A simplification of Laplace's method: Applications to the gamma function and Gauss hypergeometric function

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ABSTRACT

The main difficulties in the Laplace's method of asymptotic expansions of integrals are originated by a change of variables. We propose a variant of the method which avoids that change of variables and simplifies the computations. On the one hand, the calculation of the coefficients of the asymptotic expansion is remarkably simpler. On the other hand, the asymptotic sequence is as simple as in the standard Laplace's method: inverse powers of the asymptotic variable. New asymptotic expansions of the gamma function $\Gamma(z)$ for large z and the Gauss hypergeometric function ${}_2F_1(a,b,c;z)$ for large b and c are given as illustration. An explicit formula for the coefficients of the classical Stirling expansion of $\Gamma(z)$ is also given.

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1. Introduction

Consider integrals of the form

$$F(x) \equiv \int_a^b e^{-xf(t)}g(t)dt, \tag{1}$$

where (a,b) is a real interval (finite or infinite), x is a large positive parameter and $f(t)$ and $g(t)$ are smooth enough functions in (a,b). Long ago, Laplace made the observation that the major contribution to the integral (1) comes from the neighborhoods of the points where $f(t)$ attains its smallest value. For instance, if $f(t)$ has its minimum value only at a point $t_0 \in (a,b)$ where $f(t)$ and $g(t)$ are analytic and $f'(t_0)=0$ and $f''(t_0)>0$, then Laplace's result is

$$F(x) \sim g(t_0)\sqrt{\frac{2\pi}{f''(t_0)x}}e^{-xf(t_0)}, \qquad x \to \infty. \tag{2}$$

The right hand side above is the first term of a complete asymptotic expansion that can be obtained in the following way [[8], Chap. 2, Sec. 1]. Perform in (1) the change of integration variable $t \to u$ defined by $f(t) - f(t_0) = u^2$:

$$F(x) = e^{-xf(t_0)} \int_{-\sqrt{f(a)-f(t_0)}}^{\sqrt{f(b)-f(t_0)}} e^{-xu^2} \frac{g(t(u))}{f'(t(u))} 2u du. \tag{3}$$

If the function

$$h(u) \equiv 2u \frac{g(t(u))}{f'(t(u))} \tag{4}$$

has a Taylor expansion at $u = 0$,

$$h(u) \sim \sum_{n=0}^{\infty} c_n u^n, \tag{5}$$

then we can apply Watson's Lemma to the integral (3). Replace (5) in the right hand side of (3) and interchange sum and integral [[8], Chap. 2, Theorem 1]:

$$F(x) \sim e^{-xf(t_0)} \sum_{n=0}^{\infty} c_{2n} \Phi_n(x), \qquad x \to \infty, \tag{6}$$

where the coefficients c_n are the Taylor coefficients of the function $h(u)$ at $u = 0$ (see eq. (5)) and the asymptotic sequence $\Phi_n(x)$ is

$$\Phi_n(x) \equiv \int_{-\infty}^{\infty} e^{-xu^2} u^{2n} du = \frac{\Gamma(n + 1/2)}{x^{n+1/2}}. \tag{7}$$

We see that in the standard Laplace's method, the analytic computation of the asymptotic sequence $\Phi_n(x)$ is straightforward. On the other hand, traditional text books about asymptotics do not give an explicit and general analytic formula for c_n, but only indications to compute them "at hand" or indications to compute them in specific examples using an algebraic manipulator. Recently, a general analytic formula for c_n has been derived in [[7], eqs. (3.6) or (3.12)]. The derivation of that formula is based on a careful examination of the computation of the coefficients of the standard Laplace's method which discovers the importance that in the analysis has the Faá di Bruno's formula for the $n-$th derivative of a composite function. It is given in terms of combinatorial functions whose complexity increases with n and can be obtained from a recurrence.

In [4], a modification of the Laplace's method is proposed which simplifies the computation of the coefficients c_n of the expansion, complicating, on the other hand, the computation of the asymptotic sequence $\Phi_n(x)$. It is shown there that the change of variable $f(t) - f(t_0) = u^2$ in (1) is not necessary to obtain an asymptotic expansion of (1). It is just necessary to expand $g(t)$ in (1) at the critical point t_0 of $f(t)$ and interchange sum and integral. If $g(t)$ has a Taylor expansion at $t = t_0$,

$$g(t) \sim \sum_{n=0}^{\infty} g_n (t - t_0)^n, \tag{8}$$

then, replace this expansion in (1) and interchange sum and integral. We obtain

$$F(x) \sim \sum_{n=0}^{\infty} g_n \tilde{\Phi}_n(x), \qquad x \to \infty, \tag{9}$$

with

$$\tilde{\Phi}_n(x) \equiv \int_{-\infty}^{\infty} e^{-xf(t)} (t - t_0)^n dt.$$

Now the asymptotic sequence $\tilde{\Phi}_n(x)$ is more complicated than the sequence $\Phi_n(x)$ given in (7) (it depends on the complexity of the function $f(t)$) and has not an universal form but it depends on the example on hand, on the function $f(t)$. On the other hand, the computation of the coefficients g_n is straightforward. This modification of the Laplace's method has permitted to derive new asymptotic expansions of several special functions such as the incomplete gamma functions [2] or the Gauss hypergeometric function [3].

In this paper we investigate a different modification of the Laplace's method which gives a new asymptotic expansion for (1), simplifying the computation of the coefficients of the expansion without complicating the computation of the asymptotic sequence. The idea is inspired in the procedure used in [5] and [[2], Sec. 4], where asymptotic expansions of the incomplete gamma functions are obtained; this is shown in Section 2. In Section 3 we apply the idea to two important special functions: the gamma function and the Gauss hypergeometric function, obtaining new asymptotic expansions of these functions. Section 4 contains some final remarks.

2. The modified Laplace's method

By subdividing the range of integration in (1) if necessary, we may assume that $f(t)$ has only one minimum at $t = t_0$ in $[a, b]$. Suppose that both, $f(t)$ and $g(t)$, have a Taylor expansion at t_0 with a common radius of convergence r. This condition may be relaxed and require only that both, $f(t)$ and $g(t)$ have an asymptotic expansion at t_0. But for the sake of clarity in the exposition we require the analyticity of $f(t)$ and $g(t)$ at $t = t_0$. On the other hand, this is the usual situation in most of the practical examples. When $t_0 = a$, we let $g(t)$ to possess perhaps an algebraic branch point at $t = a$, that is, $(t-a)^{-s}g(t)$ is analytic at $t = a$, with $s \in (-1, 0]$.

We distinguish two cases: (i) $f(t)$ attains its minimum at a point $t_0 \in (a, b)$. In this case $f'(t_0) = f''(t_0) = ... = f^{(m-1)}(t_0) = 0$ and $f^{(m)}(t_0) > 0$ with $m > 0$ even. (ii) $f(t)$ attains its minimum at the point $t_0 = a$ or $t_0 = b$. In this case $f'(t_0) = f''(t_0) = ... = f^{(m-1)}(t_0) = 0$ and $f^{(m)}(t_0) > 0$ if $t_0 = a$ and $(-1)^m f^{(m)}(t_0) > 0$ if $t_0 = b$, with m even or odd. In any case, consider the function

$$f_m(t) \equiv f(t) - f(t_0) - \frac{f^{(m)}(t_0)}{m!}(t-t_0)^m. \tag{10}$$

This function and the function $e^{-xf_m(t)} = e^{xf(t_0)}e^{-xf(t)+x\frac{f^{(m)}(t_0)}{m!}(t-t_0)^m}$ have a Taylor expansion at $t = t_0$:

$$e^{-xf(t)} = \sum_{n=0}^{\infty} A_n(x)(t-t_0)^n, \qquad |t-t_0| < r,$$

$$e^{x\frac{f^{(m)}(t_0)}{m!}(t-t_0)^m} = \sum_{n=0}^{\infty} \frac{x^n}{n!}\left(\frac{f^{(m)}(t_0)}{m!}\right)^n (t-t_0)^{nm}.$$

Also, the function $(t-a)^{-s}g(t)$ has a Taylor expansion at $t = t_0$:

$$g(t) = \sum_{n=0}^{\infty} B_n(t-t_0)^{n+s}, \qquad |t-t_0| < r,$$

with $s \in (-1, 0]$ if $t_0 = a$ and $s = 0$ if $t_0 \in (a, b]$. Therefore,

$$h(t,x) \equiv e^{-xf_m(t)}g(t) = \sum_{n=0}^{\infty} a_n(x)(t-t_0)^{n+s}, \qquad |t-t_0| < r, \tag{11}$$

with

$$a_n(x) = e^{xf(t_0)} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{x^k}{k!} \left(\frac{f^{(m)}(t_0)}{m!} \right)^k \sum_{j=0}^{n-mk} A_j(x) B_{n-mk-j}. \tag{12}$$

Observe that the coefficients $A_n(x)$ are $e^{-xf(t_0)}$ times a polynomial in x of degree at most n. Therefore, the coefficients $a_n(x)$ are polynomials in x of degree at most n. This is an explicit formula for the coefficients $a_n(x)$ of the Taylor expansion of $h(t,x)$, although the asymptotic behaviour of $a_n(x)$ when $x \to \infty$ is not clear from (12). We obtain below a different formula for $a_n(x)$ which shows its asymptotic behaviour (although it is less explicit). The asymptotic behaviour of the coefficients $a_n(x)$ of the expansion (11) depends on the order of the first term of the Taylor expansion of $f_m(t)$ at $t = t_0$. From the definition of $f_m(t)$ we have that:

$$f_m(t) = \sum_{n=p}^{\infty} \frac{f^{(n)}(t_0)}{n!} (t-t_0)^n, \qquad |t-t_0| < r,$$

with $p \geq m+1$. Then, for $|t-t_0| < r$,

$$\begin{aligned} e^{-xf_m(t)} =& 1 + \sum_{n=1}^{\infty} \frac{(-x)^n}{n!} \left[\frac{f^{(p)}(t_0)}{p!} (t-t_0)^p + \frac{f^{(p+1)}(t_0)}{(p+1)!} (t-t_0)^{p+1} + \ldots \right]^n \\ =& \sum_{n=0}^{\infty} c_n(x) (t-t_0)^n, \end{aligned} \tag{13}$$

with $c_0(x) = 1$, $c_1(x) = c_2(x) = \ldots c_{p-1}(x) = 0$ and $c_n(x) = \mathcal{O}(x^{\lfloor n/p \rfloor})$ for $n \geq p$. Then, apart from formula (12), coefficients $a_n(x)$ can also be written in the form

$$a_n(x) = \sum_{k=0}^{n} c_k(x) B_{n-k} = \mathcal{O}\left(x^{\lfloor n/p \rfloor} \right) \qquad \text{as} \qquad x \to \infty. \tag{14}$$

This formula shows that the coefficients $c_n(x)$, and then the coefficients $a_n(x)$, are polynomials in x of degree $[n/p]$. With these preliminaries, we can write the integral (1) in the form

$$F(x) = e^{-xf(t_0)} \int_a^b e^{-x \frac{f^{(m)}(t_0)}{m!} (t-t_0)^m} h(t,x) dt. \tag{15}$$

The Taylor series in the right hand side of (11) converges uniformly and absolutely to the function $h(t,x)$ in the disk $|t-t_0| < r$. Therefore, if the integration interval (a,b) is contained in that disk, we can replace that expansion in (15) and interchange sum and integral:

$$F(x) = e^{-xf(t_0)} \sum_{n=0}^{\infty} a_n(x) \Phi_n(x), \tag{16}$$

where

$$\begin{aligned} \Phi_n(x) \equiv& \int_a^b e^{-x \frac{f^{(m)}(t_0)}{m!} (t-t_0)^m} (t-t_0)^{n+s} dt = \int_{a-t_0}^{b-t_0} e^{-x \frac{f^{(m)}(t_0)}{m!} u^m} u^{n+s} du \\ =& \frac{1}{m} \left| \frac{m!}{f^{(m)}(t_0) x} \right|^{(n+s+1)/m} \left\{ \gamma \left(\frac{n+s+1}{m}, \frac{f^{(m)}(t_0)(b-t_0)^m}{m!} x \right) \right. \\ & \left. + (-1)^n \gamma \left(\frac{n+s+1}{m}, \frac{f^{(m)}(t_0)(a-t_0)^m}{m!} x \right) \right\}. \end{aligned} \tag{17}$$

Using the asymptotic behaviour of the incomplete gamma function we have that

$$\Phi_n(x) = \frac{\alpha + (-1)^n \beta}{m} \left| \frac{m!}{f^{(m)}(t_0) x} \right|^{(n+s+1)/m} \Gamma\left(\frac{n+s+1}{m} \right) + \text{exponentially small terms}, \tag{18}$$

with

$$\alpha = \begin{cases} 1 & \text{if} \quad t_0 \in [a,b) \\ 0 & \text{if} \quad t_0 = b \end{cases}, \qquad \beta = \begin{cases} 1 & \text{if} \quad t_0 \in (a,b] \\ 0 & \text{if} \quad t_0 = a. \end{cases}$$

Therefore,

$$F(x) \sim e^{-xf(t_0)} \sum_{n=0}^{\infty} a_n(x) \frac{\alpha + (-1)^n \beta}{m} \Gamma\left(\frac{n+s+1}{m}\right) \left|\frac{m!}{f^{(m)}(t_0)x}\right|^{(n+s+1)/m}. \tag{19}$$

We can see that $\Phi_n(x)$ is of the order $\mathcal{O}\left(x^{-(n+s+1)/m}\right)$ as $x \to \infty$. Therefore, the terms of the expansion (16) satisfy $a_n(x)\Phi_n(x) = \mathcal{O}(x^{\lfloor n/p \rfloor - (n+s+1)/m})$. Then, (16) (and (19)) is not a genuine asymptotic expansion. But we can group the terms of (16) in such a way that we get a genuine asymptotic expansion:

$$F(x) = e^{-xf(t_0)} \sum_{n=0}^{\infty} \Psi_n(x), \tag{20}$$

with

$$\begin{aligned} \Psi_n(x) \equiv & \sum_{k=\lfloor (n-m)/(p-m) \rfloor}^{\lfloor n/(p-m) \rfloor} a_{n+mk}(x) \Phi_{n+mk}(x) \\ = \sum_{k=\lfloor (n-m)/(p-m) \rfloor}^{\lfloor n/(p-m) \rfloor} & a_{n+mk}(x) \frac{\alpha + (-1)^{n+mk}\beta}{m} \left|\frac{m!}{f^{(m)}(t_0)x}\right|^{k+\frac{n+s+1}{m}} \Gamma\left(k + \frac{n+s+1}{m}\right) \\ & + \text{exponentially small terms} = \mathcal{O}\left(x^{-(n+s+1)/m}\right), \qquad x \to \infty. \end{aligned} \tag{21}$$

In this sum, $\lfloor -n \rfloor$ with n natural must be understood as zero. Observe that we have grouped the terms $a_k(x)\Phi_k(x)$ of (16) in blocks $\Psi_n(x)$ of $\lfloor n/(p-m) \rfloor - \lfloor (n-m)/(p-m) \rfloor + 1$ terms $a_k(x)\Phi_k(x)$. The table below represents the orders of the first few terms $a_k(x)\Phi_k(x)$ of the expansion (16) (multiplied by $x^{(s+1)/m}$) for the particular case $m = 3$ and $p = 5$.

Ψ_0	Ψ_1	Ψ_2	Ψ_3	Ψ_4	Ψ_5	Ψ_6	Ψ_7	Ψ_8	Ψ_9
0, x^0	1, $x^{-1/3}$	2, $x^{-2/3}$ 5, $x^{-2/3}$	3, x^{-1} 6, x^{-1}	4, $x^{-4/3}$ 7, $x^{-4/3}$ 10, $x^{-4/3}$	8, $x^{-5/3}$ 11, $x^{-5/3}$	9, x^{-2} 12, x^{-2} 15, x^{-2}	13, $x^{-7/3}$ 16, $x^{-7/3}$	14, $x^{-8/3}$ 17, $x^{-8/3}$ 19, $x^{-8/3}$	18, x^{-3} 20, x^{-3}

Table 1. *Every entry in this table represents (k, $\mathcal{O}(a_k\Phi_k)$) for the particular example $m = 3$ and $p = 5$. We group the terms $a_k\Phi_k$ of the expansion (16) in blocks of $p = 5$ terms. Every row of the table contains one of those blocks. In every column, the terms $a_k\Phi_k$ are of the same order, and sum up the corresponding Ψ_n. From the third column to the right, odd columns (odd n) contain $\lfloor n/2 \rfloor - \lfloor (n-3)/2 \rfloor + 1 = 2$ terms (Ψ_n is the sum of 2 terms $a_k\Phi_k$ for odd n) and even columns (even n) contain $\lfloor n/2 \rfloor - \lfloor (n-3)/2 \rfloor + 1 = 3$ terms (Ψ_n is the sum of 3 terms $a_k\Phi_k$ for even n).*

If the integration interval (a,b) is not contained in the disk $D = \{t, |t - t_0| < r\}$ of convergence of the Taylor series of $(t-a)^{-s}h(t,x)$ at $t = t_0$, we cannot replace expansion (11) in (15) and interchange sum and integral as we did to get equation (16). This happens for example when a and/or b are the point at infinity and $(t-a)^{-s}h(t,x)$ is not an entire function. Equation (16) does not hold but the right hand side of (16) is still an asymptotic expansion of $F(x)$. To see this, divide the integration interval (a,b) in two pieces, the one contained in the disk D: $D_{in}\equiv(a,b)\bigcap D$ and its complement not contained in D: $D_{out}\equiv(a,b)\setminus D_{in}$. We can write (1) in the form

$$F(x)=\left\{\int_{D_{in}}e^{-xf(t)}g(t)dt+\int_{D_{out}}e^{-xf(t)}g(t)dt\right\}. \tag{22}$$

As the point $t=t_0$ is the only minimum of $f(t)$ in $[a,b]$, $\exists\ \epsilon>0$ such that $f(t)\geq f(t_0)+\epsilon$ for any $t\in D_{out}$. Then we have

$$\int_{D_{out}}e^{-xf(t)}g(t)dt=e^{-x(f(t_0)+\epsilon)}\int_{D_{out}}e^{-x(f(t)-f(t_0)-\epsilon)}g(t)dt.$$

The last integral above is of the order $\mathcal{O}(1)$ when $x\to\infty$ and then, the integral on D_{out} is exponentially small compared with the integral on D_{in}:

$$\begin{aligned}F(x)&=\int_{D_{in}}e^{-xf(t)}g(t)dt+\mathcal{O}\left(e^{-x(f(t_0)+\epsilon)}\right)\\&=e^{-xf(t_0)}\left[\int_{D_{in}}e^{-x\frac{f^{(m)}(t_0)}{m!}(t-t_0)^m}h(t,x)dt+\mathcal{O}\left(e^{-x\epsilon}\right)\right].\end{aligned}$$

Now, we can proceed with the above integral over D_{in} as we proceeded in the case in which (a,b) was contained in D. We obtain:

$$F(x)=e^{-xf(t_0)}\left[\sum_{n=0}^{\infty}a_n(x)\Phi_n(x)+\text{exponentially small terms}\right], \tag{23}$$

where

$$\Phi_n(x)\equiv\int_{D_{in}}e^{-x\frac{f^{(m)}(t_0)}{m!}(t-t_0)^m}(t-t_0)^{n+s}dt=\int_{\tilde{a}}^{\tilde{b}}e^{-x\frac{f^{(m)}(t_0)}{m!}u^m}u^{n+s}du, \tag{24}$$

$\tilde{a}=a-t_0$ or $\tilde{a}=-r$ and $\tilde{b}=b-t_0$ or $\tilde{b}=r$. In any case, we obtain again (19).

We can resume the above discussion in the following theorem.

Theorem 1. *Let the functions $f(t)$ and $g(t)$ in (1) be continuous on (a,b), with (a,b) finite or infinite and suppose that the integral (1) exists for $x\geq x_0$. Let t_0 be the unique minimum of $f(t)$ on $[a,b]$ and let $f(t)$ and $g(t)$ be analytic at $t=t_0$. If $t_0=a$, we let $g(t)$ to possess a power branch point at $t=a$ in such a way that $(t-a)^{-s}g(t)$ is analytic at $t=a$, with $s\in(-1,0]$ if $t_0=a$ and $s=0$ if $t_0\in(a,b]$. Then,*

$$F(x)\sim e^{-xf(t_0)}\sum_{n=0}^{\infty}\Psi_n(x),\qquad\text{as}\quad x\to\infty, \tag{25}$$

with

$$\Psi_n(x)\equiv\sum_{k=\lfloor\frac{n-m}{p-m}\rfloor}^{\lfloor\frac{n}{p-m}\rfloor}a_{n+mk}(x)\frac{\alpha+(-1)^{n+mk}\beta}{m}\Gamma\left(k+\frac{n+s+1}{m}\right)\left|\frac{m!}{f^{(m)}(t_0)x}\right|^{k+\frac{n+s+1}{m}}. \tag{26}$$

In this sum, $\lfloor -n\rfloor$ with n natural must be understood as zero and

$$\alpha=\begin{cases}1 & \text{if}\quad t_0\in[a,b)\\ 0 & \text{if}\quad t_0=b\end{cases},\qquad\qquad \beta=\begin{cases}1 & \text{if}\quad t_0\in(a,b]\\ 0 & \text{if}\quad t_0=a.\end{cases}$$

The integer m is the degree of the first non vanishing derivative of $f(t)$ at $t=t_0$ and p is the degree of the next non vanishing derivative. For $n=0,1,2,...$,

$$a_n(x)\equiv e^{xf(t_0)}\sum_{k=0}^{\lfloor n/m\rfloor}\frac{x^k}{k!}\left(\frac{f^{(m)}(t_0)}{m!}\right)^k\times\sum_{j=0}^{n-mk}A_j(x)B_{n-mk-j}, \tag{27}$$

where $A_j(x)$ and B_j are the Taylor coefficients at $t = t_0$ of $e^{-xf(t)}$ and $(t-a)^{-s}g(t)$ respectively:

$$e^{-xf(t)} = \sum_{n=0}^{\infty} A_n(x)(t-t_0)^n, \qquad g(t) = \sum_{n=0}^{\infty} B_n(t-t_0)^{n+s}, \qquad |t-t_0| < r.$$

The terms of the expansion are of the order $\Psi_n(x) = \mathcal{O}\left(x^{-(n+s+1)/m}\right)$ as $x \to \infty$.

Observation 1. The terms $a_n(x)$ are polynomials of degree $\lfloor n/p \rfloor$. Then, from (26) we see that the asymptotic sequence $\Psi_n(x)$ is a sum of $\lfloor n/(p-m) \rfloor - \lfloor (n-m)/(p-m) \rfloor + 1$ negative powers of x. Therefore, the expansion (16) or (20) are rearrangements of the standard Laplace's expansion and "vice-versa". This means that the coefficients of the standard Laplace's expansion can be explicitly obtained from the coefficients of (16) after an appropriate rearrangement. We will show this fact in the particular example of $\Gamma(z)$, obtaining an explicit formula for the coefficients of the Stirling expansion.

3. Examples

3.1. The gamma function $\Gamma(x)$ for large x

From [[8], p. 60, eq. (1.27)] we have

$$\Gamma(x+1) = e^{-x}x^{x+1}\int_{-1}^{\infty} e^{-xf(t)}\,dt, \qquad \text{with} \qquad f(t) = t - \log(1+t).$$

This integral has the form (1) with the above mentioned $f(t)$ and $g(t) = 1$. We consider $x > 0$ large. The functions $f(t)$ and $g(t)$ are differentiable on $(-1, \infty)$. The unique critical point t_0 of $f(t)$ is $t_0 = 0$. We have $f(0) = 0$, $f''(0) = 1$ and $f'''(0) \neq 0$. With the notation of the preceding section we have $m = 2$ and $p = 3$.

The function $e^{-xf_3(t)}$ has a Taylor expansion at $t = 0$ (with $s = 0$):

$$e^{-xf_3(t)} = e^{xt^2/2}e^{-xt}(1+t)^x = \sum_{n=0}^{\infty} a_n(x)t^n.$$

From equation (27) or directly from the above equality:

$$\begin{aligned} a_n(x) =& (-1)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{n-2k} \frac{x^{k+j}(-x)_{n-2k-j}}{2^k k! j! (n-2k-j)!} \\ =& (-1)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{x^k(-x)_{n-2k}}{2^k k!(n-2k)!}\,{}_1F_1(2k-n; 2k-n+1+x; x). \end{aligned} \tag{28}$$

We know that $a_n(x)$ is a polynomial of degree $\lfloor n/3 \rfloor$ in x. To see this explicitly we write

$$(-x)_n = \sum_{m=0}^{n} (-1)^{n+m} S_n^{(m)} (-x)^m,$$

where $S_n^{(m)}$ are the Stirling numbers of the first kind [[1], eq. 24.1.3 (B)], and replace this in (28):

$$a_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{n-2k} \frac{x^{k+j}}{2^k k! j! (n-2k-j)!} \sum_{m=0}^{n-2k-j} (-1)^m S_{n-2k-j}^{(n-2k-j-m)} (-x)^{n-2k-j-m}.$$

If we interchange the order of the sums in j and m and then we make a change of summation variables $(j,m)\to(j,k+m)$ we get

$$a_n(x)=\sum_{m=0}^{\lfloor n/3\rfloor} A_m^n x^m,$$

with

$$A_m^n\equiv\sum_{k=0}^{n-m}\sum_{j=0}^{m-k}\frac{(-1)^j S_{n-2k-j}^{(m-k-j)}}{2^k k!j!(n-2k-j)!}.$$

From (17), the asymptotic sequence $\Phi_n(x)$ is

$$\Phi_n(x)=\frac{2^{(n-1)/2}}{x^{(n+1)/2}}\left[(1+(-1)^n)\Gamma\left(\frac{n+1}{2}\right)-(-1)^n\Gamma\left(\frac{n+1}{2},\frac{x}{2}\right)\right].$$

Therefore, up to exponentially small terms, $\Phi_{2n+1}(x)=0$ and

$$\Phi_{2n}(x)=\left(\frac{2}{x}\right)^{n+1/2}\Gamma\left(n+\frac{1}{2}\right).$$

From (25) and (26) we have that an asymptotic expansion of $\Gamma(x)$ for large x is

$$\Gamma(x+1)\sim e^{-x}x^x\sqrt{2\pi x}\sum_{n=0}^{\infty}\sum_{k=2n-2}^{2n} a_{2n+2k}(x)\frac{\Gamma(n+k+1/2)}{\Gamma(1/2)}\left(\frac{2}{x}\right)^{n+k}. \tag{29}$$

The terms $a_n(x)$ are polynomials of degree $\lfloor n/3\rfloor$ in x and therefore, this expansion is a rearrangement of the Stirling formula [[1], eq. 6.1.37]:

$$\Gamma(x+1)\sim e^{-x}x^x\sqrt{2\pi x}\left[1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}+...\right]$$

Collecting equal powers of x from (29) we obtain an explicit and new formula for the coefficients of the Stirling formula:

$$\Gamma(x+1)\sim e^{-x}x^x\sqrt{2\pi x}\sum_{n=0}^{\infty}\frac{c_n}{x^n},$$

with

$$c_n\equiv\sum_{m=0}^{2n}\frac{2^{n+m}\Gamma(n+m+1/2)}{\Gamma(1/2)}\sum_{k=0}^{m}\frac{1}{k!2^k}\sum_{j=0}^{m-k}\frac{(-1)^j S_{2n+2m-2k-j}^{(m-k-j)}}{j!(2n+2m-2k-j)!}.$$

3.2. The Gauss hypergeometric function ${}_2F_1(a,b,c;z)$ for large b and c

The Gauss hypergeometric function may be written in the form [[6], p. 110, eq. (5.4)]

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt, \qquad \Re c>\Re b>0.$$

Define $x=b-1$ and $\sigma=(c-b-1)/(b-1)$. Then, this integral can be put in the form (1):

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 e^{-xf(t)}g(t)\,dt,$$

with

$$f(t) = -\log t - \sigma \log(1-t) \qquad \text{and} \qquad g(t) = (1-zt)^{-a}.$$

We consider $\sigma > 0$ fixed and $x > 0$ large, which means that both, b and c are large and of the same order, with $c > b$. The functions $f(t)$ and $g(t)$ are differentiable on $(0,1)$ for $z \in \mathbb{C} \setminus [1,\infty)$. The unique critical point t_0 of $f(t)$ is

$$t_0 = \frac{1}{\sigma+1} \in (0,1).$$

Both functions $e^{-xf(t)}$ and $g(t)$ have a Taylor expansion at $t = t_0$ (with $s = 0$):

$$\begin{aligned} e^{-xf(t)} =& e^{-xf(t_0)} \left(1 + \frac{t-t_0}{t_0}\right)^x \left(1 + \frac{t-t_0}{t_0-1}\right)^{\sigma x} \\ &= e^{-xf(t_0)} \sum_{n=0}^{\infty} \left[(\sigma+1)^n \sum_{j=0}^{n} \frac{(-1)^j (-x)_j (-\sigma x)_{n-j}}{j!(n-j)!\sigma^{n-j}} \right] (t-t_0)^n, \end{aligned}$$

$$g(t) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n \left[1 - \frac{z}{\sigma+1}\right]^{-a-n} \left(t - \frac{1}{\sigma+1}\right)^n, \qquad \left|t - \frac{1}{\sigma+1}\right| < r,$$

where the common radius of convergence r depends on σ and z. We have $f(t_0) = -\sigma \log \sigma + (\sigma+1)\log(\sigma+1)$, $f''(t_0) = (\sigma+1)^3/\sigma$ and $f'''(t_0) \neq 0$. With the notation of the preceding section we have $m = 2$ and $p = 3$. Equation (27) reads

$$a_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{n-2k} \sum_{l=0}^{j} \frac{x^k (-x)_l (-\sigma x)_{j-l} (\sigma+1)^{k+a+n} (\sigma+1-z)^{j+2k-a-n} (a)_{n-2k-j}}{(-1)^l l! k! (j-l)! (n-2k-j)! 2^k \sigma^{k+j-l} z^{2k+j-n}}. \tag{30}$$

The asymptotic sequence $\Phi_n(x)$ is

$$\begin{aligned} \Phi_n(x) =& \frac{1}{2} \left(\frac{2\sigma}{(\sigma+1)^3 x}\right)^{\frac{n+1}{2}} \left\{ \gamma\left(\frac{n+1}{2}, \frac{(\sigma+1)^3 \sigma}{2} x\right) + (-1)^n \gamma\left(\frac{n+1}{2}, \frac{(\sigma+1)}{2\sigma} x\right) \right\} \\ &\sim \frac{1+(-1)^n}{2} \left(\frac{2\sigma}{(\sigma+1)^3 x}\right)^{(n+1)/2} \Gamma\left(\frac{n+1}{2}\right). \end{aligned}$$

Therefore, from (25) and (26) we have that an asymptotic expansion of ${}_2F_1(a,b;c;z)$ for large b and c is

$$\begin{aligned} {}_2F_1(a,b;c;z) \sim & \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{\sigma^{\sigma x}}{(\sigma+1)^{(\sigma+1)x}} \sum_{n=0}^{\infty} \left[\sum_{k=\mathrm{Max}\{2n-2,0\}}^{2n} a_{2n+2k}(x) \right. \\ & \left. \times \left(\frac{2\sigma}{(\sigma+1)^3 x}\right)^{n+k+1/2} \Gamma\left(n+k+\frac{1}{2}\right) \right], \end{aligned} \tag{31}$$

with $x = b-1$, $\sigma = (c-b-1)/(b-1)$ and $a_k(x)$ given in (30). The quantity between brackets is $\Psi_n(x)$. The first few terms of the expansion are

$$\begin{aligned} {}_2F_1(a,b;c;z) \sim & \frac{\sqrt{2\pi}\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{(c-b-1)^{c-b-1/2}}{(c-2)^{c-1/2}} (b-1)^{b-1/2} \left(\frac{c-2}{z-bz+c-2}\right)^a \\ & \times \left\{ 1 + \left[\frac{(a)_2 z^2 b(c-b)}{2(c-bz)^2} + \left(\frac{3(3b^2+c^2-3bc)}{4b(b-c)} + \frac{az(2b-c)}{(zb-c)} \right) \right. \right. \\ & \left. \left. + \frac{5(4b^4c - 8b^3c^2 + 5b^2c^3 - bc^4)}{6b^2c(b-c)^2} \right] \frac{1}{c} + \mathcal{O}\left(\frac{1}{c^2}\right) \right\}. \end{aligned}$$

b, c	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
10, 20	-0.03544405	0.01050693	-0.00352388	0.00229437	-0.00309512
20, 40	-0.01680944	0.00231339	-0.00018853	0.00000325	-0.00001323
40, 100	-0.00526004	0.00037352	0.00004229	-5.0953 E(-6)	-5.8478 E(-7)
100, 200	-0.00322682	0.00008399	-0.00000027	-1.3501 E(-8)	5.6717 E(-10)
900, 1500	-0.00050506	2.6907 E(-6)	1.7740 E(-8)	6.6844 E(-11)	1.1346 E(-14)

The table below shows a numerical experiment about the accuracy of the above approximation.

Table 2. *All the columns represent the relative error in the approximation of* ${}_2F_1(1, b, c; 1/2)$ *taking the first* n *terms of the expansion (31) for the given values of* b *and* c.

4. Concluding remarks

The standard Laplace's method of asymptotic expansions of integrals requires a change of variables which, up to now, has made the analytic computation of the coefficients of the expansion very complicated. Usually, an analytic general formula was only possible for a first few coefficients. Recently, a general formula has been given in [[7], eqs. (3.6) or (3.12)] which gives an analytic expression for all of the coefficients c_n of the expansion. It is given in terms of combinatorial functions that can be obtained from a recurrence [[7], eq. (3.13)].

We have proposed in this paper a modification of the Laplace's method which gives a new asymptotic expansion for Laplace integrals (1) and makes the computation of the complete expansion straightforward. With this new procedure, the phase function $f(t)$ is always a power of $t - t_0$, where t_0 is the minimum of the phase function. On the other hand, the remaining integrand $h(t, x)$ is a function not only of the integration variable t, but also of the asymptotic variable x. As a consequence, not only the asymptotic sequence $\Phi_n(x)$, but also the coefficients of the expansion $a_n(x)$ depend on x. The expansion obtained in this way, $\sum_n a_n(x)\Phi_n(x)$, is not a genuine asymptotic expansion. Nevertheless, the terms of the expansion $a_n(x)\Phi_n(x)$ can be grouped in new terms $\Psi_n(x)$ and the new expansion $\sum_n \Psi_n(x)$ is a generalized asymptotic expansion: $\Psi_n(x) = \mathcal{O}(x^{-(n+s+1)/m})$ as $x \to \infty$.

The asymptotic sequence $\Phi_n(x)$ given in (18) is as simple as in the standard Laplace's method: a constant times an inverse power of x. The coefficients $a_n(x)$ are given explicitly in (27) as a double sum of the coefficients of the expansions of $\exp(-xf(t))$ and $g(t)$.

The expansions (16) or (20) are rearrangements of the standard Laplace's expansion and "vice versa". This means that the coefficients of the standard Laplace's expansion can be explicitly obtained from the coefficients of (16) after an appropriate rearrangement. This means that formula (27) and formulas [[7], eqs. (3.6) or (3.12)] should be connected by that rearrangement.

We have applied the method to the two important examples of the gamma function and the Gauss hypergeometric function. We have obtained new and complete asymptotic expansions of these functions when some of their variables are large. As a consequence, we have derived a new and explicit formula for the coefficients of the classical Stirling expansion of the gamma function.

The standard saddle point method for asymptotic expansions of integrals has, not only the complication of the change of variable (as well as Laplace's method), but also the many times complicated task of finding the steepest descent paths. We believe that the idea presented here for the Laplace's technique can be translated to the saddle point method. This is subject of future investigation.

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