\gamma} = \sum_{n=0}^{\infty} C_n^\gamma(x) w^n, \quad -1 \le x \le 1, \quad |w| < 1, \tag{1.2}$$

have the well-known limits (cf. Temme (1996, p. 168, 169))

$$\lim_{\gamma\to\infty} \frac{C_n^\gamma(x)}{C_n^\gamma(1)} = x^n, \tag{1.3}$$

$$\lim_{\gamma\to\infty} \gamma^{-n/2} C_n^\gamma(x/\sqrt{\gamma}\,) = \frac{1}{n!} H_n(x). \tag{1.4}$$

These limits give insight in the location of the zeros of $C_n^\gamma(x)$ for large values of the order γ. The first limit shows that the zeros of $C_n^\gamma(x)$ tend to the origin if the order γ tends to infinity. The second limit is more interesting; it gives the relation with the Hermite polynomials if the order becomes large and the argument x is properly scaled.

For the Laguerre polynomials, which are defined by the generating function

$$(1-w)^{-\alpha-1} e^{-wx/(1-w)} = \sum_{n=0}^{\infty} L_n^\alpha(x)\, w^n, \quad \alpha, x \in \mathbb{C}, \quad |w| < 1, \tag{1.5}$$

similar results are

$$\lim_{\alpha\to\infty} \alpha^{-n} L_n^\alpha(\alpha x) = \frac{(1-x)^n}{n!}, \tag{1.6}$$

$$\lim_{\alpha\to\infty} \alpha^{-n/2} L_n^\alpha(x\sqrt{\alpha} + \alpha) = \frac{(-1)^n\, 2^{-n/2}}{n!}\, H_n(x/\sqrt{2}\,). \tag{1.7}$$

This again gives insight in the location of the zeros for large values of the order α, and the relation with the Hermite polynomials if the order becomes large and x is properly scaled.

In this paper we describe the asymptotics that governs the above limits with the Hermite polynomials. We consider large values of orders α and γ, and obtain asymptotic representations of $C_n^\gamma(x)$ and $L_n^\alpha(x)$ from which the above limits can be derived as special cases.

For large values of the degree n and fixed values of the order α the Laguerre polynomials $L_n^\alpha(x)$ are considered in Frenzen & Wong (1988); see also Wong (1989). In our present paper we keep n fixed, and we do not use the complicated analysis of uniform expansions. Our results are rather simple to derive, and can be considered as first approximations before considering uniform expansions.

We also discuss Tricomi-Carlitz polynomials, which have been considered recently by Goh & Wimp (1994) and (1997). These polynomials are related with Laguerre polynomials $L_n^\alpha(x)$ with negative order α.

In the following section we give the principles of the Hermite-type asymptotic approximations used in this paper. In later sections we give expansions for the Gegenbauer, the Laguerre, the Jacobi and the Tricomi-Carlitz polynomials. The same method can be used for many other classes of polynomials.

2. Expansions in terms of Hermite polynomials

The Hermite polynomials follow from the generating function

$$e^{2xw-w^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} w^n, \quad x, w \in \mathbb{C}, \tag{2.1}$$

which gives the Cauchy-type integral

$$H_n(x) = \frac{n!}{2\pi i} \int_{\mathcal{C}} e^{2xz-z^2} z^{-n-1} \, dz, \tag{2.2}$$

where $\mathcal{C}$ is a circle around the origin and the integration is in positive direction.

2.1. An expansion in terms of Hermite polynomials

Many special functions satisfy a relation in the form of a generating series, which usually has the form

$$F(x, w) = \sum_{n=0}^{\infty} p_n(x) \, w^n, \tag{2.3}$$

F is a given function, which is analytic with respect to w in a domain that contains the origin, and p_n is independent of w. Examples are the generating functions given in (1.2), (1.5) and (2.1).

The relation (2.3) gives for the special function p_n the Cauchy-type integral

$$p_n(x) = \frac{1}{2\pi i} \int_{\mathcal{C}} F(x, w) \, \frac{dw}{w^{n+1}}, \tag{2.4}$$

where $\mathcal{C}$ is a circle around the origin inside the domain where F is analytic (as a function of w).

We write

$$F(x, w) = e^{Aw-Bw^2} f(x, w), \tag{2.5}$$

where A and B do not depend on w, and can be chosen arbitrarily. This gives

$$p_n(x) = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{Aw-Bw^2} f(x, w) \, \frac{dw}{w^{n+1}}. \tag{2.6}$$

Because f is also analytic (as a function of w), we can expand

$$f(x, w) = \sum_{k=0}^{\infty} c_k w^k \tag{2.7}$$

and substitute this in (2.6). By (2.2), the result is the finite expansion

$$p_n(x) = z^n \sum_{k=0}^{n} \frac{c_k}{z^k} \frac{H_{n-k}(\zeta)}{(n-k)!}, \quad z = \sqrt{B}, \quad \zeta = \frac{A}{2\sqrt{B}}, \tag{2.8}$$

because terms with $k > n$ do not contribute in the integral in (2.6). The quantities A and B may depend on x, and if B happens to be zero for a special $x-$value x_0, say, we write

$$p_n(x_0) = A^n \sum_{k=0}^{n} \frac{c_k}{A^k\,(n-k)!}. \tag{2.9}$$

In the examples considered in the following sections, the choice of A and B is based on our requirement that $c_1 = c_2 = 0$. This happens if we take

$$A = p_1(x), \quad B = \tfrac{1}{2}p_1^2(x) - p_2(x), \tag{2.10}$$

if we assume that $F(x,0) = p_0(x) = 1$ (which implies $c_0 = 1$). This is easily verified from (2.3) by writing

$$\ln[F(x,w)] = p_1(x)w + \left[p_2(x) - \tfrac{1}{2}p_1^2(x)\right] w^2 + \mathcal{O}(w^3), \quad w \to 0.$$

This choice of A and B makes the matching at the origin of the exponential function in (2.5) with $F(x,w)$ as best as possible.

We will show in later sections that for several interesting cases the finite sum in (2.8) gives the desired asymptotic representations, from which well-known limits can be derived. The special choice of A and B is crucial for obtaining asymptotic properties.

3. Gegenbauer polynomials

¿From (1.2) we obtain the following Cauchy-type integral

$$C_n^\gamma(x) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{dw}{(1-2xw+w^2)^\gamma\, w^{n+1}}, \tag{3.1}$$

where $\mathcal{C}$ is a circle around the origin with radius less than unity. Initially we assume that $x \in (-1,1)$, but later we do not need this restriction. We assume that $(1-2xw+w^2)^\gamma$ assumes real values for real values of x, w and γ.

We have

$$C_0^\gamma(x) = 1, \quad C_1^\gamma(x) = 2\gamma x, \quad C_2^\gamma(x) = 2\gamma(\gamma+1)x^2 - \gamma. \tag{3.2}$$

Hence, by (2.10),

$$A = 2x\gamma, \quad B = \gamma(1-2x^2).$$

It follows that

$$C_n^\gamma(x) = z^n \sum_{k=0}^{n} \frac{c_k}{z^k}\, \frac{H_{n-k}(\zeta)}{(n-k)!}, \tag{3.3}$$

where

$$z = \sqrt{B} = \sqrt{\gamma(1-2x^2)}, \quad \zeta = \frac{A}{2\sqrt{B}} = \frac{x\gamma}{z}.$$

The coefficients follow from (cf. (2.7))

$$f(x,w) = e^{-2\gamma xw+\gamma(1-2x^2)w^2}(1-2xw+w^2)^{-\gamma} = \sum_{k=0}^{\infty} c_k w^k. \tag{3.4}$$

We have

$$c_0 = 1, \quad c_1 = c_2 = 0, \quad c_3 = \tfrac{2}{3}\gamma x(4x^2-3), \quad c_4 = \tfrac{1}{2}\gamma\left[1+8x^2(x^2-1)\right].$$

Higher coefficients follow from the recursion relation

$$kc_k = 2x(k-1)c_{k-1} - (k-2)c_{k-2} + 2\gamma x(4x^2-3)c_{k-3} + 2\gamma(1-2x^2)c_{k-4}. \tag{3.5}$$

This relation follows from substituting the Maclaurin series of f (see (3.4)) into the differential equation

$$(1-2xw+w^2)\frac{df}{dw} = 2\gamma(-3x+4x^3+w-2x^2w)w^2\, f.$$

In (3.3) no restrictions on x and γ are needed, although ζ and $1/z$ become infinite if $x^2 = \frac{1}{2}$; this singularity is removable because of the term z^n in front of the series. It follows from (2.9) that, if $x_0^2 = 1/2$, we have

$$C_n^\gamma(x_0) = (2x_0\gamma)^n \sum_{k=0}^{n} \frac{c_k}{(2x_0\gamma)^k\,(n-k)!}, \tag{3.6}$$

where the c_k are as in (3.4) and (3.5) with $x = x_0$.

3.1. Asymptotic properties of the expansion (3.3)

To verify the asymptotic character of (3.3), we observe that the sequence $\{\Phi_k\}$ with $\Phi_k = c_k/z^k$ has the following (somewhat irregular $^{*)}$) asymptotic structure:

$$\Phi_k = \mathcal{O}\left(\gamma^{\lfloor k/3\rfloor - k/2}\right), \quad k = 0,1,2,\ldots, \tag{3.7}$$

where $\lfloor x\rfloor$ means the integer part of x.

More important, the successive Hermite polynomials $H_n(\zeta), H_{n-1}(\zeta),\ldots,$ in (3.3) are of lower degree with respect to γ. This means that, using (3.7),

$$\frac{c_k}{z^k}\, H_{n-k}(\zeta) = \mathcal{O}\left(\gamma^{n/2+\lfloor k/3\rfloor-k}\right), \quad \gamma\to\infty.$$

This gives the asymptotic nature of the terms in (3.3) for large values of γ, with x and n fixed. We can also estimate the remainder.

$^{*)}$ It is irregular because, for example, Φ_{12} has the same estimate $\mathcal{O}(\gamma^{-2})$ as Φ_{10}, whereas Φ_{11}, has the estimate $\mathcal{O}(\gamma^{-5/2})$.

Let, for $n_0 = 0, 1, \ldots, n,$, the remainder Δ_{n_0} be defined by

$$\Delta_{n_0} := \gamma^{-n} \left[C_n^\gamma(x) - z^n \sum_{k=0}^{n_0} \frac{c_k}{z^k} \frac{H_{n-k}(\zeta)}{(n-k)!} \right] = \gamma^{-n} z^n \sum_{k=n_0+1}^{n} \frac{c_k}{z^k} \frac{H_{n-k}(\zeta)}{(n-k)!}.$$

Then we can estimate Δ_{n_0} for large γ. For example, for $n = 20$,we have the following results:

$$\begin{array}{rl}
n_0 = 0, 1, 2, & \Delta_{n_0} = \mathcal{O}\left(\gamma^{-2}\right), \\
n_0 = 3, & \Delta_{n_0} = \mathcal{O}\left(\gamma^{-3}\right), \\
n_0 = 4, 5, & \Delta_{n_0} = \mathcal{O}\left(\gamma^{-4}\right), \\
n_0 = 6, & \Delta_{n_0} = \mathcal{O}\left(\gamma^{-5}\right), \\
n_0 = 7, 8, & \Delta_{n_0} = \mathcal{O}\left(\gamma^{-6}\right), \\
n_0 = 9, & \Delta_{n_0} = \mathcal{O}\left(\gamma^{-7}\right), \\
n_0 = 10, 11, & \Delta_{n_0} = \mathcal{O}\left(\gamma^{-8}\right), \\
n_0 = 12, & \Delta_{n_0} = \mathcal{O}\left(\gamma^{-9}\right), \\
n_0 = 13, 14, & \Delta_{n_0} = \mathcal{O}\left(\gamma^{-10}\right), \\
n_0 = 15, & \Delta_{n_0} = \mathcal{O}\left(\gamma^{-11}\right), \\
n_0 = 16, 17, & \Delta_{n_0} = \mathcal{O}\left(\gamma^{-12}\right), \\
n_0 = 18, & \Delta_{n_0} = \mathcal{O}\left(\gamma^{-13}\right), \\
n_0 = 19, & \Delta_{n_0} = \mathcal{O}\left(\gamma^{-14}\right), \\
n_0 = 20, & \Delta_{n_0} = 0.
\end{array}$$

In fact, we have

$$\Delta_{n_0} = \mathcal{O}\left(\gamma^{\lfloor (n_0+1)/3 - n_0 - 1 \rfloor}\right), \quad \text{as} \quad \gamma \to \infty.$$

For proofs of the asymptotic properties we refer to the following subsection.

It is not difficult to verify that the limits given in (1.3) and (1.4) follow from (3.3).

3.2. Proofs of the asymptotic properties

To prove the asymptotic properties of the expansion we first show that the coefficients c_k in formula (3.4) satisfy

$$c_k = \mathcal{O}\left(\gamma^{\lfloor k/3 \rfloor}\right), \quad \gamma \to \infty. \tag{3.8}$$

The proof follows by induction with respect to k, by using the recurrence relation (3.5), that we write in the form

$$c_k = a c_{k-1} + b c_{k-2} + c \gamma c_{k-3} + d \gamma c_{k-4}, \tag{3.9}$$

where a, b, c, d are functions of k and x generically different from 0 ($a \neq 0$ for $x \neq 0$, $b \neq 0\ \forall x$, $c \neq 0$ for $x \neq 0$ and $x \neq \pm\sqrt{3}/2$ and $d \neq 0$ for $x \neq \pm 1/\sqrt{2}$).

i) From the coefficients given after (3.4) it follows that (3.8) is true for $k = 0, 1, 2, 3, 4$.

ii) Suppose that (3.8) is true for a given $k \geq 4$. Then, using (3.9),

$$\begin{aligned} c_{k+1} =& a\mathcal{O}(\gamma^{\lfloor k/3 \rfloor}) + b\mathcal{O}(\gamma^{\lfloor (k-1)/3 \rfloor}) + c\mathcal{O}(\gamma^{1+\lfloor (k-2)/3 \rfloor}) + d\mathcal{O}(\gamma^{1+\lfloor (k-3)/3 \rfloor}) = \\ & (a+d)\mathcal{O}(\gamma^{\lfloor k/3 \rfloor}) + b\mathcal{O}(\gamma^{\lfloor (k-1)/3 \rfloor}) + c\mathcal{O}(\gamma^{\lfloor (k+1)/3 \rfloor}) = c\mathcal{O}(\gamma^{\lfloor (k+1)/3 \rfloor}). \end{aligned}$$

Therefore, (3.8) is proved, and by using $\Phi_k = c_k/z^k$ and $z = \mathcal{O}(\gamma^{1/2})$, it follows that (3.7) holds true.

The above derivation fails when $c = 0$. Certainly, it does not fail, we should say: the estimate of the order in γ can be improved when $c = 0$. That is, when $x = 0$ or $x = \pm\sqrt{3}/2$, we can show

$$c_k = \mathcal{O}(\gamma^{\lfloor k/4 \rfloor}). \tag{3.10}$$

In this case the recurrence (3.5) reads

$$c_k = ac_{k-1} + bc_{k-2} + d\gamma c_{k-4}, \tag{3.11}$$

(where a is also zero if $x = 0$, but never mind). We can repeat the steps i), ii) in the above induction proof:

i) is true for $k = 0, 1, 2, 3, 4$ ($c_3 = 0$ for $x = 0$ or $x = \pm\sqrt{3}/2$).

ii) Suppose (3.10) is true for $k \geq 4$. Then,

$$\begin{aligned} c_{k+1} =& a\mathcal{O}(\gamma^{\lfloor k/4 \rfloor}) + b\mathcal{O}(\gamma^{\lfloor (k-1)/4 \rfloor}) + d\mathcal{O}(\gamma^{1+\lfloor (k-3)/4 \rfloor}) = \\ & a\mathcal{O}(\gamma^{\lfloor k/4 \rfloor}) + b\mathcal{O}(\gamma^{\lfloor (k-1)/4 \rfloor}) + d\mathcal{O}(\gamma^{\lfloor (k+1)/4 \rfloor}) = d\mathcal{O}(\gamma^{\lfloor (k+1)/4 \rfloor}). \end{aligned}$$

And so, (3.10) is proved. Therefore, for $c = 0$,

$$\Phi_k = \mathcal{O}(\gamma^{\lfloor k/4 \rfloor - k/2}), \quad k = 0, 1, 2, \ldots \tag{3.12}$$

This derivation of (3.10) fails when $d = 0$. But in this case $c \neq 0$ and (3.7) holds.

3.3. Approximating the zeros

When computing approximations of the zeros of the Gegenbauer polynomials for large values of γ we start with the zeros of the Hermite polynomial $H_n(\zeta)$ in (3.3).

Let $g_{n,m}, h_{n,m}$ be the $m-$th zero of $C_n^\gamma(x)$, $H_n(x)$, respectively, $m = 1, 2, \ldots, n$. Then, for given γ and n we take the relation for ζ used in (3.3) to compute a first approximation of $g_{n,m}$ by writing

$$\frac{\gamma g_{n,m}}{\sqrt{\gamma(1 - 2g_{n,m}^2)}} \sim h_{n,m}.$$

Inverting this relation we obtain

$$g_{n,m} \sim \frac{h_{n,m}}{\sqrt{\gamma + 2h_{n,m}^2}}, \quad m = 1, 2, \ldots, n. \tag{3.13}$$

The accuracy is rather limited, unless γ is very large. For example, if $\gamma = 1000, n = 20$, the best relative accuracy in the zeros is about 1/1000, but the worst result (for the largest zero) is 0.016. In the next section we give more details on how to obtain better approximations from the representation like (3.3) for the case of the Laguerre polynomials.

4. Laguerre polynomials

We take as generating function (see (1.5))

$$F(x,w) = (1+w)^{-\alpha-1} e^{wx/(1+w)} = \sum_{n=0}^{\infty} p_n(x)\, w^n, \tag{4.1}$$

with

$$p_n(x) = (-1)^n L_n^{\alpha}(x). \tag{4.2}$$

We have

$$L_0^{\alpha}(x) = 1, \quad L_1^{\alpha}(x) = \alpha + 1 - x, \quad L_2^{\alpha}(x) = \tfrac{1}{2}[(\alpha+1)(\alpha+2) - 2(\alpha+2)x + x^2].$$

This gives

$$A = x - \alpha - 1, \quad B = x - \tfrac{1}{2}(\alpha+1).$$

Writing

$$f(x,w) = F(x,w)\, e^{-Aw+Bw^2} = \sum_{k=0}^{\infty} c_k w^k, \tag{4.3}$$

we obtain

$$c_0 = 1, \quad c_1 = c_2 = 0, \quad c_3 = \tfrac{1}{3}(3x - \alpha - 1), \quad c_4 = \tfrac{1}{4}(-4x + \alpha + 1).$$

and the recursion relation

$$kc_k = -2(k-1)c_{k-1} - (k-2)c_{k-2} + (3x - \alpha - 1)c_{k-3} + (2x - \alpha - 1)c_{k-4}. \tag{4.4}$$

This relation follows from substituting the Maclaurin series of f into the differential equation

$$(1+w)^2 \frac{df}{dw} = [3x - \alpha - 1 + (2x - 1 - a)w]\; w^2\, f.$$

It follows that

$$L_n^{\alpha}(x) = (-1)^n\, z^n \sum_{k=0}^{n} \frac{c_k}{z^k}\, \frac{H_{n-k}(\zeta)}{(n-k)!}, \tag{4.5}$$

where

$$z = \sqrt{x - (\alpha+1)/2}\,, \quad \zeta = \frac{x - \alpha - 1}{2z}. \tag{4.6}$$

The representation in (4.5) holds for $n = 0, 1, 2, \ldots$, and all complex values of x and α. If $z = 0$ it is more convenient to write

$$L_n^\alpha(x_0) = x_0^n \sum_{k=0}^{n} \frac{(-1)^k\, c_k}{x_0^k\,(n-k)!},$$

where c_k follow from (4.3) and (4.4), with $x = x_0 = \frac{1}{2}(\alpha+1)$.

The representation in (4.5) has an asymptotic character for large values of $|\alpha| + |x|$; the degree n should be fixed. To verify the asymptotic character, we write $\gamma = \alpha+1, x = \gamma\xi$. We observe that the sequence $\{\Phi_k\}$ with $\Phi_k = c_k/z^k$ has the following asymptotic structure:

$$\Phi_k = \mathcal{O}\left[\gamma^{-\lfloor k/3\rfloor - k/2}\right], \quad k = 0, 1, 2, \ldots. \tag{4.7}$$

as $\gamma \to \infty$. The derivation of (4.7) runs as in the case of the Gegenbauer polynomials (Section 3.1). The only difference is that, in this case in the recurrence relation (3.9) we have $c = 3\xi - 1$ and $d = 2\xi - 1$. Therefore, condition $c \neq 0$ reads $\xi \neq 1/3$ and $d \neq 0$ reads $\xi \neq 1/2$. Also, in this case a never vanishes.

Again, the successive Hermite polynomials $H_n(\zeta), H_{n-1}(\zeta), \ldots$, in (4.5) are of lower degree with respect to γ. This, together with (4.7), explains the asymptotic nature of the representation in (4.4) for large values of $|\alpha| + |x|$, with n fixed.

It is not difficult to verify that the limits given in (1.6) and (1.7) follow from (4.5).

4.1. Approximating the zeros

Let $l_{n,m}, h_{n,m}$ be the $m-$th zero of $L_n^\alpha(x)$, $H_n(x)$, respectively, $m = 1, 2, \ldots, n$. Then, for given α and n we use the relation for ζ in (4.6) to compute a first approximation of $l_{n,m}$ by writing

$$\frac{l_{n,m} - \alpha - 1}{2\sqrt{l_{n,m} - \frac{1}{2}(\alpha+1)}} \sim h_{n,m}.$$

Inverting this relation we obtain

$$l_{n,m} \sim \alpha + 1 + 2h_{n,m}^2 + h_{n,m}\sqrt{2(\alpha+1) + 4h_{n,m}^2}\,. \tag{4.8}$$

In Calogero (1978) the following asymptotic result has been given:

$$l_{n,m} = \alpha + \sqrt{2\alpha}\, h_{n,m} + \tfrac{1}{3}(1 + 2n + 2h_{n,m}^2) + \mathcal{O}\left(\alpha^{-\frac{1}{2}}\right), \tag{4.9}$$

as $\alpha \to \infty$. This result does not follow from (4.8) but we can derive (4.9) from (4.5). We give a few steps of this method.

Using the recursion relations

$$2nH_{n-1}(x) = 2xH_n(x) - H_{n+1}(x) = \frac{d}{dx}H_n(x),$$

we obtain

$$H_{n-3}(\zeta) = \frac{1}{4n(n-1)(n-2)} \left[(2\zeta^2 - n + 1)H_n'(\zeta) - 2\zeta n H_n(\zeta)\right]. \tag{4.10}$$

Hence, the first two nonvanishing terms in (4.5) yield

$$\frac{1}{n!}H_n(\zeta) + \frac{c_3}{(n-3)!z^3}H_{n-3}(\zeta) = \frac{1}{n!}\left[F(\zeta)H_n(\zeta) + G(\zeta)H_n'(\zeta)\right], \tag{4.11}$$

where

$$F(\zeta) = 1 - \frac{\zeta n c_3}{2z^3}, \quad G(\zeta) = \frac{c_3(2\zeta^2 - n + 1)}{4z^3},$$

with c_3 given after (4.3); x can be written as a function of ζ by inverting the relations used in (4.6), that is,

$$x = \alpha + 1 + 2\zeta^2 + \zeta\sqrt{2(\alpha+1) + 4\zeta^2}\,. \tag{4.12}$$

Now, let h be a zero of $H_n(\zeta)$. To solve $F(\zeta)H_n(\zeta) + G(\zeta)H_n'(\zeta) = 0$, we substitute $\zeta = h + \varepsilon$ and expand in powers of ε. Neglecting powers $\varepsilon^n, n \ge 2$, we obtain

$$\begin{aligned}&[G(h) + \varepsilon\{F(h) + G'(h)\}]H_n'(h) + \varepsilon G(h)H_n''(h) = \\ &[G(h) + \varepsilon\{F(h) + G'(h) + 2hG(h)\}]H_n'(h) \sim 0,\end{aligned}$$

where we have replaced $H_n''(h)$ with $2hH_n'(h)$ by using the differential equation of the Hermite polynomials.

Solving for ε we find

$$\varepsilon \sim \frac{-G(h)}{F(h) + G'(h) + 2hG(h)},$$

and expanding the result for large α we obtain

$$\varepsilon = \tfrac{1}{3}\sqrt{\tfrac{2}{\alpha}}\,(n - 1 - 2h^2) + \mathcal{O}\left(\alpha^{-1}\right). \tag{4.13}$$

Substituting $\zeta = h + \varepsilon$ with the approximation (4.13) in (4.12) and expanding again, we find

$$x = \alpha + \sqrt{2\alpha}\,h + \tfrac{1}{3}(1 + 2n + 2h^2) + \mathcal{O}\left(\alpha^{-\frac{1}{2}}\right), \quad \alpha \to \infty,$$

which is the same as Calogero's result (4.9).

In Table 4.1 we show for $n = 10$ the relative accuracy in the approximation (4.9) for several values of α. That is, we show

$$\left|\frac{l_{10,m} - \widetilde{l}_{10,m}}{l_{10,m}}\right|, \quad m = 1, 2, \ldots, 10,$$

where $\widetilde{l}_{10,m}$ are the approximations obtained by (4.9).

m \ α	10	50	100	250	500	1000
1	.17e+1	.11e-0	.35e-1	.77e-2	.25e-2	.84e-3
2	.73e-0	.63e-1	.21e-1	.49e-2	.16e-2	.56e-3
3	.34e-0	.36e-1	.12e-1	.30e-2	.10e-2	.36e-3
4	.15e-0	.18e-1	.65e-2	.16e-2	.57e-3	.20e-3
5	.40e-1	.54e-2	.20e-2	.52e-3	.18e-3	.64e-4
6	.25e-1	.41e-2	.16e-2	.45e-3	.17e-3	.60e-4
7	.67e-1	.12e-1	.48e-2	.13e-2	.50e-3	.18e-3
8	.95e-1	.18e-1	.77e-2	.22e-2	.83e-3	.31e-3
9	.12e-0	.24e-1	.11e-1	.31e-2	.12e-2	.45e-3
10	.13e-0	.31e-1	.14e-1	.42e-2	.16e-2	.62e-3

Table 4.1. Relative accuracy in computed zeros of $L_{10}^{\alpha}(x)$ by using approximation (4.9).

5. The Tricomi-Carlitz polynomials

The Tricomi-Carlitz polynomials are defined by

$$t_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{x-\alpha}{k} \frac{x^{n-k}}{(n-k)!}. \tag{5.1}$$

We have the relation with the Laguerre polynomials:

$$t_n^{(\alpha)}(x) = (-1)^n \, L_n^{(x-\alpha-n)}(x). \tag{5.2}$$

The polynomials satisfy the recurrence

$$(n+1)t_{n+1}^{(\alpha)}(x) - (n+\alpha)\, t_n^{(\alpha)}(x) + x\, t_{n-1}^{(\alpha)}(x) = 0, \quad n \geq 1, \tag{5.3}$$

with initial values $t_0^{(\alpha)}(x) = 1,\ t_1^{(\alpha)}(x) = \alpha$. A few other values are

$$t_2^{(\alpha)}(x) = \tfrac{1}{2}\left(\alpha + \alpha^2 - x\right), \quad t_3^{(\alpha)}(x) = \tfrac{1}{6}\left(2\alpha + 3\alpha^2 + \alpha^3 - 2x - 3x\alpha\right). \tag{5.4}$$

Tricomi (1948) introduced the polynomials. He observed that $\{t_n^{(\alpha)}(x)\}$ is not a system of orthogonal polynomials, the recurrence relations failing to have the required form (cf. Szegö (1975, page 43)). However, Carlitz (1958) discovered that if one sets

$$f_n^{(\alpha)}(x) = x^n\, t_n^{(\alpha)}(x)(x^{-2}), \tag{5.5}$$

then $\{f_n^{(\alpha)}(x)\}$ satisfies

$$(n+1)f_{n+1}^{(\alpha)}(x) - (n+\alpha)\, x\, f_n^{(\alpha)}(x) + f_{n-1}^{(\alpha)}(x) = 0, \quad n \geq 1, \tag{5.6}$$

with initial values $f_0^{(\alpha)}(x) = 1,\ f_1^{(\alpha)}(x) = \alpha\, x$. A few other values are

$$f_2^{(\alpha)}(x) = \tfrac{1}{2}\left[\alpha(1+\alpha)x^2 - 1\right], \quad f_3^{(\alpha)}(x) = \tfrac{1}{6}\, x\, \left(-2 + 2\alpha x^2 - 3\alpha + 3\alpha^2 x^2 + \alpha^3 x^2\right).$$

There is a generating function for $f_n^{(\alpha)}(x)$:

$$F(x,w) = e^{w/x + (1-\alpha x^2)/x^2\, \ln(1-xw)} = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) w^n. \quad |wx| < 1. \tag{5.7}$$

If $x = 0$ this reduces to

$$e^{-\frac{1}{2}w^2} = \sum_{n=0}^{\infty} f_{2n}^{(\alpha)}(0)\, w^{2n},$$

giving

$$f_{2n}^{(\alpha)}(0) = (-1)^n\, 2^{-n}/n!, \quad f_{2n+1}^{(\alpha)}(0) = 0, \quad n = 0, 1, 2, \ldots .$$

Carlitz proved that for $\alpha > 0$, $\{f_n^{(\alpha)}(x)\}$ satisfies the orthogonality relation

$$\int_{-\infty}^{\infty} f_m^{(\alpha)}(x)\, f_n^{(\alpha)}(x)\, d\psi^{(\alpha)}(x) = \frac{2\, e^{\alpha}}{(n+\alpha)\, n!}\, \delta_{mn}, \tag{5.8}$$

where $\psi^{(\alpha)}(x)$ is the step function whose jumps are

$$d\,\psi^{(\alpha)}(x) = \frac{(k+\alpha)^{k-1}\, e^{-k}}{k!} \quad \text{at} \quad x = x_k = \pm \frac{1}{\sqrt{k+\alpha}}, \quad k = 0, 1, 2, \ldots . \tag{5.9}$$

The values x_k play a special role in the generating function because for these $x-$values we have

$$e^{w/x_k}\, (1 - x_k w)^k = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x_k)\, w^n,$$

and now the series converges for all values of w.

For further generalizations of the Tricomi-Carlitz polynomials the reader is referred to Askey & Ismail (1984) and Chihara & Ismail (1982); Chihara (1978) gives a brief treatment of the polynomials $t_n^{(\alpha)}(x)$. Goh & Wimp (1994 and 1997) establish the asymptotic behavior of the Tricomi-Carlitz polynomials and discuss their zero distribution. They observe that the polynomials $f_n(x/\sqrt{\alpha}\,)$ have all zeros in the interval $[-1, 1]$. They use in their second paper a probabilistic approach for improving their earlier results concerning the asymptotic distribution of the zeros of the polynomials $f_n^{(\alpha)}(x)$. Saddle point methods are used to study the asymptotics for $f_n^{(\alpha)}(x)$ in the complex plane.

In this section we use the method of Section 2 for obtaining an asymptotic representation (for large values of α) of the Tricomi-Carlitz polynomials in terms of the Hermite polynomial.

The role of the Hermite polynomials can be shown by observing that

$$\lim_{\alpha\to\infty} f_n^{(\alpha)}\left(\frac{x\sqrt{2}}{\alpha}\right) = \frac{2^{-n/2}}{n!}\, H_n(x). \tag{5.10}$$

This follows from the results given below.

5.1. Hermite-type representation of the Tricomi-Carlitz polynomials

¿From the first polynomials given after (5.6) we obtain

$$A = \alpha x, \quad B = \tfrac{1}{2}(1-\alpha x^2).$$

Hence,

$$f_n^{(\alpha)}(x) = z^n \sum_{k=0}^{n} \frac{c_k}{z^k}\, \frac{H_{n-k}(\zeta)}{(n-k)!}, \tag{5.11}$$

where

$$z = \sqrt{(1-\alpha x^2)/2}\,, \quad \zeta = \frac{\alpha x}{\sqrt{2(1-\alpha x^2)}}.$$

For the special value $x = x_0$ that satisfies $\alpha x_0^2 = 1$, it is better to write an expansion of the form (2.9).

The coefficients c_k in (5.11) are are defined by

$$f(x,w) = F(x,w)\, e^{-Aw+Bw^2} = \sum_{k=0}^{\infty} c_k w^k.$$

where $F(x,w)$ is given in (5.7). We have

$$c_0 = 1, \quad c_1 = c_2 = 0, \quad c_3 = \tfrac{1}{3}x(\alpha x^2-1), \quad c_4 = \tfrac{1}{4}x^2(1-\alpha x^2).$$

The coefficients can easily be computed from the differential equation

$$(xw-1)\frac{df}{dw} = x(1-\alpha x^2)w^2\, f,$$

which gives the recursion relation

$$kc_k = x(k-1)c_{k-1} + x(\alpha x^2-1)c_{k-3}, \quad k = 3,4,\ldots. \tag{5.12}$$

Observe that $c_0 = f(0,w) = 1$, and that $c_k = 0, k \ge 1$ if $x = 0$.

To verify the asymptotic character of (5.11), we observe that the sequence $\{\Phi_k\}$ with $\Phi_k := c_k/z^k$ has the following asymptotic structure:

$$\Phi_k = \mathcal{O}\left[\alpha^{-\lfloor k/3\rfloor - k/2}\right], \quad k = 0,1,2,\ldots,$$

as $\alpha \to \infty, x \neq 0$. The main step in verifying this estimate is the proof of

$$c_k = \mathcal{O}\left(\alpha^{\lfloor k/3 \rfloor}\right), \quad \alpha \to \infty,$$

Which follows from and that the successive Hermite polynomials $H_n(\zeta), H_{n-1}(\zeta), \ldots,$ in (5.11) are of lower degree with respect to α. This explains the asymptotic nature of the representation in (5.11) for large values of α, with $x \neq 0$ and n fixed.

Observe that the limit in (5.10) indeed follows from (5.11).

5.2. Approximating the zeros

Let $f_{n,m}, h_{n,m}$ be the $m-$th zero of $L_n^\alpha(x)$, $H_n(x)$, respectively, $m = 1, 2, \ldots, n$. Then, for given α and n we use the relation for ζ in (5.11) to compute a first approximation of $f_{n,m}$ by writing

$$\frac{\alpha f_{n,m}}{\sqrt{2(1-\alpha f_{n,m}^2)}} \sim h_{n,m}.$$

Inverting this relation we obtain

$$f_{n,m} \sim h_{n,m} \sqrt{\frac{2}{\alpha^2 + 2\alpha h_{n,m}^2}}\,. \tag{5.13}$$

We can use the method of §4.1 for obtaining better approximations.

6. Jacobi polynomials

We give a few steps for the Jacobi case, which is quite complicated because of the many parameters involved. Consider the generating function

$$F(x,w) = \frac{2^{\alpha+\beta}}{R}(1-w+R)^{-\alpha}(1+w+R)^{-\beta} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) w^n, \tag{6.1}$$

where

$$R = \sqrt{1-2xw+w^2}\,.$$

We have

$$\begin{aligned}
P_0^{(\alpha,\beta)}(x) &= 1,\\
P_1^{(\alpha,\beta)}(x) &= \tfrac{1}{2}[\alpha-\beta+(\alpha+\beta+2)x],\\
P_2^{(\alpha,\beta)}(x) &= \tfrac{1}{8}\{(1+\alpha)(2+\alpha)+(1+\beta)(2+\beta)-2(\alpha+2)(\beta+2)+\\
&\qquad 2x[(1+\alpha)(2+\alpha)-(1+\beta)(2+\beta)]+\\
&\qquad x^2[(1+\alpha)(2+\alpha)+(1+\beta)(2+\beta)+2(\alpha+2)(\beta+2)]\},
\end{aligned}$$

from which A and B follow:

$$
\begin{aligned}
A &= \tfrac{1}{2}[\alpha - \beta + (\alpha + \beta + 2)x], \\
B &= \tfrac{1}{8}[\alpha + \beta + 4 + 2x(\beta - \alpha) - x^2(3\alpha + 3\beta + 8)].
\end{aligned}
\tag{6.2}
$$

(0.8;0cm)fig61.ps **Figure 6.1.** Graphs of the Hermite and Jacobi polynomials that occur in (6.7) (with $\alpha = 50, \beta = 40$). The graphs coincide when $\alpha, \beta \to \infty$.

We obtain the expansion

$$
P_n^{(\alpha,\beta)}(x) = z^n \sum_{k=0}^{n} \frac{c_k}{z^k} \frac{H_{n-k}(\zeta)}{(n-k)!}, \tag{6.3}
$$

where

$$
z = \sqrt{B}, \quad \zeta = \frac{A}{2\sqrt{B}}, \tag{6.4}
$$

and c_k are the coefficients defined by

$$
f(x,w) = F(x,w)e^{-Aw+Bw^2} = \sum_{k=0}^{\infty} c_k w^k, \tag{6.5}
$$

with $F(x,w)$ given in (6.1). We have

$$
\begin{aligned}
c_0 &= 1, \quad c_1 = c_2 = 0, \\
c_3 &= \tfrac{1}{12}[\beta - \alpha - 3x(\alpha + \beta + 4) + 3x^2(\alpha - \beta) + x^3(5\alpha + 5\beta + 16)].
\end{aligned}
\tag{6.6}
$$

For large values of the parameters α and β we can prove that the quantities $\Phi_k := c_k/z^k$ have the asymptotic behavior as shown in (3.7).

The following limit is of interest:

$$
\lim_{\alpha,\beta\to\infty} \left(\tfrac{8}{\alpha+\beta}\right)^{n/2} P_n^{(\alpha,\beta)}\left(x\sqrt{\tfrac{2}{\alpha+\beta}} - \tfrac{\alpha-\beta}{\alpha+\beta}\right) = \frac{1}{n!} H_n(x), \tag{6.7}
$$

under the conditions that

$$
\frac{\alpha - \beta}{\alpha + \beta} = o(1), \quad x = \mathcal{O}(1), \quad \text{as} \quad \alpha, \beta \to \infty. \tag{6.8}
$$

Details on this limit will be given in the next section. Graphs of the polynomials of (6.7) (with $\alpha = 50, \beta = 40$) are given in Figure 6.1.

6.1. Asymptotic properties of the expansion (6.3)

We want to show formula (3.8) for the Jacobi case. It is very similar to the other cases, although somewhat more complicated.

We can derive a linear second order differential equation $pf'' + qf' + rf = 0$ (where derivatives are with respect to w) of the function $f(x,w)$ defined in (6.5). The quantities $p, q,$and r are polynomials in α, β, x and w, and are available on request from the authors (together they fill several pages of Maple output). They have the following structure

$$p = \sum_{k=1}^{4} p_k w^k, \quad q = \sum_{k=0}^{5} q_k w^k, \quad r = \sum_{k=2}^{6} r_k w^k.$$

Let $\gamma = \max(\alpha, \beta)$. From the Maple output we observe that, for all occurring values of the index k,

$$p_k = \mathcal{O}(\gamma), \quad q_k = \mathcal{O}(\gamma^2), \quad r_k = \mathcal{O}(\gamma^3).$$

By substituting this information into the differential equation $pf'' + qf' + rf = 0$, we obtain for the coefficients c_k of (6.5) a recurrence relation of the form

$$c_{k+1} = a\,c_k + b\,c_{k-1} + c\,c_{k-2} + d\,c_{k-3} + e\,c_{k-4} + f\,c_{k-5} + g\,c_{k-6}. \tag{6.9}$$

We denote $h \equiv (k+1)(kp_1 + q_0) = \mathcal{O}(\gamma^2)$. Then, we have

$$\begin{aligned} a &= \frac{k((k-1)p_2 + q_1)}{h} = \mathcal{O}(1), \quad b = \frac{(k-1)((k-2)p_3 + q_2)}{h} = \mathcal{O}(1), \\ c &= \frac{(k-2)(k-3)p_4 + (k-2)q_3 + r_2}{h} = \mathcal{O}(\gamma), \quad d = \frac{(k-3)q_4 + r_3}{h} = \mathcal{O}(\gamma), \\ e &= \frac{(k-4)q_5 + r_4}{h} = \mathcal{O}(\gamma), \quad f = \frac{r_5}{h} = \mathcal{O}(\gamma), \quad g = \frac{r_6}{h} = \mathcal{O}(\gamma). \end{aligned}$$

For the first four c_k we have (cf. (6.6))

$$c_0 = 1 = \mathcal{O}(1), \quad c_1 = c_2 = 0 = \mathcal{O}(1), \quad c_3 = \mathcal{O}(\gamma).$$

By computing the next coefficients we find that

$$c_4 = -(r_3 + 3q_1c_3 + 6p_2c_3)/(12p_1 + 4q_0) = \mathcal{O}(\gamma),$$

and similarly

$$c_5 = \mathcal{O}(\gamma), \quad c_6 = \mathcal{O}(\gamma^2), \quad c_7 = \mathcal{O}(\gamma^2).$$

Therefore, equation (3.8) holds for $k = 0, ..., 7$. Now let us suppose that (3.8) holds for a certain $k \geq 7$. Then, using the recurrence (6.9) and the orders of $a, ..., g$, we have

$$\begin{aligned} c_{k+1} =& a\mathcal{O}(\gamma^{\lfloor k/3 \rfloor}) + b\mathcal{O}(\gamma^{\lfloor (k-1)/3 \rfloor}) + \frac{c}{\gamma}\mathcal{O}(\gamma^{1+\lfloor (k-2)/3 \rfloor}) + \frac{d}{\gamma}\mathcal{O}(\gamma^{1+\lfloor (k-3)/3 \rfloor}) + \\ & \frac{e}{\gamma}\mathcal{O}(\gamma^{1+\lfloor (k-4)/3 \rfloor}) + \frac{f}{\gamma}\mathcal{O}(\gamma^{1+\lfloor (k-5)/3 \rfloor}) + \frac{g}{\gamma}\mathcal{O}(\gamma^{1+\lfloor (k-6)/3 \rfloor}) = \\ & \frac{c}{\gamma}\mathcal{O}(\gamma^{\lfloor (k+1)/3 \rfloor}) = \mathcal{O}(\gamma^{\lfloor (k+1)/3 \rfloor}), \end{aligned}$$

unless $c = 0$. But, for generic x, $c \neq 0$. The order of c is governed by r_2, that is a polynomial of degree 3 in x that vanishes, at most, for three real values of x. For them, for the real roots of r_2, the order (3.8) of c_k could be improved as in the other cases. Details on these special cases will not be given because they are not very essential.

We conclude with giving details on the limit in (6.7). For large α and β the quantities A and B of (6.2) behave as follows:

$$A \sim \tfrac{1}{2}\,(\alpha+\beta)\,(x+\rho), \quad B \sim \tfrac{1}{8}\,(\alpha+\beta)\,(1+2x\rho-3x^2), \quad \rho = \frac{\alpha-\beta}{\alpha+\beta}. \tag{6.10}$$

Hence,

$$z \sim \sqrt{\tfrac{\alpha+\beta}{8}}\,\sqrt{1+2x\rho-3x^2}\,, \quad \zeta \sim \sqrt{\tfrac{\alpha+\beta}{2}}\,\frac{\rho+x}{\sqrt{1+2x\rho-3x^2}}. \tag{6.11}$$

If $x\rho$ and x^2 become small for large values of α and β, we can easily invert the relation between ζ and x, giving

$$x \sim \sqrt{\tfrac{2}{\alpha+\beta}}\,\zeta - \rho, \quad z \sim \sqrt{\tfrac{\alpha+\beta}{8}}\,. \tag{6.12}$$

Hence, taking only the term $k = 0$ in (6.3), we obtain

$$P_n^{(\alpha,\beta)}\left(\sqrt{\tfrac{2}{\alpha+\beta}}\,\zeta - \tfrac{\alpha-\beta}{\alpha+\beta}\right) \sim \left(\tfrac{\alpha+\beta}{8}\right)^{n/2}\frac{H_n(\zeta)}{n!}, \quad \text{as} \quad \alpha,\beta\to\infty. \tag{6.13}$$

Replacing x and ζ, we obtain (6.7) under the conditions given in (6.8). We can relax this condition, and derive more complicated limits. But the simple form of (6.7) is rather attractive, and will get lost under more general conditions on x and $\alpha-\beta$.

Again, an approximation of the zeros $j_{n,m}$ of $P_n^{(\alpha,\beta)}$ follows from (6.13):

$$j_{n,m} \sim \sqrt{\tfrac{2}{\alpha+\beta}}\,h_{n,m} - \frac{\alpha-\beta}{\alpha+\beta}, \quad m = 1,2,\ldots,n,$$

where $h_{n,m}$ are the zeros of $H_n(\zeta)$. When we use the method of §4.1 we obtain the following:

$$j_{n,m} = -\frac{\alpha-\beta}{\alpha+\beta} + \frac{2\sqrt{2\alpha\beta}\,h_{n,m}}{(\alpha+\beta)^{\frac{3}{2}}} + \frac{2(\alpha-\beta)(2n+1+2h_{n,m}^2)}{(\alpha+\beta)^2} + \mathcal{O}\left(\alpha^{-\frac{3}{2}}\right), \tag{6.14}$$

for $m = 1,2,\ldots,n$. This expansion is derived under the conditions $\alpha\to\infty, \beta = b\alpha$, b fixed, not with the approximations given in (6.10) - (6.12), but by using the original values of A, B, z, ζ and c_3 given in (6.2), (6.4) and (6.6). The derivation of (6.14) is quite straightforward, but rather complicated. For example, the first step is to write x as a function of ζ by inverting the relation in (6.4) with A, B given in (6.2). This gives

$$\begin{aligned}
x &= \frac{V+4\zeta\sqrt{W}}{2U},\\
U &= (\alpha+\beta+2)^2 + 2\zeta^2[3(\alpha+\beta)+8],\\
V &= 2(\alpha-\beta)[\alpha+\beta+2(1+\zeta^2)],\\
W &= 2(\alpha+1)(\beta+1)(\alpha+\beta+4) + 4\zeta^2[\alpha^2+\alpha\beta+\beta^2+5(\alpha+\beta)+8].
\end{aligned}$$

With this x the quantities z and c_3 of (6.4) and (6.6) should be used, to construct $F(\zeta)$ and $G(\zeta)$ of (4.11). Further details will not be given.

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