The Askey Scheme for Hypergeometric Orthogonal Polynomials Viewed from Asymptotic Analysis

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ABSTRACT

Many limits are known for hypergeometric orthogonal polynomials that occur in the Askey scheme. We show how asymptotic representations can be derived by using the generating functions of the polynomials. For example, we discuss the asymptotic representation of the Meixner-Pollaczek Jacobi, Meixner, and Krawtchouk polynomials in terms of Laguerre polynomials.

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1. Introduction

It is well known that the Hermite polynomials play a crucial role in certain limits of the classical orthogonal polynomials. For example, the ultraspherical (Gegenbauer) polynomials $C_n^\gamma(x)$, which are defined by the generating function

$$(1-2xw+w^2)^{-\gamma} = \sum_{n=0}^{\infty} C_n^\gamma(x) w^n, \quad -1 \le x \le 1, \quad |w| < 1, \tag{1.1}$$

have the well-known limit

$$\lim_{\gamma\to\infty} \gamma^{-n/2} C_n^\gamma(x/\sqrt{\gamma}) = \frac{1}{n!} H_n(x). \tag{1.2}$$

For the Laguerre polynomials, which are defined by the generating function

$$(1-w)^{-\alpha-1} e^{-wx/(1-w)} = \sum_{n=0}^{\infty} L_n^\alpha(x)\, w^n, \quad |w| < 1, \tag{1.3}$$

$\alpha, x \in \mathbb{C}$, a similar results reads

$$\lim_{\alpha\to\infty} \alpha^{-n/2} L_n^\alpha\left(x\sqrt{\alpha}+\alpha\right) = \frac{(-1)^n\, 2^{-n/2}}{n!}\, H_n\left(x/\sqrt{2}\right). \tag{1.4}$$