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ASYMPTOTIC EXPANSIONS OF SYMMETRIC STANDARD ELLIPTIC INTEGRALS *

JOSÉ L. LÓPEZ†

Abstract. Symmetric standard elliptic integrals are considered when one of their parameters is larger than the others. Distributional approach is used for deriving five convergent expansions of these integrals in inverse powers of the respective five possible asymptotic parameters. Four of these expansions involve also a logarithmic term in the asymptotic variable. Coefficients of these expansions are obtained by recurrence. For the first four expansions these coefficients are expressed in terms of elementary functions, whereas coefficients of the fifth expansion involve non-elementary functions. Convergence speed of any of these expansions increases for increasing difference between the asymptotic variable and the remaining ones. All the expansions are accompanied by an error bound at any order of the approximation.

Key words. Elliptic integrals, asymptotic expansions, distributional approach

AMS subject classifications. 41A60, 33E05

1. Introduction. Elliptic integrals (EI) are integrals of the type $\int R(x,y)dx$, where $R(x,y)$ is a rational function of x and y, with y^2 a polynomial of the third or fourth degree in x. When the polynomial y^2 has not a repeated factor and $R(x,y)$ contains some odd power of y, EI cannot, in general, be expressed in terms of elementary functions. Legendre showed that all EI can be expressed in terms of three standard EI (Legendre's normal EI) [14].

The three complete EI of the first, second and third kind are particularly important cases of the respective three standard EI. These integrals and the three standard EI are special non-elementary functions that play an important role in several mathematical problems: the first complete elliptic integral appears as a certain limit in the theory of iterated number sequences based on the arithmetic geometric mean [18, sec. 12.1.2]. Standard EI are related to Theta functions and the Weierstrass' elliptic function [18, sec. 12.3]. EI constitute a basic ingredient of certain geometrical [11] and statistical [17] problems.

EI are also involved in several physical problems: the period of a simple pendulum in a constant gravitational field can be expressed in terms of the first complete elliptic integral [18, sec. 12.1.1]. The zeros of EI can be used for determining an upper bound for the number of limit cycles of certain hamiltonian systems [19]. EI are related to certain problems of electromagnetism [20].

A survey of properties of the standard EI can be found, for example, in [1, chap. 17], [2] or [18, chap. 12]. However, as it has been shown by Carlson [5-9], for numerical computations it is more convenient to use symmetric standard EI instead of Legendre's normal EI. (Legendre's normal EI are connected with the symmetric standard EI by means of simple formulas [18, eq. 12.33].) A very complete table of the three

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†Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Zaragoza, 50013-Zaragoza, Spain. (jllopez@posta.unizar.es).

symmetric standard EI can be found in [5-9]. They are defined as follows,

$$R_F(x,y,z)=\frac{1}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}},$$

$$R_D(x,y,z)=\frac{3}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)^3}},$$

$$R_J(x,y,z,p)=\frac{3}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}(t+p)},$$

where we assume that the parameters x, y, z are nonnegative. We assume also that they are distinct (otherwise these integrals reduce to elementary functions). If the fourth argument of R_J is negative, the Cauchy principal value of R_J can be written in terms of R_F and R_J with all the arguments nonnegative [10]. Therefore, we will consider $p>0$ and $p\neq x$, y, z (otherwise R_J reduces to R_D).

On the other hand, the asymptotic approximation of EI has not been exhaustively investigated: classical methods for approximation of integrals cannot be applied. Some result concerning approximations of EI can be found for example in [2] and [13]. Although the more recent results about the asymptotic behaviour of these integrals have been obtained by Carlson, Gustafson and Wong: R_F, R_D and R_J may be written as a convolution and the method of regularization [22, chap. 6, sec. 7] can be applied.

When one of the parameters of the integrals tends to zero or infinity, the first (and sometimes the second too) term of the asymptotic expansion of R_F, R_D and R_J, as well as a quite accurate bound for the first error term has been obtained by Gustafson [12]. Higher terms of the expansion and higher error bounds are not explicitly derived in that work because of the complexity of the Mellin transforms involved in their calculation. Using a very clever analytical trick [10], Carlson and Gustafson have sharpened the bounds for the first error terms obtained in [12] in the case of one parameter going to infinity. Besides, they supply in [10] very accurate bounds for the first error term of the totally symmetric elliptic integral of the second kind. Moreover, for all the symmetric EI, they consider also the case of several parameters going to infinity.

Complete convergent expansions of R_F, R_D and R_J (and not only first terms) have been obtained by Carlson using also Mellin transforms techniques [3]. Although these expansions have an attractively simple structure, explicit computation of the terms of the expansions is not straightforward and the upper bound on the truncation error is not quite satisfactory [3, sec. 5]. Carlson and Gustafson have solved this problem for $R_F(x,y,z)$ in [4], where an algorithm for computing the coefficients of the convergent expansion of $R_F(x,y,z)$ in terms of Legendre functions and their derivatives is derived. Moreover, accurate error bounds are given too at any order of the approximation.

In this paper we try to solve for R_D and R_J the problem that Carlson and Gustafson have solved for $R_F(x,y,z)$. That is, we consider complete convergent expansions for R_D and R_J when one of their parameters x, y, z or p is large. Then, we face the challenge of obtaining easy algorithms for computing the coefficients of these expansions (in terms of elementary functions when it is possible) and simple expressions for the error bounds at any order of the approximation. For completeness, we include also R_F in this project.

For this purpose, in section 2, we make a review of the asymptotic expansions of Stieltjes [22, chap. 6, sec. 2] and generalized Stieltjes transforms (see [21, theorem 2 and example 1]): distributional approach is used in lemmas 1 and 2 and theorems 1 and

2 for deriving complete expansions of a certain family of integrals which contains R_F, R_D and R_J. On the other hand, using lemmas 3 and 4, we obtain simple expressions for the error bounds in the expansions of this family of integrals in propositions 1 and 2. In section 3 we apply the results of section 2 for deriving complete convergent expansions of $R_F(x,y,z)$, $R_D(x,y,z)$, $R_D(x,z,y)$, $R_J(x,y,z,p)$ and $R_J(x,y,p,z)$ for large z. They are presented in corollaries 1-5 accompanied by error bounds at any order of the approximation. Numerical examples are shown as an illustration. A brief summary and a few comments are postponed to section 4.

2. Distributional approach.

The procedure for deriving convergent expansions of the integrals R_F, R_D and R_J are based on the distributional approach. It requires the concepts of rapidly decreasing functions and tempered distributions.

Definition 1. *We denote by $\mathcal{S}$ the space of rapidly decreasing functions (infinitely differentiable functions $\varphi(t)$ defined on $[0,\infty)$ that, together with their derivatives, approach zero more rapidly than any power of t^{-1} as $t\to\infty$)*

Definition 2. *We denote by $<\Lambda,\varphi>$ the image of a tempered distribution Λ (a continuous linear functional defined over $\mathcal{S}$) acting over a function $\varphi\in\mathcal{S}$. Recall that we can associate to any locally integrable function $g(t)$ on $[0,\infty)$ a tempered distribution Λ_g defined by*

$$<\Lambda_g,\varphi>=\int_0^\infty g(t)\varphi(t)dt.$$

Definition 3. *For a locally integrable function $f(t)$ on $(0,\infty)$, we denote by $M[f;w]$ the Mellin transform of $f(t)$ or its analytic continuation. It is defined by*

$$M[f;w]=\int_0^\infty t^{w-1}f(t)dt \tag{1}$$

when the integral converges.

Derivation of convergent expansions of $R_J(x,y,z,p)$ for large p is based on the following theorem proved in [22, chap. 6, theorem 1].

Theorem 1. *Let $f(t)$ a locally integrable function on $[0,\infty)$, $\{A_k\}$ a sequence of complex numbers and let $f(t)$ satisfy, for $n=1,2,3,...$,*

$$f(t)=\sum_{k=0}^{n-1}\frac{A_k}{t^{k+\alpha}}+f_n(t),$$

where $f_n(t)=\mathcal{O}(t^{-n-\alpha})$ as $t\to\infty$ and $0<\alpha<1$. Then, for $p>0$ and $n=1,2,3,...$,

$$\int_0^\infty\frac{f(t)}{t+p}dt=\frac{\pi}{\sin(\alpha\pi)}\sum_{k=0}^{n-1}(-1)^k\frac{A_k}{p^{k+\alpha}}+\sum_{k=0}^{n-1}(-1)^k\frac{M[f;k+1]}{p^{k+1}}+R_n(p). \tag{2}$$

The remainder term satisfies

$$R_n(p)=n!\int_0^\infty\frac{f_{n,n}(t)dt}{(t+p)^{n+1}}, \tag{3}$$

where $f_{n,n}(t)$ is defined by

$$f_{n,n}(t)=\frac{(-1)^n}{(n-1)!}\int_t^\infty(u-t)^{n-1}f_n(u)du. \tag{4}$$

Convergent expansions of $R_F(x,y,z)$, $R_D(x,y,z)$, $R_D(x,z,y)$ and $R_J(x,y,z,p)$ for large z can be derived from [21, theorem 2] (see also example 1 there). This result has been proved by using Mellin transform techniques and, as it is suggested by Wong [21, example 1], it can also be proved by using the distributional approach. We carry out Wong's proposal in the following two lemmas and theorem 2. The first lemma is proved in [22, chap. 6, lemma 2].

LEMMA 1. *Let $f(t)$ as in theorem 1 but with $\alpha = 1$. Then, for any integer $n \geq 1$ and for any function $\varphi \in \mathcal{S}$ we have*

$$< f, \varphi > = -\sum_{k=0}^{n-1} \frac{A_k}{k!} < \log(t), \varphi^{(k+1)} > + \sum_{k=0}^{n-1} \frac{B_k}{k!} < \delta, \varphi^{(k)} > + (-1)^n < f_{n,n}, \varphi^{(n)} >,$$

where f, $f_{n,n}$ and $\log(t)$ denote the tempered distributions associated to the locally integrable functions $f(t)$, $f_{n,n}(t)$ and $\log(t)$ respectively, δ is the delta distribution in the origin and

$$\begin{aligned} B_k =& A_k \sum_{j=1}^{k} \frac{1}{j} + \lim_{w \to k+1} \left\{ M[f;w] + \frac{A_k}{w-k-1} \right\} \\ =& A_k \sum_{j=1}^{k} \frac{1}{j} + \int_0^1 t^k f_k(t) dt + \int_1^\infty t^k f_{k+1}(t) dt, \end{aligned} \tag{5}$$

empty sums being understood as zero.

LEMMA 2. *Let $f(t)$ as in theorem 1 with $0 < \alpha \leq 1$. Define, for $t \in [0, \infty)$, $z > 0$, $\eta > 0$ and $\alpha + \rho > 1$*

$$\varphi_\eta(t) = \frac{e^{-\eta t}}{(t+z)^\rho} \in \mathcal{S}.$$

Then, for $k = 0, 1, 2, ...$ and $n = 1, 2, 3, ...$, the following identities hold,

$$\lim_{\eta \to 0} < f, \varphi_\eta > = \int_0^\infty \frac{f(t)}{(t+z)^\rho} dt,$$

$$\lim_{\eta \to 0} < \delta, \varphi_\eta^{(k)} > = \frac{(-1)^k (\rho)_k}{z^{k+\rho}},$$

where $(\rho)_k$ denotes the Pochhammer's symbol,

$$\lim_{\eta \to 0} < \log(t), \varphi_\eta^{(k+1)} > = \frac{(-1)^{k+1}}{z^{k+\rho}} (\rho)_k \big(\log(z) - \gamma - \psi(k+\rho) \big),$$

where γ is the Euler constant and ψ the digamma function and

$$\lim_{\eta \to 0} < f_{n,n}, \varphi_\eta^{(n)} > = (-1)^n (\rho)_n \int_0^\infty \frac{f_{n,n}(t)}{(t+z)^{n+\rho}} dt.$$

Proof. The first identity is trivial by using the dominated convergence theorem. The second one follows after a simply computation. On the other hand,

$$< \log(t), \varphi_\eta^{(k+1)} > = (-1)^{k+1} \sum_{j=0}^{k+1} \binom{k+1}{j} \eta^j (\rho)_{k+1-j} \int_0^\infty \frac{e^{-\eta t} \log(t)}{(t+z)^{k+\rho+1-j}} dt.$$

For $j \leq k$ or $j = k+1$ and $\rho > 1$, the integrand of each integral in the right hand side of the above equation is absolutely dominated by the integrable function $\log(t)(t+z)^{j-k-\rho-1}$ $\forall\, \eta, t \geq 0$ and therefore, finite. For $j = k+1$ and $\rho \leq 1$, we divide the interval $[0,\infty)$ in the above integral at the point $t = 1$. On the interval $[0,1]$ the integral is finite for $\eta \geq 0$. In the interval $[1,\infty)$ we use the bound $\log(t) \leq \log(t+z)$, perform the change of variable $\eta t = u$ and divide again the resulting $u-$interval $[\eta,\infty)$ at the point $u = 1-\eta z$ (assume $\eta \leq (1+z)^{-1}$). In the $u-$interval $[\eta, 1-\eta z]$ we use the bound $(u+\eta z)^\rho \geq u+\eta z$. After straightforward operations we obtain that the integral on the $t-$interval $[1,\infty)$ is $\mathcal{O}\left(\eta^{\rho-1}\log^2(\eta)\right)$ as $\eta \to 0$. Therefore,

$$\lim_{\eta\to 0} < \log(t), \varphi_\eta^{(k+1)} >= (-1)^{k+1}(\rho)_{k+1}\int_0^\infty \frac{\log(t)}{(t+z)^{k+\rho+1}}dt.$$

Using now formula [16, p. 489, eq. 7] we obtain the third identity. The fourth identity follows from the dominated convergence theorem, the local integrability of $f_{n,n}(t)$ on $[0,\infty)$ and the behaviour $f_{n,n}(t) = \mathcal{O}(t^{-\alpha})$ as $t \to \infty$ [22, p. 296]. ∎

Theorem 2. *Let $f(t)$ a locally integrable function on $[0,\infty)$, $\{A_k\}$ a sequence of complex numbers and let $f(t)$ have the following asymptotic expansion for large t and $n = 1,2,3,...$,*

$$f(t) = \sum_{k=0}^{n-1}\frac{A_k}{t^{k+1}} + f_n(t), \tag{6}$$

where $f_n(t) = \mathcal{O}(t^{-n-1})$ as $t \to \infty$. Then, for $z, \rho > 0$ and $n = 1,2,3,...$,

$$\int_0^\infty \frac{f(t)}{(t+z)^\rho}dt = \sum_{k=0}^{n-1}\frac{(-1)^k}{k!z^{k+\rho}}(\rho)_k\left[A_k\big(\log(z)-\gamma-\psi(k+\rho)\big)+B_k\right]+R_n(z), \tag{7}$$

where, for $k = 0,1,2,...$, the coefficients B_k are given by

$$\begin{aligned} B_k =& A_k\sum_{j=1}^{k}\frac{1}{j} + \lim_{w\to k+1}\left\{M[f;w] + \frac{A_k}{w-k-1}\right\} \\ =& A_k\sum_{j=1}^{k}\frac{1}{j} + \lim_{T\to\infty}\left\{\int_0^T t^k f(t)dt - \sum_{j=0}^{k-1}A_j\frac{T^{k-j}}{k-j} - A_k\log(T)\right\}, \end{aligned} \tag{8}$$

empty sums being understood as zero. The remainder term is given by

$$R_n(z) = (\rho)_n\int_0^\infty \frac{f_{n,n}(t)}{(t+z)^{n+\rho}}dt, \tag{9}$$

where $f_{n,n}(t)$ is defined in (4).

Proof. From lemmas 1 and 2 we obtain immediately formulas (7), (9) and the first line in (8). Introducing

$$f_k(t) = f(t) - \sum_{j=0}^{k-1}\frac{A_j}{t^{j+1}}$$

in the second line of (5) and after simple manipulations we obtain the second line in (8). ☐

A bound for the error term in the expansions given in theorems 1 and 2 will be obtained in propositions 1 and 2 respectively when the function $f(t)$ has the form

$$f(t)=\prod_{k=1}^{m}\frac{1}{(t+x_k)^{\mu_k}}, \tag{10}$$

where $m\in\mathbb{N}$, $x_1,...,x_m$ are nonnegative parameters at least one different from zero and $\mu_1,...,\mu_m>0$. Define

$$\mu=\sum_{k=1}^{m}\mu_k>0.$$

For $\mu\notin\mathbb{N}$, the asymptotic expansion of $f(t)$ in $t=\infty$ is given, for $n=1,2,3,...$, by

$$f(t)=\sum_{k=0}^{n-1}\frac{A_k}{t^{k+\mu-\lfloor\mu\rfloor}}+f_n(t), \tag{11}$$

where

$$A_0=A_1=...=A_{\lfloor\mu\rfloor-1}=0 \qquad \text{if } \lfloor\mu\rfloor\geq 1,$$

$$A_{k+\lfloor\mu\rfloor}=\lim_{u\to 0}\frac{1}{k!}\frac{d^k}{du^k}\left(u^{-\mu}f(u^{-1})\right) \qquad \text{for } k=0,1,2,... \tag{12}$$

and $f_n(t)=\mathcal{O}(t^{-n-\mu+\lfloor\mu\rfloor})$ as $t\to\infty$. Then , we have the following

LEMMA 3. *For $\mu\notin\mathbb{N}$ and $\forall\ t\in[0,\infty)$, the remainder term $f_n(t)$ and the coefficients A_n in the expansion (11)-(12) of the function $f(t)$ defined in (10) verify*

$$|f_n(t)|\leq\frac{|A_n|}{t^{n+\mu-\lfloor\mu\rfloor}} \quad \text{for } n\geq\lfloor\mu\rfloor, \qquad |f_n(t)|\leq\frac{|A_{n-1}|}{t^{n+\mu-\lfloor\mu\rfloor-1}} \quad \text{for } n\geq\lfloor\mu\rfloor+1 \tag{13}$$

and sign($f_n(t)$) =sign(A_n) =sign($(-1)^{n-\lfloor\mu\rfloor}$) for $n\geq\lfloor\mu\rfloor$.

Proof. The Taylor expansion of $u^{-\mu}f(u^{-1})$ at $u=0$ is given by

$$u^{-\mu}f(u^{-1})\equiv\prod_{k=1}^{m}(1+x_ku)^{-\mu_k}=\sum_{k=0}^{n-\lfloor\mu\rfloor-1}A_{k+\lfloor\mu\rfloor}u^k+u^{-\mu}f_n(u^{-1}).$$

Applying the binomial formula for the derivative of a product we realize that the $n-$esim $u-$derivative of $u^{-\mu}f(u^{-1})$ has the same sign as $(-1)^n$ $\forall u\in[0,\infty)$. Then, $\text{sign}(A_n)=\text{sign}(-1)^{n-\lfloor\mu\rfloor}$ for $n\geq\lfloor\mu\rfloor$ and, by the Lagrange formula for the remainder $u^{-\mu}f_n(u^{-1})$ we obtain that $\text{sign}(f_n(t))=\text{sign}(-1)^{n-\lfloor\mu\rfloor}$ for $n\geq\lfloor\mu\rfloor$ and $\forall t\in[0,\infty)$. Therefore, two consecutive error terms $f_n(t)$ and $f_{n+1}(t)$ in the expansion of $f(t)$ have opposite sign. After applying the error test (see for example [15, p. 68] or [22, p. 38]) we obtain the first inequality in (13). The second inequality follows from the first one and

$$f_n(t)=f_{n-1}(t)-\frac{A_{n-1}}{t^{n+\mu-\lfloor\mu\rfloor-1}}.$$

☐

On the other hand, for $\mu \in \mathbb{N}$, the asymptotic expansion in $t = \infty$ of the function $f(t)$ defined in (10) is given, for $n = 1, 2, 3, ...$, by

$$f(t) = \sum_{k=0}^{n-1} \frac{A_k}{t^{k+1}} + f_n(t), \tag{14}$$

where

$$A_0 = A_1 = ... = A_{\mu-2} = 0 \qquad \text{if} \;\; \mu \geq 2,$$

$$A_{k+\mu-1} = \lim_{u\to 0} \frac{1}{k!} \frac{d^k}{du^k} \left(u^{-\mu} f(u^{-1})\right) \qquad \text{for} \;\; k = 0, 1, 2, ... \tag{15}$$

and $f_n(t) = \mathcal{O}(t^{-n-1})$ as $t \to \infty$. Then, we have the following

LEMMA 4. *For $\mu \in \mathbb{N}$ and $\forall\, t \in [0, \infty)$, the remainder term $f_n(t)$ and the coefficients A_n in the expansion (14)-(15) of the function $f(t)$ defined in (10) verify*

$$|f_n(t)| \leq \frac{|A_n|}{t^{n+1}} \qquad \text{for} \;\; n \geq \mu - 1, \qquad |f_n(t)| \leq \frac{|A_{n-1}|}{t^n} \qquad \text{for} \;\; n \geq \mu \tag{16}$$

and sign$(f_n(t))$ =sign(A_n) =sign$((-1)^{n-\mu+1})$ for $n \geq \mu - 1$.

Proof. Similar to the proof of lemma 3 replacing $\lfloor\mu\rfloor$ by $\mu - 1$. ∎

PROPOSITION 1. *If the function $f(t)$ of theorem 1 has the form (10) with $\mu \notin \mathbb{N}$ then, $\forall\, p > 0$ and $n \geq \lfloor\mu\rfloor$, the error term $R_n(p)$ in the expansion (2) satisfies*

$$0 \leq (-1)^{\lfloor\mu\rfloor} R_n(p) \leq \frac{\pi |A_n|}{|\sin(\pi\mu)| p^{n+\mu-\lfloor\mu\rfloor}}, \tag{17}$$

providing the expansion (2) of an asymptotic character for large p.

Proof. The parameter α in theorem 1 equals $\mu - \lfloor\mu\rfloor$ in lemma 3. Using now sign$(f_n(u))$ =sign$((-1)^{n-\lfloor\mu\rfloor})$ $\forall u \in [0, \infty)$ in (4) and (3) we obtain $(-1)^{\lfloor\mu\rfloor} R_n(p) \geq 0$. Introducing the first bound of (13) in the right hand side of (4) and performing the change of variable $u = tv$ we obtain

$$|f_{n,n}(t)| \leq \frac{\Gamma(\mu - \lfloor\mu\rfloor)}{\Gamma(n + \mu - \lfloor\mu\rfloor)} \frac{|A_n|}{t^{\mu-\lfloor\mu\rfloor}} \qquad \forall \quad t \in [0, \infty).$$

Introducing this bound in (3) and after the change of variable $t = pu$ we obtain (17). ∎

PROPOSITION 2. *If the function $f(t)$ of theorem 2 has the form (10) with $\mu \in \mathbb{N}$ then, $\forall\, z > 0$ and $n \geq \mu$, the error term $R_n(z)$ in the expansion (7) satisfies the bounds*

$$0 \leq -(-1)^{\mu} R_n(z) \leq \frac{\pi\Gamma(n + \rho - 1/2)}{\Gamma(\rho)\Gamma(n + 1/2)} \frac{\bar{A}_n}{z^{n+\rho-1/2}}, \tag{18}$$

where $\bar{A}_n$ =max$\{|A_n|, |A_{n-1}|\}$, and

$$|R_n(z)| \leq \left[na|A_{n-1}| + |A_n|\big(S_n(z, a, \rho) + T_n(z, a, \rho)\big)\right] \frac{(\rho)_n}{n! z^{n+\rho}}, \tag{19}$$

where a is an arbitrary positive number,

$$S_n(z,a,\rho)=\min\left\{\frac{nz\left[(a+z)^{n+\rho-1}-z^{n+\rho-1}\right]}{a(n+\rho-1)(a+z)^{n+\rho-1}},\psi(n+1)+\gamma\right\} \tag{20}$$

and

$$\begin{aligned} T_n(z,a,\rho)&=\frac{z^{n+\rho}}{(n+\rho)(a+z)^{n+\rho}}F\left(n+\rho,1;n+\rho+1;\frac{z}{a+z}\right)\\ &\leq\left(\frac{z}{a+z}\right)^{\rho}\left(\log\left(1+\frac{z}{a}\right)-\sum_{k=1}^{n-1}\frac{z^k}{k(z+a)^k}\right), \end{aligned} \tag{21}$$

where $F(b,c;d;z)$ is the hypergeometric function. For large z and fixed n, the optimum value for a is given by

$$a=\frac{|A_n|}{n|A_{n-1}|}. \tag{22}$$

Any of these bounds provide the expansion (7) of an asymptotic character for large z.

Proof. From lemma 4, sign$(f_n(u))$ =sign$((-1)^{n-\mu+1})$ $\forall u\in[0,\infty)$. Introducing this in (4) and (9) we obtain $(-1)^{\mu}R_n(z)\leq 0$. For obtaining the bound (19) we divide the integral in the right hand side of (4) by a fixed point $u=a\geq t$ and use the second bound of (16) in the integral over $[t,a]$ and the first bound of (13) in the integral over $[a,\infty)$. Using $u-t\leq u$ in the integral over $[t,a]$ we obtain

$$\begin{aligned} |f_{n,n}(t)|&\leq\frac{1}{(n-1)!}\left[|A_{n-1}|\log\left(\frac{a}{t}\right)+\frac{|A_n|}{nt}\left(1-\left(1-\frac{t}{a}\right)^n\right)\right]\\ &\leq\frac{1}{(n-1)!}\left[|A_{n-1}|\log\left(\frac{a}{t}\right)+\frac{|A_n|}{a}\right]\qquad\forall\ t\in[0,a],\qquad a>0. \end{aligned} \tag{23}$$

On the other hand, $\forall\, t\in[0,\infty)$ we introduce the first bound of (13) in the right hand side of (4) and perform the change of variable $u=tv$. We obtain

$$|f_{n,n}(t)|\leq\frac{|A_n|}{n!}\frac{1}{t}\qquad\forall\quad t\in[0,\infty). \tag{24}$$

We divide the integral in the right hand side of (9) at the point $t=a$ and use the bound (24) in the integral over $[a,\infty)$. Now, if we use the second bound of (23) in the integral over $[0,a]$ we obtain

$$\begin{aligned} |R_n(z)|\leq\frac{(\rho)_n}{n!}&\left[\frac{|A_n|S_n(z,a,\rho)}{z^{n+\rho}}+n|A_{n-1}|\int_0^a\frac{\log(a/t)}{(t+z)^{n+\rho}}dt+\right.\\ &\left.|A_n|\int_a^{\infty}\frac{dt}{t(t+z)^{n+\rho}}\right], \end{aligned} \tag{25}$$

where $S_n(z,a,\rho)$ is given by the first quantity between the brackets in (20). If instead of this, we use the first bound of (23) in the integral over $[0,a]$, expand $(1-t/a)^n$, use the bound $(t+z)\geq z$ and [1, eq. 6.3.6] we obtain again (25), but with $S_n(z,a,\rho)$ replaced by $\psi(n+1)+\gamma$.

A bound for the first integral in the right hand side of (25) is given by $a/z^{n+\rho}$. After the change of variable $t = a/u$ in the second integral and using [18, eqs. (5.4)-(5.5)], we obtain (19) with $T_n(z, a, \rho)$ given by the right hand side of the first line in (21). If, instead of computing exactly the second integral in (25), we use the bound $(t+z)^\rho \geq (a+z)^\rho \; \forall \; t \geq a$ and equality [16, p. 31, eq. 4] we obtain the second line in (21).

Finally, if we get rid of irrelevant terms for large z, the right hand side of (19), as function of a, has a minimum for a given in (22).

For obtaining the second inequality in (18), using lemma 4 we have, for $n \geq \mu$, $|f_n(t)| \leq |A_n| t^{-n-1/2}$ if $t \geq 1$ and $|f_n(t)| \leq |A_{n-1}| t^{-n-1/2}$ if $t \leq 1$. Therefore, $|f_n(t)| \leq \bar{A}_n t^{-n-1/2} \; \forall \; t \in [0, \infty)$ and $n \geq \mu$. Then, $f_n(t)$ satisfies the first bound of (13) with μ replaced by $1/2$ and $|A_n|$ by $\bar{A}_n$. Repeating now the calculations of the proof of proposition 1 we obtain the second inequality in (18). ☐

Remark 1. For large n and fixed z, the bound (19) (with a given in (22)) 'contains an extra asymptotic factor $\log(n)$' with respect to the bound (18), whereas for large z and fixed n, it 'contains an extra asymptotic factor $\log(z)/\sqrt{z}$'. Therefore, (19) is more suitable for large z and (18) is more suitable for large n.

3. Expansions of the symmetric standard elliptic integrals.

Convergent expansions of R_F, R_D and R_J for large values of one of their parameters are obtained as corollaries of theorems 1 or 2. Error bounds for the remainder terms in these expansions follow from propositions 1 and 2. We derive the explicit expansions and error bounds for the remainders in the following subsections.

3.1. Expansion of $R_F(x, y, z)$ for large z.

Corollary 1. *A uniformly convergent expansion of $R_F(x, y, z)$ for $0 \leq x < y \leq z$ is given, for $n = 1, 2, 3, \ldots$, by*

$$R_F(x,y,z) = \frac{1}{2\sqrt{z}} \sum_{k=0}^{n-1} \frac{(-1)^k}{k! z^k} \left(\frac{1}{2}\right)_k \left[A_k^F(x,y) \left(\log(z) - \gamma - \psi\left(k + \frac{1}{2}\right)\right) + B_k^F(x,y) \right] + R_n^F(x,y,z), \tag{26}$$

where, for $k = 0, 1, 2, \ldots$,

$$A_k^F(x,y) = (-1)^k \sum_{j=0}^{k} \frac{(1/2)_j (1/2)_{k-j}}{j!(k-j)!} x^j y^{k-j} \tag{27}$$

and coefficients $B_k^F(x,y)$ are given by the recurrence

$$B_{k+2}^F = \frac{A_{k+2}^F}{k+1} + \frac{xyA_k^F + (x+y)A_{k+1}^F + 2A_{k+2}^F}{k+2} + \frac{2k+3}{2k+4}(x+y)\left[\frac{A_{k+1}^F}{k+1} - B_{k+1}^F\right] - \frac{k+1}{k+2} xy B_k^F, \tag{28}$$

empty sums being understood as zero and

$$B_0^F = -2\log\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right), \qquad B_1^F = (x+y)\log\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right) - \sqrt{xy}. \tag{29}$$

For $n=1,2,3,...$, the remainder $R_n^F(x,y,z)$ is positive and a bound for $R_n^F(x,y,z)$ is given by the right hand side of (18) or (19) putting $\rho\equiv 1/2$ and $A_n\equiv A_n^F(x,y)$ given above. In particular, two error bounds are given by

$$
\begin{aligned}
R_n^F(x,y,z) \leq & \frac{1}{2n!}\left(\frac{1}{2}\right)_n\left[1+\psi(n+1)+\gamma+\log\left(1+\frac{nz|A_{n-1}^F|}{|A_n^F|}\right)\right]\frac{|A_n^F|}{z^n\sqrt{z}}, \\
R_n^F(x,y,z) \leq & \frac{\sqrt{\pi}(n-1)!}{\Gamma(n+1/2)}\frac{\bar{A}_n^F}{z^n},
\end{aligned}
\tag{30}
$$

where $\bar{A}_n^F = max\{|A_n^F|,|A_{n-1}^F|\}$.

Proof. The integral $2R_F(x,y,z)$ has the form considered in theorem 2 with

$$
f(t)\equiv f^F(t)=\frac{1}{\sqrt{(t+x)(t+y)}}=\sum_{k=0}^{n-1}\frac{A_k^F}{t^{k+1}}+f_n^F(t), \tag{31}
$$

where $f_n^F(t)=\mathcal{O}(t^{-n-1})$ as $t\to\infty$ and $\rho=1/2$. Therefore, the asymptotic expansion of $2R_F(x,y,z)$ for large z follows from eq. (7) in theorem 2. Coefficients $A_k\equiv A_k^F(x,y)$ in eq. (6) are trivially given by formula (27).

For calculating $B_k\equiv B_k^F(x,y)$ we consider the second line in (8). Define, for $k=0,1,2,...$,

$$
I_k^F(x,y,T)\equiv\int_0^T t^k f^F(t)dt\equiv\int_0^T\frac{t^k}{\sqrt{(t+x)(t+y)}}dt.
$$

and

$$
\sigma_k^F(x,y)\equiv\lim_{T\to\infty}\left\{I_k^F(x,y,T)-\sum_{j=0}^{k-1}A_j^F\frac{T^{k-j}}{k-j}-A_k^F\log(T)\right\}. \tag{32}
$$

Integrals $I_k^F(x,y,T)$ satisfy the recurrence

$$
I_{k+2}^F=\frac{1}{2(k+2)}\left[2T^{k+1}\sqrt{(T+x)(T+y)}-(2k+3)(x+y)I_{k+1}^F-2(k+1)xyI_k^F\right]. \tag{33}
$$

On the other hand, from the differential equation $2(t+x)(t+y)(f^F)'+(2t+x+y)f^F=0$, we obtain, for $k=0,1,2,...$,

$$
2(k+2)A_{k+2}^F+(2k+3)(x+y)A_{k+1}^F+2(k+1)xyA_k^F=0. \tag{34}
$$

Now we substitute $I_{k+2}^F(x,y,T)$ in the definition (32) of $\sigma_{k+2}^F(x,y)$ by the right hand side of (33), expand the term $\sqrt{(T+x)(T+y)}$ in inverse powers of T and use recurrence (34). We obtain

$$
2(k+2)\sigma_{k+2}^F=2xyA_k^F+2(x+y)A_{k+1}^F+2A_{k+2}^F-(2k+3)(x+y)\sigma_{k+1}^F-2(k+1)xy\sigma_k^F,
$$

from which (28) follows easily by using the second line in (8) and (34). Integrals $I_0^F(x,y,T)$ and $I_1^F(x,y,T)$ may be calculated by using formula [16, p. 53, eqs. 3,8]. Then, from the second line in (8) and using $A_0^F=1$ and $A_1^F=-(x+y)/2$ we obtain (29).

Function $f^F(t)$ satisfies the conditions of proposition 2 with $\mu = 1$. Therefore, $R_n^F(x,y,z) \geq 0$ and the bounds (18) and (19) hold for $2R_n^F(x,y,z)$ setting $\rho \equiv 1/2$ and $A_n \equiv A_n^F(x,y)$ given in (27). In particular, introducing (22) in (19) we obtain the first line of (30).

Introducing the bound $|A_n^F| \leq y^n$ in the second line of (30) we obtain, for $n \geq 1$,

$$R_n^F(x,y,z) \leq C(y,z)\frac{y^n}{z^n\sqrt{n}}, \tag{35}$$

where $C(y,z)$ is independent of n. Therefore, expansion (26) is uniformly convergent for $y \leq z$. ☐

Remark 2. An alternative (and explicit) expression for the coefficients $B_k^F(x,y)$ can be obtained from the first line in (8). Using equality [16, p. 303, eq. 24] and the reflection formula of the gamma function [1, eq. 6.1.17] we have, for $w \notin \mathbb{Z}$,

$$M[f^F;w] = \frac{\pi}{\sin(\pi w)}\frac{x^w}{\sqrt{xy}}F\left(w,\frac{1}{2};1;1-\frac{x}{y}\right) \qquad \text{if } \ y > x > 0,$$

$$M[f^F;w] = \frac{\sqrt{\pi}}{\sin(\pi w)}\frac{\Gamma(w-1/2)}{\Gamma(w)}y^{w-1} \qquad \text{if } \ y > x = 0 \ \text{ and } \ \frac{3}{2} - w \notin \mathbb{N}.$$

Subtracting the pole $-A_k/(w-k-1)$ and taking the limit $w \to k+1$ we obtain

$$B_k^F(x,y) = A_k^F(x,y)\sum_{j=1}^{k}\frac{1}{j} - (-1)^k C_k^F(x,y), \tag{36}$$

where

$$C_k^F(0,y) = \frac{(1/2)_k y^k}{k!}\left(\psi\left(k+\frac{1}{2}\right) - \psi(k+1) + \log(y)\right) \tag{37}$$

and

$$C_k^F(x,y) = x^k\sqrt{\frac{x}{y}}\left(\log(x)F\left(k+1,\frac{1}{2};1;1-\frac{x}{y}\right) + F'\left(k+1,\frac{1}{2};1;1-\frac{x}{y}\right)\right) \tag{38}$$

for $x > 0$, where $F'(a,b;c;z)$ denotes the first derivative of $F(a,b;c;z)$ with respect to the argument a.

Remark 3. Formulas (36)-(38) provide coefficients B_k^F in expansion (26) of an explicit expression which may be useful for analytical purposes. On the other hand, recurrence (28)-(29) involve only elementary functions and may be more appropriate for numerical computations. Similar comments can be made about the coefficients B_k^D in the expansion of $R_D(x,z,y)$ for large z given in section 3.3.

Table 1 shows a numerical example of the approximation supplied by expansion (26).

3.2. Expansion of $R_D(x,y,z)$ for large z.

COROLLARY 2. *A uniformly convergent expansion of $R_D(x,y,z)$ for $0 \leq x < y < z$ is given, for $n = 1,2,3,...,$ by*

$$\begin{aligned} R_D(x,y,z) = \frac{3}{2\sqrt{z^3}}\sum_{k=0}^{n-1}\frac{(-1)^k}{k!z^k}\left(\frac{3}{2}\right)_k\bigg[A_k^F(x,y)\left(\log(z) - \gamma - \psi\left(k+\frac{3}{2}\right)\right) + \\ B_k^F(x,y)\bigg] + R_n^D(x,y,z), \end{aligned} \tag{39}$$

Table 1

Second, third and sixth columns represent $R_F(1,2,z)$, approximation (26) for $n=1$ and approximation (26) for $n=2$ respectively. Fourth and seventh columns represent the respective relative error $-R_n^F(1,2,z)/R_F(1,2,z)$. Fifth and last columns represent the respective error bounds given by eq. (19).

z	$R_F(1,2,z)$	First order approx.	Relative error	Relative error bound	Second order approx.	Relative error	Relative error bound
10	.5537947453	.5237406385	-.0543	.0725789476	.5504844438	-.00598	.0100213601
20	.4609268635	.4478367679	-.0284	.0366674515	.4601982389	-.00158	.0024544658
50	.3522219102	.3480283802	-.0119	.0148009679	.3521274856	-.000268	.0003865126
100	.2824793637	.2807505868	-.00612	.0074304273	.2824597696	-.0000694	.0000958172
200	.2237272736	.2230270972	-.00313	.0037243942	.2237232837	-.0000178	.0000237907

Table 2

Second, third and sixth columns represent $R_D(1,2,z)$, approximation (39) for $n=1$ and approximation (39) for $n=2$ respectively. Fourth and seventh columns represent the respective relative error $-R_n^D(1,2,z)/R_D(1,2,z)$. Fifth and last columns represent the respective error bounds given by eq. (19).

z	$R_D(1,2,z)$	First order approx.	Relative error	Relative error bound	Second order approx.	Relative error	Relative error bound
10	.0835011776	.0622538617	-.254	.3565374834	.0792081617	-.0514	.0913314070
20	.0384213534	.0336344955	-.125	.1669454033	.0379393692	-.0125	.0203200273
50	.0130325135	.0123964214	-.0488	.0624511404	.0130069811	-.00196	.0029107969
100	.0055565283	.0054225176	-.0241	.0299926153	.0055538440	-.000483	.0006835181
200	.0023123734	.0022847462	-.0120	.0145049853	.0023120972	-.000119	.0001625233

where $A_k^F(x,y)$ and $B_k^F(x,y)$ are given in (27) and (28)-(29) (or (36)-(38)) respectively.

For $n=1,2,3,\ldots$, the remainder term $R_n^D(x,y,z)$ is positive and a bound for $(2/3)R_n^D(x,y,z)$ is given by the right hand side of (18) or (19) putting $\rho\equiv 3/2$ and $A_n\equiv A_n^F(x,y)$ given in (27). In particular, two error bounds are given by

$$\begin{aligned} R_n^D(x,y,z) \leq& \frac{3}{2n!}\left(\frac{3}{2}\right)_n\left[1+\psi(n+1)+\gamma+\log\left(1+\frac{nz|A_{n-1}^F|}{|A_n^F|}\right)\right]\frac{|A_n^F|}{z^n\sqrt{z^3}}, \\ R_n^D(x,y,z) \leq& \frac{2\sqrt{\pi}n!}{\Gamma(n+1/2)}\frac{\bar{A}_n^F}{z^{n+1}}. \end{aligned}$$

Proof. The integral $(2/3)R_D(x,y,z)$ has the form considered in theorem 2 with $f(t)\equiv f^F(t)$ given in (31) and $\rho=3/2$. The remaining proof follows as in corollary 1 except that, in this case, eq. (35) reads

$$R_n^D(x,y,z)\leq C(y,z)\frac{y^n\sqrt{n}}{z^n}$$

and convergence of expansion (39) is restricted to $y<z$. □

Table 2 shows a numerical example of the approximation supplied by expansion (39).

3.3. Expansion of $R_D(x,z,y)$ for large z.

COROLLARY 3. *A uniformly convergent expansion of $R_D(x,z,y)$ for $0 \le x < y < z$ or $0 \le y < x < z$ is given, for $n = 1, 2, 3, \ldots$, by*

$$R_D(x,z,y) = \frac{3}{2\sqrt{z}} \sum_{k=0}^{n-1} \frac{(-1)^k}{k! z^k} \left(\frac{1}{2}\right)_k \left[A_k^D(x,y)\left(\log(z) - \gamma - \psi\left(k+\frac{1}{2}\right)\right) + B_k^D(x,y)\right] + \bar{R}_n^D(x,z,y), \tag{40}$$

where $A_0^D(x,y) = 0$ and, for $k = 1, 2, 3, \ldots$,

$$A_k^D(x,y) = (-1)^{k-1} \sum_{j=0}^{k-1} \frac{(1/2)_j (3/2)_{k-j-1}}{j!(k-j-1)!} x^j y^{k-j-1}. \tag{41}$$

Coefficients $B_k^D(x,y)$, for $k = 0, 1, 2, \ldots$, are given by the recurrence

$$B_{k+2}^D = \frac{A_{k+2}^D}{k+2} + \frac{xyA_k^D + (x+y)A_{k+1}^D + 2A_{k+2}^D}{k+1} + \left[\frac{x-y}{2k+2} - x - y\right]\left[B_{k+1}^D - \frac{A_{k+1}^D}{k+1}\right] - xyB_k^D, \tag{42}$$

empty sums being understood as zero and

$$B_0^D = \frac{2}{y-x}\left(1 - \sqrt{\frac{x}{y}}\right), \qquad B_1^D = 1 + \frac{2y}{x-y}\left(1 - \sqrt{\frac{x}{y}}\right) - 2\log\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right). \tag{43}$$

For $n = 1, 2, 3, \ldots$, the remainder term $\bar{R}_n^D(x,z,y)$ is negative,

$$|\bar{R}_1^D(x,z,y)| \le \frac{3}{2\sqrt{z}(r-z)} + \frac{3}{4\sqrt{(z-r)^3}} \log\left[\frac{\sqrt{z}+\sqrt{z-r}}{\sqrt{z}-\sqrt{z-r}}\right] \tag{44}$$

if $r \equiv \min\{x,y\} > 0$,

$$|\bar{R}_1^D(x,z,y)| \le \frac{3}{y\sqrt{z}} - \frac{3\pi}{4\sqrt{z^3}} F\left(\frac{3}{2}, \frac{3}{2}; 2; 1 - \frac{y}{z}\right) \qquad \text{if } \ y > x \ge 0 \tag{45}$$

and, for $n = 2, 3, 4, \ldots$, a bound for $|\bar{R}_n^D(x,z,y)|$ is given by the right hand side of (18) or (19) setting $\rho \equiv 1/2$ and $A_n \equiv A_n^D(x,y)$ given above. In particular, two error bounds are given, for $n \ge 2$, by

$$\begin{aligned} |\bar{R}_n^D(x,z,y)| \le & \frac{3}{2n!}\left(\frac{1}{2}\right)_n \left[1 + \psi(n) + \gamma + \log\left(1 + \frac{nz|A_{n-1}^D|}{|A_n^D|}\right)\right] \frac{|A_n^D|}{z^n\sqrt{z}}, \\ |\bar{R}_n^D(x,z,y)| \le & \frac{\sqrt{\pi}(n-1)!}{\Gamma(n+1/2)} \frac{\bar{A}_n^D}{z^n}, \end{aligned} \tag{46}$$

where $\bar{A}_n^D = max\{|A_n^D|, |A_{n-1}^D|\}$.

Proof. The integral $(2/3)R_D(x,z,y)$ has the form required in theorem 2 with

$$f(t) \equiv f^D(t) = \frac{1}{\sqrt{(t+x)(t+y)^3}} = \sum_{k=0}^{n-1} \frac{A_k^D}{t^{k+1}} + f_n^D(t),$$

where $f_n^D(t) = \mathcal{O}(t^{-n-1})$ as $t \to \infty$ and $\rho = 1/2$. Therefore, the asymptotic expansion of $(2/3)R_D(x,z,y)$ follows from eq. (7) in theorem 2. Trivially, the coefficients $A_k \equiv A_k^D(x,y)$ in formula (6) satisfy $A_0^D = 0$ and, for $k = 1, 2, 3, \ldots$, they are given by (41).

Recurrence (42)-(43) for $B_k^D(x,y)$ follows from the second line in (8). Its derivation follows the pattern of derivation of formulas (28)-(29) in corollary 1. We define, for $k = 0, 1, 2, \ldots$,

$$I_k^D(x,y,T) \equiv \int_0^T f^D(t)dt \equiv \int_0^T \frac{t^k}{\sqrt{(t+x)(t+y)^3}}dt.$$

and

$$\sigma_k^D(x,y) \equiv \lim_{T\to\infty}\left\{ I_k^D(x,y,T) - \sum_{j=0}^{k-1} A_j^D \frac{T^{k-j}}{k-j} - A_k^D \log(T)\right\}. \tag{47}$$

Integrals $I_k^D(x,y,T)$ satisfy the recurrence

$$I_{k+2}^D = \frac{T^{k+1}}{k+1}\sqrt{\frac{T+x}{T+y}} + \left(\frac{x-y}{2k+2} - x - y\right) I_{k+1}^D - xyI_k^D. \tag{48}$$

On the other hand, from the differential equation $2(t+x)(t+y)(f^D)' + (4t+3x+y)f^D = 0$, we obtain, for $k = 0, 1, 2, \ldots$,

$$2(k+1)A_{k+2}^D + (2(k+1)(x+y) + y - x)A_{k+1}^D + 2(k+1)xyA_k^D = 0. \tag{49}$$

Now we substitute $I_{k+2}^D(x,y,T)$ in the definition (47) of $\sigma_{k+2}^D(x,y)$ by the right hand side of (48), expand the term $\sqrt{(T+x)/(T+y)}$ in inverse powers of T and use the recurrence (49). We obtain

$$\begin{aligned} 2(k+1)\sigma_{k+2}^D =& 2xyA_k^D + 2(x+y)A_{k+1}^D + 2A_{k+2}^D + (x-y-2(k+1)(x+y))\sigma_{k+1}^D - \\ & 2(k+1)xy\sigma_k^D, \end{aligned}$$

from which (42) follows easily by using the second line in (8) and (49). Integrals $I_0^D(x,y,T)$ and $I_1^D(x,y,T)$ can be calculated by using formulas [16, p. 53, eqs. 6,8]. Then, from the second line in (8), $A_0^D = 0$ and $A_1^D = 1$ we obtain (43).

Function $f^D(t)$ has the form required in proposition 2 with $\mu = 2$. Therefore, $\bar{R}_n^D(x,z,y) \le 0$ and the bounds (18) and (19) hold for $(2/3)\bar{R}_n^D(x,z,y)$ setting $\rho = 1/2$ and $A_n \equiv A_n^D(x,y)$ given in (41) for $n = 2, 3, 4, \ldots$. In particular, the first line of (46) follows after introducing (22) in inequality (19). On the other hand, $A_0^D = 0$ means $f_1^D(t) = f^D(t)$. Introducing the bounds $f^D(u) \le (u+r)^{-2}$ if $r = \min\{x,y\} > 0$ or $f^D(u) \le \sqrt{u}(u+y)^{-3/2}$ if $y > x \ge 0$ in the definition (4) of $f_{1,1}(t)$ and using [16, p. 52, eq. 6] or [16, p. 53, eq. 4] respectively, we obtain (44) and (45).

Using the second line of (46) and $|A_{n+1}^D(x,y)| \le (n+1)s^n$, where $s = \max\{x,y\}$, we obtain, for $n \ge 1$,

$$|\bar{R}_n^D(x,z,y)| \le C(s,z)\frac{s^n\sqrt{n}}{z^n},$$

where $C(s,z)$ is independent of n. Therefore, expansion (40) is uniformly convergent for $s < z$. $\square$

TABLE 3

Second, third and sixth columns represent $R_D(1,z,2)$, approximation (40) for $n=1$ and approximation (40) for $n=2$ respectively. Fourth and seventh columns represent the respective relative error $-\bar{R}_n^D(1,z,2)/R_D(1,z,2)$. Fifth and last columns represent the respective error bounds given by eq. (19).

z	$R_D(1,z,2)$	First order approx.	Relative error	Relative error bound	Second order approx.	Relative error	Relative error bound
10	.2390532443	.2778629050	.162	.2021314366	.2508051822	.0492	.0870637346
20	.1784332265	.1964787444	.101	.1221700356	.1811001790	.0149	.0238026222
50	.1180725686	.1242640688	.0524	.0612672806	.1184298191	.00303	.0043925602
100	.0852097498	.0878679657	.0312	.0357104912	.0852853865	.000888	.0012264633
200	.0610194998	.0621320343	.0182	.0205219425	.0610351563	.000257	.0003410757

Remark 4. An alternative (and explicit) expression for the coefficients $B_k^D(x,y)$ can be obtained by using the first equality in (8) and

$$M[f^D;w]=\frac{\pi(1-w)x^w}{\sin(\pi w)\sqrt{xy^3}}F\left(w,\frac{3}{2};2;1-\frac{x}{y}\right)\qquad \text{if } x,y>0,\ \ w\notin \mathbb{Z}$$

or

$$M[f^D;w]=\frac{2\sqrt{\pi}y^{w-2}}{\sin(\pi(w-1))}\frac{\Gamma(w-1/2)}{\Gamma(w-1)}\qquad \text{if } y>x=0,\ \ w\notin \mathbb{Z},\ \ \frac{3}{2}-w\notin\mathbb{N}.$$

Derivation of these formulas is similar to the derivation of $M[f^F;w]$ in remark 1. Subtracting the pole $-A_k^D/(w-k-1)$ and taking the limit $w\to k+1$ we obtain, for $k=0,1,2,...,$

$$B_k^D(x,y)=A_k^D(x,y)\sum_{j=1}^{k}\frac{1}{j}+(-1)^kC_k^D(x,y), \tag{50}$$

where $C_0^F(0,y)=2/y$,

$$C_k^F(0,y)=\frac{2k(1/2)_k y^{k-1}}{k!}\big(\psi(k+1/2)-\psi(k)+\log(y)\big)\qquad \text{for } k=1,2,3,...$$

and, for $k=0,1,2,...$ and $x,y>0$,

$$C_k^F(x,y)=x^k\sqrt{\frac{x}{y^3}}\left[(1+k\log(x))F\left(k+1,\frac{3}{2};2;1-\frac{x}{y}\right)+ kF'\left(k+1,\frac{3}{2};2;1-\frac{x}{y}\right)\right].$$

Table 3 shows a numerical example of the approximation supplied by expansion (40).

3.4. Expansion of $R_J(x,y,z,p)$ for large z.

COROLLARY 4. *A uniformly convergent expansion of $R_J(x,y,z,p)$ for $0<p<z$ and $0\le x<y<z$ is given, for $n=1,2,3,...$, by*

(51)

$$R_J(x,y,z,p)=\frac{3}{2\sqrt{z}}\sum_{k=0}^{n-1}\frac{(-1)^k}{k!z^k}\left(\frac{1}{2}\right)_k\left[A_k^J(x,y,p)\left(\log(z)-\gamma-\psi\left(k+\frac{1}{2}\right)\right)+ B_k^J(x,y,p)\right]+R_n^J(x,y,z,p),$$

where $A_0^J(x,y,p)=0$ *and, for* $k=1,2,3,...,$

$$A_k^J(x,y,p)=\sum_{j=0}^{k-1}(-p)^{k-j-1}A_j^F(x,y), \tag{52}$$

where $A_j^F(x,y)$ *are given in (27). Coefficients* $B_k^J(x,y,p)$ *are given by the recurrence*

$$B_{k+1}^J=B_k^F-pB_k^J+A_{k+1}^J\sum_{j=1}^{k+1}\frac{1}{j}+(pA_k^J-A_k^F)\sum_{j=1}^{k}\frac{1}{j}, \qquad k=0,1,2,..., \tag{53}$$

where $B_k^F(x,y)$ *are given by (28)-(29) (or (36)-(38)), empty sums must be understood as zero and*

$$B_0^J=\frac{2}{\sqrt{|(p-x)(p-y)|}}\log\left[\frac{\sqrt{p(p-x)}+\sqrt{p(p-y)}}{\sqrt{y(p-x)}+\sqrt{x(p-y)}}\right] \qquad \text{if} \;\; p>x,y, \tag{54}$$

$$B_0^J=\frac{2}{\sqrt{|(p-x)(p-y)|}}\log\left[\frac{\sqrt{x(y-p)}+\sqrt{y(x-p)}}{\sqrt{p(x-p)}+\sqrt{p(y-p)}}\right] \qquad \text{if} \;\; p<x,y, \tag{55}$$

$$\begin{aligned}B_0^J=\frac{1}{\sqrt{|(p-x)(p-y)|}}&\left[\sin^{-1}\left(\frac{x+y-2p}{y-x}\right)-\right.\\ &\left.\sin^{-1}\left(\frac{2xy-p(x+y)}{p(y-x)}\right)\right] \qquad \text{if} \;\; x<p<y.\end{aligned} \tag{56}$$

For $n=1,2,3,...,$ *the remainder term* $R_n^J(x,y,z,p)$ *is negative,*

$$|R_1^J(x,y,z)|\le\frac{3}{2\sqrt{z}(r-z)}+\frac{3}{4\sqrt{(z-r)^3}}\log\left[\frac{\sqrt{z}+\sqrt{z-r}}{\sqrt{z}-\sqrt{z-r}}\right] \tag{57}$$

if $r\equiv\min\{x,y,p\}>0,$

$$|R_1^J(x,y,z)|\le\frac{3}{r\sqrt{z}}-\frac{3\pi}{4\sqrt{z^3}}F\left(\frac{3}{2},\frac{3}{2};2;1-\frac{r}{z}\right) \tag{58}$$

if $r\equiv\min\{y,p\}>0$ *or* $r\equiv\min\{x,p\}>0$ *and, for* $n=2,3,4,...,$ *a bound for* $(2/3)|R_n^J(x,y,z,p)|$ *is given by the right hand side of (18) or (19) putting* $\rho\equiv 1/2$ *and* $A_n\equiv A_n^J(x,y,p)$ *given above. In particular, two error bounds are given, for* $n\ge 2$, *by*

$$\begin{aligned}|R_n^J(x,y,z,p)|\le&\frac{3}{2n!}\left(\frac{1}{2}\right)_n\left[1+\psi(n+1)+\gamma+\log\left(1+\frac{nz|A_{n-1}^J|}{|A_n^J|}\right)\right]\frac{|A_n^J|}{z^n\sqrt{z}},\\ |R_n^J(x,y,z,p)|\le&\frac{\sqrt{\pi}(n-1)!}{\Gamma(n+1/2)}\frac{\bar{A}_n^J}{z^n},\end{aligned} \tag{59}$$

where $\bar{A}_n^J=max\{|A_n^J|,|A_{n-1}^J|\}$.

TABLE 4

Second, third and sixth columns represent $R_J(1,2,z,3)$, approximation (51) for $n=1$ and approximation (51) for $n=2$ respectively. Fourth and seventh columns represent the respective relative error $-R_n^J(1,2,z,3)/R_J(1,2,z,3)$. Fifth and last columns represent the respective error bounds given in eq. (19).

z	$R_J(1,2,z,3)$	First order approx.	Relative error	Relative error bound	Second order approx.	Relative error	Relative error bound
10	.1877070842	.2227576125	.187	.2574232928	.2013272410	.0726	.1357463320
20	.1409922070	.1575134184	.117	.1546127555	.1441244221	.0222	.0367400044
50	.0938633074	.0996202328	.0613	.0770693616	.0942893087	.00454	.0067564415
100	.0679464537	.0704421421	.0367	.0447835297	.0680375155	.00134	.0018883978
200	.0487571525	.0498101164	.0216	.0256831788	.0487761541	.000390	.0005263141

Proof. The integral $(2/3)R_J(x,y,z,p)$ has the form required in theorem 2 with

$$f(t)\equiv f^J(t)=\frac{1}{\sqrt{(t+x)(t+y)}(t+p)}=\sum_{k=0}^{n-1}\frac{A_k^J}{t^{k+1}}+f_n^J(t),$$

where $f_n^J(t)=\mathcal{O}(t^{-n-1})$ as $t\to\infty$ and $\rho=1/2$. The asymptotic expansion of $(2/3)R_J(x,y,z,p)$ for large z follows from eq. (7) in theorem 2. Trivially, $A_0^J=0$ and, for $k=1,2,3,...$, coefficients A_k^J are given by (52).

Recurrence (53) for $B_k^J(x,y,p)$ follows from the second line in (8). We define, for $k=0,1,2,...$,

$$I_k^J(x,y,p,T)\equiv\int_0^T t^k f^D(t)dt\equiv\int_0^T\frac{t^k}{\sqrt{(t+x)(t+y)}(t+p)}dt.$$

and

$$\sigma_k^J(x,y,p)\equiv\lim_{T\to\infty}\left\{I_k^J(x,y,p,T)-\sum_{j=0}^{k-1}A_j^J\frac{T^{k-j}}{k-j}-A_k^J\log(T)\right\}. \tag{60}$$

Integrals $I_k^J(x,y,p,T)$ satisfy the recurrence $I_{k+1}^J=I_k^F-pI_k^J$. Then, (53) follows easily by using (8), (32), (52) and (60). Integral $I_0^J(x,y,p,T)$ may be calculated by using [16, pp. 53,54, eqs. 8-11]. Then, from the second line in (8) and $A_0^J=0$ we obtain (54)-(56).

Function $f^J(t)$ has the form required in proposition 2 with $\mu=2$. Therefore, $R_n^J(x,y,z,p)\le 0$ and the bounds (18) and (19) hold for $(2/3)R_n^J(x,y,z,p)$ setting $\rho=1/2$ and $A_n\equiv A_n^J(x,y,p)$ given in (52) for $n=2,3,4,....$ In particular, the first line of (59) follows after introducing (22) in inequality (19). On the other hand, $A_0^J=0$ means $f_1^J(t)=f^J(t)$. Then, after a similar calculation to the one used for deriving (44) and (45), we obtain (57) and (58).

Using the second line of (59) and the bound $|A_{n+1}^J(x,y,p)|\le(n+1)s^n$, where $s=\max\{x,y,p\}$, we obtain, for $n\ge 1$,

$$|R_n^J(x,y,z,p)|\le C(s,z)\frac{s^n\sqrt{n}}{z^n},$$

where $C(s,z)$ is independent of n. Therefore, expansion (51) is uniformly convergent for $s<z$. $\square$

Table 4 shows a numerical example of the approximation supplied by expansion (51).

3.5. Expansion of $R_J(x,y,z,p)$ for large p.

Corollary 5. *A uniformly convergent expansion of $R_J(x,y,z,p)$ for $0 \le x < y < z < p$ is given, for $n = 1,2,3,\ldots$, by*

$$R_J(x,y,z,p) = \frac{3}{2}\sum_{k=0}^{n-1}\frac{D_k^J(x,y,z)}{p^{k+1}} - \frac{3\pi}{2\sqrt{p}}\sum_{k=0}^{n-1}\frac{C_k^J(x,y,z)}{p^k} + \bar{R}_n^J(x,y,z,p), \tag{61}$$

where $C_0^J(x,y,z) = 0$ and, for $k = 1,2,3,\ldots$,

$$C_k^J(x,y,z) = \sum_{j=0}^{k-1}\sum_{s=0}^{k-j-1}\frac{(1/2)_j(1/2)_s(1/2)_{k-j-s-1}}{j!s!(k-j-s-1)!}x^j y^s z^{k-j-s-1}. \tag{62}$$

Coefficients $D_k^J(x,y,z)$ are given by the recurrence

$$\begin{aligned} D_{k+3}^J = \frac{1}{2k+5}&\left[2(k+2)(x+y+z)D_{k+2}^J - (2k+3)(xy+xz+yz)D_{k+1}^J + \right.\\ &\left. 2(k+1)xyzD_k^J\right], \end{aligned} \tag{63}$$

$$D_0^J(x,y,z) = 2R_F(x,y,z), \tag{64}$$

$$D_1^J(x,y,z) = \frac{2}{3}(z-x)(y-z)R_D(x,y,z) + 2zR_F(x,y,z) + 2\sqrt{\frac{xy}{z}} \tag{65}$$

and

$$D_2^J(x,y,z) = \frac{2}{3}(x+y+z)D_1^J(x,y,z) - \frac{2}{3}(xy+xz+yz)R_F(x,y,z) - \frac{2}{3}\sqrt{xyz}. \tag{66}$$

For $n = 1,2,3,\ldots$, the remainder term $\bar{R}_n^J(x,y,z,p)$ is negative and a bound is given by

$$|\bar{R}_n^J(x,y,z,p)| \le \frac{3\pi C_n^J(x,y,z)}{2p^n\sqrt{p}}. \tag{67}$$

Proof. The integral $(2/3)R^J(x,y,z,p)$ has the form required in theorem 1 with

$$f(t) \equiv \bar{f}^J(t) = \frac{1}{\sqrt{(t+x)(t+y)(t+z)}} = -\sum_{k=0}^{n-1}\frac{(-1)^k C_k^J}{t^{k+1/2}} + \bar{f}_n(t), \tag{68}$$

where $\bar{f}_n(t) = \mathcal{O}(t^{-n-1/2})$ as $t \to \infty$ and $\alpha = 1/2$. Therefore, the asymptotic expansion of $(2/3)R^J(x,y,z,p)$ for large p is given by eq. (2). Coefficients $A_k \equiv -(-1)^k C_k^J(x,y,z)$ are trivially given, for $k = 1,2,3,\ldots$, by (62) and $C_0^J(x,y,z) = 0$. Coefficients $D_k^J(x,y,z) \equiv (-1)^k M[\bar{f}^J; k+1]$ in eq. (2) represent the analytic continuation of the Mellin transform of $\bar{f}^J(t)$ evaluated in $k+1$. The Mellin transform of the function $\bar{f}^J(t)$ given in (68) is defined by formula (1) for $0 < \mathrm{Re}(w) < 3/2$. We divide the integration path in this formula at $t = 1$ and, in the integral over $[1,\infty)$, we substitute $\bar{f}^J(t)$ by the right hand side of (68) with n replaced by $n+1$,

$$M[\bar{f}^J; w] = \int_0^1 t^{w-1}\bar{f}^J(t)dt + \sum_{k=0}^{n}\frac{(-1)^k C_k^J(x,y,z)}{w-k-1/2} + \int_1^\infty t^{w-1}\bar{f}_{n+1}^J(t)dt.$$

The first integral is an analytic function of w for $\mathrm{Re}(w) > 0$. The second integral is analytic for $\mathrm{Re}(w) < n + 3/2$. Therefore, this formula gives the analytic continuation of $M[\bar{f}^J; w]$ to the strip $0 < \mathrm{Re}(w) < n+3/2$ (which have simple poles at $w = k+1/2$, $k = 0, 1, 2, ..., n$). We evaluate $M[\bar{f}^J; w]$ above at the point $w = k + 1$ and replace $\bar{f}^J_{n+1}(t)$ in the last integral by $\bar{f}^J(t) + \sum_{k=0}^{n}(-1)^k C_k^J t^{-k-1/2}$. After straightforward operations we obtain

$$D_k^J(x,y,z) = (-1)^k \lim_{T\to\infty}\left\{\int_0^T t^k \bar{f}^J(t)dt + \sum_{j=0}^{k}\frac{(-1)^j C_j^J T^{k-j+1/2}}{k-j+1/2}\right\}. \tag{69}$$

We define, for $s > 0$,

$$\alpha_k(x,y,z,s,T) \equiv \sqrt{s}\int_0^T \frac{t^k}{\sqrt{(t+x)(t+y)(t+z)(t+s)}}dt$$

and

$$I_k(x,y,z,T) \equiv \int_0^T \frac{t^k}{\sqrt{(t+x)(t+y)(t+z)}}dt = \lim_{s\to\infty}\alpha_k(x,y,z,s,T).$$

Integrals $\alpha_k(x,y,z,s,T)$ satisfy the recurrence

$$\begin{aligned}2T^{k+1}\sqrt{s(T+x)(T+y)(T+z)(T+s)} = 2(k+3)\alpha_{k+4} +\\ (2k+5)(x+y+z+s)\alpha_{k+3} + 2(k+2)(xy+xz+xs+yz+ys+zs)\alpha_{k+2} +\\ (2k+3)(xyz+xys+xzs+yzs)\alpha_{k+1} + 2(k+1)xyzs\alpha_k.\end{aligned}$$

Taking the limit $s\to\infty$ we obtain that the integrals $I_k(x,y,z,T)$ satisfy the recurrence

$$\begin{aligned}I_{k+3} = \frac{1}{2k+5}\Big[2T^{k+1}\sqrt{(T+x)(T+y)(T+z)} - 2(k+2)(x+y+z)I_{k+2} -\\ (2k+3)(xy+xz+yz)I_{k+1} - 2(k+1)xyzI_k\Big].\end{aligned} \tag{70}$$

On the other hand, from the differential equation $2(t+x)(t+y)(t+z)(\bar{f}^J)' + (3t^2 + 2(x+y+z)t + xy + xz + yz)\bar{f}^J = 0$, we obtain, for $k = 0, 1, 2, ...,$

$$2(k+1)C_{k+2}^J - (2k+1)(x+y+z)C_{k+1}^J + 2k(xy+xz+yz)C_k^J - (2k-1)xyzC_{k-1}^J = 0. \tag{71}$$

If we expand the term $\sqrt{(T+x)(T+y)(T+z)}$ in (70) in inverse powers of T and use the recurrence (71) and the definition (69), we obtain the recurrence (63). Using (69) we see that D_0^J is trivially given by (64). Integrating $I_1(x,y,z)$ by parts in (69) and using [16, p. 71, eq 10] and [18, eq. 12.33] we obtain (65). Equality (66) follows after straightforward operations.

Function $f^J(t)$ satisfies the conditions of proposition 1 with $\mu = 3/2$. Therefore, $\bar{R}_n^J(x,y,z,p) \le 0$ and the bound (17) holds for $(2/3)\bar{R}_n^J(x,y,z,p)$ setting $A_n \equiv (-1)^n C_n^J$ and (67) follows. Using (67) and the bound $|C_{n+1}^J(x,y,z)| \le (3/2)_n z^n/n!$, we obtain, for $n \ge 1$,

$$|\bar{R}_n^J(x,y,z,p)| \le C(z,p)\frac{\sqrt{n}z^n}{p^n},$$

where $C(z,p)$ is independent of n. Therefore, expansion (61) is uniformly convergent for $z < p$. $\square$

Table 4

Second, third and sixth columns represent $R_J(1,2,3,p)$, approximation (61) for $n=2$ and approximation (61) for $n=3$ respectively. Fourth and seventh columns represent the respective relative error $-\bar{R}_n^J(1,2,3,p)/R_J(1,2,3,p)$. Fifth and last columns represent the respective error bounds given by eq. (67).

z	$R_J(1,2,3,p)$	Second order approx.	Relative error	Relative error bound	Third order approx.	Relative error	Relative error bound
10	.1237859612	.1531757825	.237	.3611528060	.1316685706	.0637	.0963074149
20	.0716068743	.0773834863	.0807	.1103653337	.0723803740	.0108	.0147153778
50	.0330037076	.0336525403	.0197	.0242311844	.0330384089	.00105	.0012923298
100	.0178156797	.0179370973	.00682	.0079352385	.0178189241	.000182	.0002116063
200	.0094259946	.0094483849	.00238	.0026513081	.0094262936	.0000317	.0000353507

Table 5 shows a numerical example of the approximation supplied by expansion (61).

Remark 5. A bound for the $n-$esim remainder term in any of the expansions given in corollaries 1-5 has the form $C(s,z)\sqrt{n}(s/z)^n$ for $n \geq 1$, where z is the asymptotic variable, s is a bound for the remaining variables and $C(s,z)$ is independent on n. Therefore, convergence rate of these expansions increases for decreasing value of the quotient s/z.

4. Conclusions. Following Wong's proposal [21, example 1], the distributional approach has been used in theorem 2 for deriving an alternative proof for the asymptotic expansion of the generalized Stieltjes transforms (see [21, theorem 2 and example 1]). Using this result we have derived convergent expansions of $R_F(x,y,z)$, $R_D(x,y,z)$, $R_D(x,z,y)$ and $R_J(x,y,z,p)$ for $x,y,p<z$ in corollaries 1-4 respectively. On the other hand, using the asymptotic expansion of the Stieltjes transforms [22, chap. 6 sec. 2, theorem 1], we have obtained a convergent expansion of $R_J(x,y,z,p)$ for $x,y,z<p$ in corollary 5. Functions $f(t)$ in the integrand of R_F, R_D and R_J (and, in general, functions $f(t)$ given in (10)) belong to a special kind of functions: the remainder terms in their asymptotic expansions in inverse powers of t satisfy the error test. This fundamental property is used in propositions 1 and 2 for deriving an error bound for the remainder in the asymptotic expansions given in theorems 1 and 2 at any order of the approximation. In particular, it has been derived for the expansions of R_F, R_D and R_J in corollaries 1-5. These bounds have been obtained from the error test and, as numerical computations show (see tables 1-5), they exhibit a remarkable accuracy. Moreover, these bounds show that the expansions are convergent when the asymptotic variable is greater than the remaining ones and that the convergence rate increases as this difference between the asymptotic variable and the remaining ones increases.

Expansions given in corollaries 1-5 are generalizations of the corresponding first order approximations given by Carlson and Gustafson [10]. Nevertheless, the complete expansion of the first elliptic integral given in corollary 1 was already obtained by Carlson and Gustafson [4] and, as well as in corollary 1, the coefficients of the expansion are given by a recurrence [4, eq. 1.7]. Complete expansions for R_D and R_J for the asymptotic parameters considered in corollaries 2-5 were also obtained by Carlson and Gustafson [3]. But a recurrence for the calculation of the coefficients is not given in [3] and error bounds supplied there are not quite satisfactory. The advantage of the approach presented here is that it supplies a simple algorithm for the calculation of the coefficients of these expansions and more accurate error bounds at any order of the approximation. This algorithm is explicitly given in corollaries 1-5. Moreover, coefficients of the expansions of $R_F(x,y,z)$, $R_D(x,y,z)$, $R_D(x,z,y)$ and $R_J(x,y,z,p)$ for large z are given in terms of elementary functions, whereas coefficients of the expansion of $R_J(x,y,z,p)$ for large p are given in terms of $R_F(x,y,z)$ and $R_D(x,y,z)$.

For large z, the error bounds supplied in corollaries 1-4 are slightly more accurate than the error bounds given in [10] for the first order approximation of $R_D(x,z,y)$, $R_F(x,y,z)$ and $R_J(x,y,z,p)$ and slightly less accurate for the first order approximation of $R_D(x,y,z)$ and the second order approximation of $R_F(x,y,z)$. When considering first order approximations to $R_J(x,y,z,p)$ for large p, a comparison between the error bounds given in corollary 5 and the error bounds given in [10] is more complicated because they are concerned with different approximations. On the other hand, at any order of the approximation, error bounds given in [4, eqs. (1.20) or (3.40)] are more accurate for large values of n than the error bound given in corollary 1 and less accurate for small n.

Distributional approach should succeed for deriving complete uniform asymptotic expansions of symmetric elliptic integrals too. This challenge is postponed for further investigations.

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