ASYMPTOTIC EXPANSION OF THE INCOMPLETE BETA FUNCTION FOR LARGE VALUES OF THE FIRST PARAMETER *

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Abstract

The Laplace transform representation of the incomplete beta function $B_x(a,b)$ allows to obtain a very simple asymptotic expansion, in ascending powers of $1/a$, whose truncation error can be easily estimated.

The incomplete beta function [1, Sect. 6.6]

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt, \qquad a>0, \quad b>0, \quad 0\le x\le 1, \tag{1}$$

and the closely related incomplete beta ratio

$$I_x(a,b) = \frac{B_x(a,b)}{B(a,b)}, \qquad B(a,b)\equiv B_1(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{2}$$

are of considerable interest because of their applications in Statistics and other scientific fields. The recent book by Temme [6] contains, besides of numerous references to previous works, a very lucid and complete discussion of the different expressions of $B_x(a,b)$, in terms of hypergeometric series, continued fractions, or asymptotic expansions, and of their numerical applicability according to the values of the parameters a and b and the variable x. The region of the parameter space in which a is large, b is small and x is close to 1 requires a careful treatment, since $B_x(a,b)$ varies most rapidly, for $b<1$, as x approaches 1. Suitable asymptotic expansions have been obtained by Molina [3], Temme [5] and, more

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recently, Doman [2]. Here we present an asymptotic expansion of $B_x(a,b)$ in powers of $1/a$, namely

$$B_x(a,b) \sim \frac{x^a(1-x)^{b-1}}{a} \sum_{n=0}^{\infty} (-1)^n \frac{f_n(b,x)}{a^n}, \tag{3}$$

where the $f_n(b,x)$ represent polynomials to be defined below in the variables b and $x/(1-x)$, easily deduced from the Laplace transform representation of $B_x(a,b)$.

Although the expansion (3) is not the most efficient for computational purposes, it presents, from the analytical point of view, the advantages of being cleaner, since the coefficients of the powers of $1/a$ do not depend on a, and avoiding to have recourse to other special functions, like the incomplete gamma function. Besides this, the error in the approximate value of $B_x(a,b)$ obtained by taking a finite number of terms in the expansion can be easily estimated, as we are going to show.

The Laplace transform representation of the incomplete beta function can be immediately derived from (1) by using

$$w = \log(x/t) \tag{4}$$

as a new integration variable, to obtain

$$B_x(a,b) = x^a \int_0^{\infty} e^{-aw} \left(1 - xe^{-w}\right)^{b-1} dw. \tag{5}$$

Now, the standard method [4, Ch.3, Sect. 2] of getting the asymptotic expansion of a Laplace integral can be used. Repeated integration by parts gives

$$\begin{aligned} B_x(a,b) &= \frac{x^a}{a} \sum_{n=0}^{N-1} \frac{1}{a^n} \frac{d^n}{dw^n} \left(\left(1 - xe^{w}\right)^{b-1} \right) \Big|_{w=0} && (6) \\ &\quad + \frac{x^a}{a^N} \int_0^{\infty} e^{-aw} \frac{d^N}{dw^N} \left(\left(1 - xe^{-w}\right)^{b-1} \right) dw, && (7) \end{aligned}$$

that shows that $B_x(a,b)$ has the asymptotic expansion (3), the coefficients being given by

$$f_n(b,x) = \frac{(-1)^n}{(1-x)^{b-1}} \frac{d^n}{dw^n} \left(\left(1 - xe^{-w}\right)^{b-1} \right) \Big|_{w=0} \tag{8}$$

and the truncation error by

$$\begin{aligned} \varepsilon_N &\equiv B_x(a,b) - \frac{x^a(1-x)^{b-1}}{a} \sum_{n=0}^{N-1} (-1)^n \frac{f_n(b,x)}{a^n} && (9) \\ &= \frac{x^a}{a^N} \int_0^{\infty} e^{-aw} \frac{d^N}{dw^N} \left(\left(1 - xe^{-w}\right)^{b-1} \right) dw, \qquad N = 1, 2, 3, \ldots. && (10) \end{aligned}$$

It is not difficult to obtain a more explicit expression of the coefficients $f_n(b, x)$. Let us define

$$F_n(b, x, w) \equiv (-1)^n \frac{d^n}{dw^n} \left(1 - xe^{-w}\right)^{b-1}, \qquad n = 0, 1, 2, \ldots. \tag{11}$$

It can be easily checked that these functions can be written in the form

$$F_n(b, x, w) = \left(1 - xe^{-w}\right)^{b-1} P_n(b, \phi(x, w)), \tag{12}$$

where we have abbreviated

$$\phi(x, w) \equiv \frac{xe^{-w}}{1 - xe^{-w}} \tag{13}$$

and $P_n(b, \phi)$ represents a polynomial of degree n in the variable ϕ, with coefficients depending on b. The set of polynomials $\{P_n\}$ obey the recurrence relations

$$P_0(b, \phi) = 1, \tag{14}$$

$$P_n(b, \phi) = \phi \left((1 - b) P_{n-1}(b, \phi) + (1 + \phi) \frac{d}{d\phi} P_{n-1}(b, \phi) \right). \tag{15}$$

Therefore, we can write

$$P_n(b, \phi) = \sum_{k=0}^{n} (1 - b)_k \alpha_{n,k} \phi^k, \tag{16}$$

with

$$\alpha_{0,0} = 1, \qquad \alpha_{n,k} = 0 \quad \text{for} \quad k < 0 \quad \text{or} \quad k > n, \tag{17}$$

$$\alpha_{n,k} = \alpha_{n-1,k-1} + k\alpha_{n-1,k} \quad \text{for} \quad 0 \leq k \leq n. \tag{18}$$

Since

$$f_n(b, x) = \frac{F_n(b, x, 0)}{(1 - x)^{b-1}}, \tag{19}$$

we obtain from (12), (13) and (16)

$$f_n(b, x) = \sum_{k=0}^{n} (1 - b)_k \alpha_{n,k} \left(\frac{x}{1 - x} \right)^k. \tag{20}$$

Notice that the coefficients $\alpha_{n,k}$ are integer and universal, i. e., independent of the variable or the parameters in $B_x(a, b)$.

In the case we are considering of $0 < b < 1$, it is not difficult to show that the truncation error ε_N alternates its sign as N takes consecutive values. Besides of the functions $F_n(b, x, w)$ introduced in (11), let us define

$$G_n(x, w) \equiv (-1)^n \frac{d^n}{dw^n} \left(\frac{xe^{-w}}{1 - xe^{-w}} \right), \qquad n = 0, 1, 2, \ldots. \tag{21}$$

One has the following recurrence relations

$$
\begin{aligned}
F_n &= (1-b)\sum_{m=0}^{n-1}\binom{n-1}{m} G_m F_{n-1-m}, \qquad n=1,2,3,\ldots, &(22)\\
G_1 &= G_0(G_0+1), &(23)\\
G_n &= \sum_{k=1}^{n-2}\binom{n-1}{k} G_k G_{n-1-k} + G_{n-1}(2G_0+1), \qquad n=2,3,\ldots. &(24)
\end{aligned}
$$

Since $F_0(b,x,w)$ and $G_0(x,w)$ are both positive for $0<x<1$ and $0\le w<\infty$, we see from (22) that, being $b<1$, all $F_n(b,x,w)$, $n=0,1,2,\ldots$, are positive. Therefore, $(1-xe^{-w})^{b-1}$ is an alternating function [4, Ch. 3, Sect. 2] of w in $[0,\infty)$ and the consecutive truncation errors ε_N, ε_{N+1},..., have opposite signs and are bounded by the first neglected term of the expansion,

$$
|\varepsilon_N| \le \frac{x^a(1-x)^{b-1}}{a^{N+1}} f_N(b,x). \tag{25}
$$

These circumstances make the expansion (3) especially interesting: the exact value of $B_x(a,b)$ is always between the two approximate values obtained by consecutive truncations of the expansion.

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References

[1] M. Abramowitz and I. A. Stegun (Eds.), *Handbook of Mathematical Functions*, Dover, New York, 1975.

[2] B. G. S. Doman, *An Asymptotic Expansion for the Incomplete Beta Function*, Math. Comp. **65** (1996), 1283–1288.

[3] E. C. Molina, *Expansions for Laplacian Integrals in terms of Incomplete Gamma Functions*, International Congress of Mathematicians, Zurich, Bell System Technical Journal **11** (1932), 563–575.

[4] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974.

[5] N. M. Temme, *Incomplete Laplace Integrals: Uniform Asymptotic Expansion with Application to the Incomplete Beta Function*, SIAM J. Math. Anal. **18** (1987), 1638–1663.

[6] N. M. Temme, *Special Functions*, John Wiley & Sons, New York, 1996.