Asymptotic Expansions of the Appell's Function F_1

Chelo Ferreira[1] and José L. López[2]

[1] *Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Zaragoza, 50013-Zaragoza, Spain. e-mail: cferrei@posta.unizar.es.*

[2] *Departamento de Matemática e Informática, Universidad Pública de Navarra, 31006-Pamplona, Spain. e-mail: jl.lopez@unavarra.es.*

ABSTRACT

The first Appell's hypergeometric function $F_1(a,b,c,d;x,y)$ is considered for large values of its variables x and/or y. An integral representation of $F_1(a,b,c,d;x,y)$ is obtained in the form of a generalized Stieltjes transform. Distributional approach is applied to this integral to derive four asymptotic expansions of this function in increasing powers of $1/(1-x)$ and/or $1/(1-y)$. For certain values of the parameters a, b, c and d, two of these expansions involve also logarithmic terms in the asymptotic variables $1-x$ and/or $1-y$. Coefficients of these expansions are given in terms of the Gauss hypergeometric function ${}_2F_1(\alpha,\beta,\gamma;x)$ and its derivative with respect to the parameter α. All the expansions are accompanied by error bounds for the remainder at any order of the approximation. These error bounds are obtained from the error test and, as numerical experiments show, they are considerably accurate.

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1. Introduction

The Appell's functions F_1, F_2, F_3 and F_4 are generalizations of the Gauss hypergeometric function ${}_2F_1$ [[8], p. 224]. In particular, the first Appell's function F_1 is defined by means of the double series

$$F_1(a,b,c,d;x,y) \equiv \sum_{k,j=0}^{\infty} \frac{(a)_{k+j}(b)_k(c)_j}{(d)_{k+j}} \frac{x^k y^j}{k!j!}, \qquad |x|<1, \quad |y|<1.$$

Appell's functions have physical applications in several problems of Quantum Mechanics. For example, they appear in the computation of transition matrices in atomic and molecular physics, such as the transitions that involve Coulombic continuum states [7] or ion-atom collisions [6]. They are also representations of the generalized Slater's and Marvin's integrals [24] and the solution of certain ordinary differential equations and partial differential equations [26]. In fact, there is an extensive mathematical literature devoted to the study of these functions: Sharma has obtained generating functions of the Appell's functions [23]. Some integral representations for F_1 and F_2 have been derived by Manocha [15] and Mittal [17]. The Laplace transforms of these functions have been obtained in [10]. Some reduction formulas for special values of the variables and contiguous relations for Appell's functions have been investigated by Buschman [1], [2]. The Lie theory of the Appell's function F_1 has been discussed in [16] and [18]. Carlson has investigated quadratic transformations of Appell's functions [4] and their role on multiple averages [3]. A numerical scheme to compute F_1 has been developed in [7] for complex values of the parameters and real values of the variables.

Complete convergent expansions of $F_1(a,b,c,d;x,y)$ for large values of x and/or y may be obtained from the expansions of the $R-$function derived by Carlson [5]. Although these expansions have an attractively simple structure, explicit computation of the terms of the expansions is not straightforward and the upper bound on the truncation error is not quite satisfactory [[5], sec. 5]. On the other hand, series for $F_1(a,b,c,d;x,y)$, except in logarithmic cases, may be obtained from the connection formulas given by Olsson [19]. The purpose of this paper is to obtain asymptotic expansions (in the form of a simple series) of $F_1(a,b,c,d;x,y)$ for large values of the variables x and/or y and any (fixed) value of a,b,c,d. We face the challenge of obtaining easy algorithms to compute the coefficients of these expansions as well as error bounds at any order of the approximation.

The starting point is the integral representation [[8], p. 230]

$$F_1(a,b,c,d;x,y) \equiv \frac{\Gamma(d)}{\Gamma(d-a)\Gamma(a)} \int_0^1 s^{a-1}(1-s)^{d-a-1}(1-sx)^{-b}(1-sy)^{-c} ds \qquad (1)$$

where $\Re(a)>0$, $\Re(d-a)>0$, $x \notin [1,\infty)$ if $b \geq 1$, and $y \notin [1,\infty)$ if $c \geq 1$. This integral defines the analytical continuation of $F_1(a,b,c,d;x,y)$ to the cut complex x or $y-$planes $\mathcal{C} \setminus [1,\infty)$ [[25], p. 30, theorem 2.3].

The first step is to write the above integral as a generalized Stieltjes transform. For that purpose we perform the change of variable $s=(1+t)^{-1}$ in (1), obtaining:

$$F_1(a,b,c,d;x,y) = \frac{\Gamma(d)}{\Gamma(d-a)\Gamma(a)} \int_0^{\infty} \frac{f_y^F(t)}{(t+1-x)^b} dt, \qquad (2)$$

or

$$F_1(a,b,c,d;x,y) = \frac{\Gamma(d)}{\Gamma(d-a)\Gamma(a)} \int_0^\infty \frac{f^F(t)}{(t+1-x)^b(t+1-y)^c} dt, \tag{3}$$

where

$$f^F(t) \equiv t^{d-a-1}(1+t)^{b+c-d}, \qquad f_y^F(t) \equiv \frac{f^F(t)}{(t+1-y)^c}. \tag{4}$$

Then, up to a factor, the Appell's function F_1 is a generalized Stieltjes transform of $f^F(t)$ or $f_y^F(t)$. For $\Re(d-a) > 0$, $f^F(t)$ is a locally integrable function on $[0,\infty)$ and satisfies

$$f^F(t) = \sum_{k=0}^{n-1} \frac{A_k^F}{t^{k-b-c+a+1}} + f_n^F(t), \tag{5}$$

where

$$A_k^F \equiv \binom{b+c-d}{k} \tag{6}$$

and $f_n^F(t) = \mathcal{O}(t^{-n+b+c-a-1})$ when $t \to \infty$.

On the other hand, for $\Re(d-a) > 0$ and $y-1 \notin \mathbb{R}^+ \cup \{0\}$ if $\Re c \geq 1$, $f_y^F(t)$ is a locally integrable function on $[0,\infty)$ and satisfies

$$f_y^F(t) = \sum_{k=0}^{n-1} \frac{B_k^F}{t^{k-b+a+1}} + f_{y,n}^F(t), \tag{7}$$

where

$$B_k^F \equiv \sum_{j=0}^{k} A_{k-j}^F \binom{-c}{j} (1-y)^j \tag{8}$$

and $f_{y,n}^F(t) = \mathcal{O}(t^{-n+b-a-1})$ when $t \to \infty$.

Carlson obtained asymptotic expansions of the integrals (2) and (3) for large x and/or y by using Mellin transform techniques. In this paper we use for the integral (2) an alternative approach based on the theory of distributions introduced by Wong [27] [[28], chaps. 5, 6], Estrada [9] and Pilipović et all [21]. This approach has been generalized in [11] and [14] to be applied to integrals of the form (3). We go a little bit forward in this paper and, in section 2, we extend the distributional asymptotic methods for generalized Stieltjes transforms from the case of real parameters to the case of complex parameters. We include in this section theorems about error bounds. In section 3 we apply these methods to the integrals (2) and (3), obtaining asymptotic expansions with error bounds for both: large x and fixed y and large x and y. Several numerical examples are shown as illustrations. A brief summary and a few comments are postponed to section 4.

2. Distributional approach with complex parameters

Let $f(t)$ be a locally integrable function on $[0,\infty)$ which satisfies

$$f(t)=\sum_{k=K}^{n-1}\frac{a_k}{t^{k+s}}+f_n(t), \tag{9}$$

where $K\in\mathcal{Z}$, $0<\Re s\le 1$, $\{a_k,\ k=K,K+1,K+2,...\}$ is a sequence of complex numbers and $f_n(t)=\mathcal{O}(t^{-n-s})$ when $t\to\infty$.

Then, asymptotic expansions (including error bounds) of the generalized Stieltjes transforms of $f(t)$,

$$S_f(w;z)\equiv\int_0^\infty\frac{f(t)}{(t+z)^w}dt,\qquad S_f(w_1,w_2;z)\equiv\int_0^\infty\frac{f(t)}{(t+xz)^{w_1}(t+yz)^{w_2}}dt, \tag{10}$$

for large z and fixed x and y may be found in [[28], chap. 6], [[21], sec 4.4], [11], [13] and [14] for real w, w_1, w_2 and s and complex x, y, z. The purpose of this section is to generalize the theorems given there to the case of complex w, w_1, w_2 and s. Therefore, in the following, we consider that the parameters w, w_1, w_2, x, y and z are complex and that $f(t)$ is a locally integrable function on $[0,\infty)$ which satisfies (9). In the following, we use the notation introduced in [28].

2.1. Asymptotic expansion of $S_f(w;z)$ and $S_f(w_1,w_2;z)$ for large z

We denote by $\mathcal{S}$ the space of rapidly decreasing functions and by $<\mathbf{\Lambda},\varphi>$ the image of a tempered distribution $\mathbf{\Lambda}$ acting over a function $\varphi\in\mathcal{S}$. Since $f(t)$ in (9)-(10) is a locally integrable function on $[0,\infty)$, it defines a distribution $\mathbf{f}$:

$$<\mathbf{f},\varphi>\equiv\int_0^\infty f(t)\varphi(t)dt.$$

The distributions associated with t^{-k-s}, $k=0,1,2,...,n-1$ are given by [[28], chap. 5]

$$<\mathbf{t}_+^{-\mathbf{k}-\mathbf{s}},\varphi>\equiv\frac{1}{(s)_k}\int_0^\infty t^{-s}\varphi^{(k)}(t)dt\qquad \text{if}\ \ 0<\Re s<1, \tag{11}$$

$$<\mathbf{t}_+^{-\mathbf{k}-\mathbf{s}},\varphi>\equiv\frac{1}{(i\Im s)_{k+1}}\int_0^\infty t^{-i\Im s}\varphi^{(k+1)}(t)dt\qquad \text{if}\ \ 1\ne s=1+i\Im s, \tag{12}$$

where $(s)_k$ denotes the Pochhammer's symbol of s, and

$$<\mathbf{t}_+^{-\mathbf{k}-\mathbf{1}},\varphi>\equiv-\frac{1}{k!}\int_0^\infty\log(t)\varphi^{(k+1)}(t)dt. \tag{13}$$

Mention must be made here to the fact that these definitions of the distributions $\mathbf{t}_+^\alpha$ are different from the standard definition given by analytic continuation [12] or by using the Hadamard finite part concept [9]. By using these definitions, asymptotic expansions of Stieltjes transforms (10)(a) for small z may be derived from the moment asymptotic expansion [[9], p. 135]. In [[21], sec. 4.4] we can find asymptotic expansions of (10)(a) for large and small z. In the remaining of the paper we use Wong's definition (11)-(13) and consider the method constructed from this definition [[28], chap. 6].

To assign a distribution to the function $f_n(t)$ introduced in (9), we first define recursively the $k-$esim integral $f_{n,k}(t)$ of $f_n(t)$ by $f_{n,0}(t) \equiv f_n(t)$ and

$$f_{n,k+1}(t) \equiv -\int_t^\infty f_{n,k}(u)du = \frac{(-1)^{k+1}}{k!}\int_t^\infty (u-t)^k f_n(u)du. \tag{14}$$

For $s \neq 1$, it is trivial to show that $f_{n,n}(t)$ is bounded on $[0,T]$ for any $T>0$ and is $\mathcal{O}(t^{-s})$ as $t\to\infty$. For $s=1$ we have $f_{n,n}(t)=\mathcal{O}(t^{-1})$ as $t\to\infty$ and $f_{n,n}(t)=\mathcal{O}(\log(t))$ as $t\to 0^+$. Therefore, for $0<\Re s\leq 1$ we can define the distribution associated to $f_n(t)$ by

$$<\mathbf{f_n},\varphi>\equiv(-1)^n<\mathbf{f_{n,n}},\varphi^{(n)}>\equiv(-1)^n\int_0^\infty f_{n,n}(t)\varphi^{(n)}(t)dt.$$

Once we have assigned a distribution to each function involved in the identity (9), we are interested in finding an identity (if any) between these distributions. In fact, this relation is established in the following two lemmas.

Lemma 1. *For $0<\Re s<1$, $n\geq K+1$, and $n\in\mathbb{N}$, the identity*

$$\mathbf{f}=\sum_{k=K}^{n-1}a_k\mathbf{t}_+^{-\mathbf{k}-\mathbf{s}}+\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}M[f;k+1]\delta^{(\mathbf{k})}+\mathbf{f_n}$$

holds for any rapidly decreasing function $\varphi\in\mathcal{S}$, where δ is the delta distribution in the origin and $M[f;k+1]$ denotes de Mellin transform of $f(t)$: $\int_0^\infty t^k f(t)dt$, or its analytic continuation.

Proof. It is a trivial generalization of [[28], chap 6, lemma 1] from real to complex values of s. □

Lemma 2. *For $\Re s=1$, $n\geq K+1$ and $n\in\mathbb{N}$, the identity*

$$\mathbf{f}=\sum_{k=K}^{n-1}a_k\mathbf{t}_+^{-\mathbf{k}-\mathbf{s}}+\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}b_{k+1}\delta^{(\mathbf{k})}+\mathbf{f_n},$$

holds for any rapidly decreasing function $\varphi\in\mathcal{S}$, where, for $n=0,1,2,\ldots,$

$$\begin{aligned} b_{n+1} \equiv& M[f;n+1] \qquad \text{if } \Im s\neq 0 \quad \text{or} \\ b_{n+1} \equiv& \lim_{z\to n}\left[M[f;z+1]+\frac{a_n}{z-n}\right]+a_n(\gamma+\psi(n+1)) \qquad \text{if } \Im s=0, \end{aligned} \tag{15}$$

where γ is the Euler constant and ψ the digamma function.

Proof. Let $f_0(t) \equiv f(t) - \sum_{k=K}^{-1} a_k t^{-k-s}$. Then, for $n = 0, 1, 2, \ldots$,

$$f_{n+1}(t) = f_n(t) - \frac{a_n}{t^{n+s}}$$

and

$$f_{n+1,n}(t) = f_{n,n}(t) - (-1)^n \frac{a_n}{(s)_n} \frac{1}{t^s}.$$

From this, by integration, it follows that

$$\int_0^t f_{n,n}(u)du = f_{n+1,n+1}(t) + (-1)^n a_n g_n(s,t) + b_{n+1},$$

where

$$g_n(s,t) \equiv \begin{cases} \log(t)/n! & \text{if} \quad \Im s = 0 \\ -t^{-i\Im s}/(i\Im s)_{n+1} & \text{if} \quad \Im s \neq 0 \end{cases}$$

and where we have defined the integration constant

$$b_{n+1} = -\lim_{t\to 0} \left[f_{n+1,n+1}(t) + (-1)^n a_n g_n(s,t) \right].$$

From here, the proof is the same as the proofs of lemma 2 and theorem 2 in [[28], chapter 6] from formulas (2.21) and (2.35) respectively: just replace $\log t$ by $n! g_n(s,t)$ and d_{n+1} by $(-1)^n b_{n+1}/n!$ in those proofs and use the formula

$$\sum_{k=0}^{n} \frac{(n+1-k)_k}{(n+s-k-1)_{k+1}} = \frac{1}{s-1},$$

which follows from [[22], p. 608, eq. 25]. □

To apply lemmas 1 and 2 to the first integral in (10) we choose a specific function in $\mathcal{S}$:

$$\varphi_\eta(t) \equiv \frac{e^{-\eta t}}{(t+z)^w} \in \mathcal{S},$$

where $\eta > 0$ and $z \notin \mathbb{R}^- \cup \{0\}$ if $\Re w \geq 1$. We will need also the following lemma.

Lemma 3. *Let $f(t)$ verify (9). Then, for $0 < \Re s \leq 1$, $k = 0, 1, 2, \ldots$ and $n = 1, 2, 3, \ldots$, the following identities hold,*

$$\lim_{\eta\to 0} < \mathbf{f}, \varphi_\eta >= \int_0^\infty \frac{f(t)}{(t+z)^w} dt \qquad \text{for } \Re(s+w) + K > 1,$$

$$\lim_{\eta\to 0} < \delta, \varphi_\eta^{(k)} >= \frac{(-1)^k (w)_k}{z^{k+w}},$$

$$\lim_{\eta\to 0} < \mathbf{t}_+^{-\mathbf{s}}, \varphi_\eta^{(k)} >= \frac{(-1)^k \Gamma(k+w+s-1)\Gamma(1-s)}{\Gamma(w) z^{k+w+s-1}} \qquad \text{for } \Re(s+w) + k > 1, \quad s \neq 1,$$

$$\lim_{\eta\to 0} < \mathbf{log}(\mathbf{t}_+), \varphi_\eta^{(k+1)} >= \frac{(-1)^{k+1}}{z^{k+w}}(w)_k\big(\log(z)-\gamma-\psi(k+w)\big) \qquad \text{for } \Re(s+w)>0,$$

$$\lim_{\eta\to 0} < \mathbf{f_{n,n}}, \varphi_\eta^{(n)} >= (-1)^n (w)_n \int_0^\infty \frac{f_{n,n}(t)}{(t+z)^{n+w}}dt \qquad \text{for } \Re(s+w)+n>1.$$

Proof. It is a straightforward generalization of the proofs of [[13], lemma 2.6] and [[14], lemma 3] from real to complex values of s and w. □

In order to apply lemmas 1 and 2 to the second integral in (10), we must choose another particular function of $\mathcal{S}$,

$$\bar{\varphi}_\eta(t) \equiv \frac{e^{-\eta t}}{(t+xz)^{w_1}(t+yz)^{w_2}} \in \mathcal{S},$$

where $xz \notin \mathbb{R}^- \cup \{0\}$ if $\Re w_1 \geq 1$, $yz \notin \mathbb{R}^- \cup \{0\}$ if $\Re w_2 \geq 1$ and $\eta > 0$. We will need also the following lemma.

Lemma 4. *Let $f(t)$ verify (9). Then, for $0 < \Re s \leq 1$, $k = 0, 1, 2, ...$ and $n = 1, 2, 3, ...$, the following identities hold,*

$$\lim_{\eta\to 0} < \mathbf{f}, \bar{\varphi}_\eta >= \int_0^\infty \frac{f(t)}{(t+xz)^{w_1}(t+yz)^{w_2}}dt \qquad \text{for } \Re(s+w_1+w_2)+K>1.$$

$$\lim_{\eta\to 0} < \delta, \bar{\varphi}_\eta^{(k)} >= \frac{(-1)^k}{z^{k+w_1+w_2}} \sum_{j=0}^k \binom{k}{j} \frac{(w_1)_j (w_2)_{k-j}}{x^{w_1+j} y^{w_2+k-j}},$$

$$\lim_{\eta\to 0} < \mathbf{t}_+^{-\mathbf{s}}, \varphi_\eta^{(k)} >= \frac{\Gamma(1-s)\Gamma(k+w_1+w_2+s-1)}{\Gamma(w_1+w_2) z^{k+w_1+w_2+s-1}} \frac{(-1)^k}{x^{w_1+k+s-1} y^{w_2}} \times$$
$$F\left(\begin{array}{c} 1-s-k, w_2 \\ w_1+w_2 \end{array} \middle| 1-\frac{x}{y}\right) \qquad \text{for } \Re(s+w_1+w_2)+k>1, \quad s\neq 1,$$

where $F\left(\begin{array}{c} \alpha, \beta \\ \delta \end{array} \middle| z\right) \equiv {}_2F_1(\alpha,\beta,\delta;z)$ *denotes the Gauss hypergeometric function,*

$$\lim_{\eta\to 0} < \mathbf{log}(\mathbf{t}_+), \bar{\varphi}_\eta^{(k+1)} >= \frac{(-1)^{k+1}}{(k+w_1+w_2) z^{k+w_1+w_2}} \sum_{j=0}^{k+1} \binom{k+1}{j} \frac{(w_1)_j (w_2)_{k+1-j}}{x^{w_1+j-1} y^{w_2+k+1-j}} \times$$
$$\left[(\log(xz)-\gamma-\psi(k+w_1+w_2))\, F\left(\begin{array}{c} 1, k+1+w_2-j \\ k+1+w_1+w_2 \end{array} \middle| 1-\frac{x}{y}\right) + \right.$$
$$\left. F'\left(\begin{array}{c} 1, k+1+w_2-j \\ k+1+w_1+w_2 \end{array} \middle| 1-\frac{x}{y}\right)\right], \qquad \text{for } \Re(s+w_1+w_2)>0,$$

where $F'\left(\begin{array}{c} \alpha, \beta \\ \delta \end{array} \middle| z\right) \equiv \frac{d}{d\alpha}\, F\left(\begin{array}{c} \alpha, \beta \\ \delta \end{array} \middle| z\right)$ *and*

$$\lim_{\eta\to 0} < \mathbf{f_{n,n}}, \bar{\varphi}_\eta^{(n)} >= (-1)^n \sum_{j=0}^n \binom{n}{j} \int_0^\infty \frac{(w_1)_j (w_2)_{n-j} f_{n,n}(t)}{(t+xz)^{j+w_1}(t+yz)^{n-j+w_2}}dt$$

for $\Re(s+w_1+w_2)+n>1$.

Proof. It is a straightforward generalization of the proofs of [[14], lemma 4] from real to complex values of s, w_1 and w_2. □

With these preparations, we are now able to obtain asymptotic expansions of the integrals (10) for large z in the following theorems.

Theorem 1. *Let* $f(t)$ *be a locally integrable function on* $[0,\infty)$ *which satisfies (9) with* $0<\Re s\le 1$, $s\neq 1$. *Then, for* $z\in\mathcal{C}\setminus\mathbb{R}^-\cup\{0\}$, $\Re(s+w)+K>1$ *and* $n=1,2,3,\ldots$,

$$\int_0^\infty \frac{f(t)}{(t+z)^w}dt=\sum_{k=K}^{n-1}\frac{(-1)^k\pi a_k\Gamma(w+s+k-1)}{\Gamma(s+k)\Gamma(w)\sin(\pi s)z^{w+s+k-1}}+ \sum_{k=0}^{n-1}\frac{(-1)^k(w)_kM[f;k+1]}{k!z^{k+w}}+R_n(w;z). \tag{16}$$

The remainder term is defined by

$$R_n(w;z)\equiv(w)_n\int_0^\infty\frac{f_{n,n}(t)dt}{(t+z)^{n+w}}, \tag{17}$$

empty sums must be understood as zero and $f_{n,n}(t)$ *is defined in (14).*

Proof. For $\Re s\neq 1$ it follows from lemmas 1 and 3 using the reflection formula of the gamma function. For $\Re s=1$, from lemmas 2 and 3 we obtain immediately formula (16). □

Theorem 2. *Let* $f(t)$ *be a locally integrable function on* $[0,\infty)$ *which satisfies (9) with* $s=1$. *Then, for* $z\in\mathcal{C}\setminus\mathbb{R}^-\cup\{0\}$, $\Re w+K>0$ *and* $n=1,2,3,\ldots$,

$$\int_0^\infty \frac{f(t)}{(t+z)^w}dt=\sum_{k=K}^{-1}a_k\frac{\Gamma(w+k)\Gamma(-k)}{\Gamma(w)z^{w+k}}+\sum_{k=0}^{n-1}\frac{(-1)^k(w)_k}{k!z^{k+w}}\times \left[a_k\big(\log(z)-\gamma-\psi(k+w)\big)+b_{k+1}\right]+R_n(w;z), \tag{18}$$

where, for $k=0,1,2,\ldots$, *the coefficients* b_{k+1} *are given by (15)(b). The remainder term* $R_n(w;z)$ *is given in (17).*

Proof. From lemmas 2 and 3 we obtain immediately formulas (17) and (18) with b_{k+1} given in formula (15)(b). □

Theorem 3. *Let* $f(t)$ *be as in theorem 1. Then, for* $xz,yz\in\mathcal{C}\setminus\mathbb{R}^-\cup\{0\}$, $\Re w_1,\Re w_2>0$, $\Re(s+w_1+w_2)+K>1$ *and* $n=1,2,3,\ldots$,

$$\int_0^\infty\frac{f(t)}{(t+xz)^{w_1}(t+yz)^{w_2}}dt=\sum_{k=K}^{n-1}\frac{A_k}{z^{w_1+w_2+s+k-1}}+\sum_{k=0}^{n-1}\frac{B_k}{z^{k+w_1+w_2}}+R_{n,s}(w_1,w_2;z), \tag{19}$$

where the coefficients A_k *and* B_k *are defined by*

$$A_k\equiv a_k\frac{\Gamma(1-s-k)\Gamma(w_1+w_2+s+k-1)}{\Gamma(w_1+w_2)x^{w_1+s+k-1}y^{w_2}}F\left(\begin{array}{c}1-s-k,w_2\\ w_1+w_2\end{array}\middle|\,1-\frac{x}{y}\right),$$

$$B_k \equiv \frac{(-1)^k M[f;k+1]}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(w_1)_j (w_2)_{k-j}}{x^{w_1+j} y^{k+w_2-j}},$$

and empty sums must be understood as zero. The remainder term is defined by

$$R_{n,s}(w_1,w_2;z) \equiv \sum_{j=0}^{n} \binom{n}{j} (w_1)_j (w_2)_{n-j} \int_0^\infty \frac{f_{n,n}(t)dt}{(t+xz)^{j+w_1}(t+yz)^{n+w_2-j}}, \tag{20}$$

where $f_{n,n}(t)$ is defined in (14).

Proof. It is similar to the proof of theorem 1, but using lemma 4 instead of lemma 3. □

Theorem 4. *Let $f(t)$ be as in theorem 2. Then, for $xz, yz \in \mathbb{C} \setminus \mathbb{R}^- \cup \{0\}$, $\Re w_1, \Re w_2 > 0$, $\Re(w_1+w_2)+K>0$ and $n = 1, 2, 3, ...,$*

$$\int_0^\infty \frac{f(t)}{(t+xz)^{w_1}(t+yz)^{w_2}} dt = \sum_{k=K}^{-1} \frac{A_k}{z^{w_1+w_2+k}} + \sum_{k=0}^{n-1} \frac{(-1)^k}{k! z^{k+w_1+w_2}} \Big[B_k \big(\log(xz) - \gamma - \psi(k+w_1+w_2) \big) + B'_k + C_k \Big] + R_{n,s}(w_1,w_2;z), \tag{21}$$

where empty sums must be understood as zero,

$$A_k \equiv a_k \frac{\Gamma(-k)\Gamma(w_1+w_2+k)}{\Gamma(w_1+w_2) x^{w_1+k} y^{w_2}} F\left(\begin{matrix} -k, w_2 \\ w_1+w_2 \end{matrix} \middle| 1-\frac{x}{y} \right),$$

$$B_k \equiv \frac{a_k}{(k+w_1+w_2)} \sum_{j=0}^{k+1} \binom{k+1}{j} \frac{(w_1)_j (w_2)_{k+1-j}}{x^{w_1+j-1} y^{w_2+k+1-j}} F\left(\begin{matrix} 1, k+1+w_2-j \\ k+1+w_1+w_2 \end{matrix} \middle| 1-\frac{x}{y} \right),$$

$$B'_k \equiv \frac{a_k}{(k+w_1+w_2)} \sum_{j=0}^{k+1} \binom{k+1}{j} \frac{(w_1)_j (w_2)_{k+1-j}}{x^{w_1+j-1} y^{w_2+k+1-j}} F'\left(\begin{matrix} 1, k+1+w_2-j \\ k+1+w_1+w_2 \end{matrix} \middle| 1-\frac{x}{y} \right),$$

and

$$C_k \equiv b_{k+1} \sum_{j=0}^{k} \binom{k}{j} \frac{(w_1)_j (w_2)_{k-j}}{x^{w_1+j} y^{k+w_2-j}},$$

where b_{k+1} is given in (15)(b). The remainder term $R_{n,s}(w_1,w_2;z)$ is given in (20).

Proof. The proof is similar to the proof of theorem 2, but using lemma 4 instead of lemma 3. □

2.2. Error bounds

In the following theorem we show that the expansions (16), (18), (19) and (21) given in the above theorems are in fact asymptotic expansions for large z.

Theorem 5. *In the region of validity of the expansions (16), (18), (19) and (21), the remainder terms $R_{n,s}(w;z)$ and $R_{n,s}(w_1,w_2;z)$ in these expansions verify,*

$$|R_{n,s}(w;z)|\le \frac{C_n}{|z|^{n+\Re s+\Re w-1}}, \qquad |R_{n,s}(w_1,w_2;z)|\le \frac{C_n}{|z|^{n+\Re(s+w_1+w_2)-1}}$$

if $0<\Re s<1$ *and*

$$|R_{n,s}(w;z)|\le \frac{C_n\log|z|}{|z|^{n+\Re w}}, \qquad |R_{n,s}(w_1,w_2;z)|\le \frac{C_n\log|z|}{|z|^{n+\Re w_1+\Re w_2}}$$

if $\Re s=1$*, where the constants* C_n *are independent of* $|z|$ *(it may depend on the remaining parameters of the problem).*

Proof. It is a straightforward generalization of the proof of [[11], theorem 5] to the case of complex parameters. □

The bounds given in theorem 5 are not useful for numerical computations unless we are able to calculate the constants C_n in terms of the data of the problem (w, w_1, w_2, x, y, $\mathrm{Arg}(z)$ and $f(t)$). The property $f_n(t)=\mathcal{O}(t^{-n-s})$ as $t\to\infty$ implies that $\exists\ t_0>0$ and $c_n>0$, $|f_n(t)|\le c_n t^{-n-\Re s}\ \forall\ t\in[t_0,\infty)$. The following two propositions show that, if the bound $|f_n(t)|\le c_n t^{-n-\Re s}$ holds $\forall\ t\in[0,\infty)$ then, the constants C_n in theorem 5 can be calculated in terms of this constant c_n.

Proposition 1. *If, for* $0<\Re s<1$*, the remainder* $f_n(t)$ *in the expansion (9) of the function* $f(t)$ *satisfies the bound* $|f_n(t)|\le c_n t^{-n-\Re s}\ \forall\ t\in[0,\infty)$ *for some positive constant* c_n *then, the remainder* $R_{n,s}(w;z)$ *in the expansion (16) satisfies*

$$|R_{n,s}(w;z)|\le \frac{c_n\pi(|w|)_n\Gamma(n+\Re w+\Re s-1)h(z,w)}{\Gamma(n+\Re s)\Gamma(n+\Re w)|\sin(\pi\Re s)||z|^{n+\Re w+\Re s-1}}\times F\left(\left.\begin{array}{c}1-\Re s, n+\Re s+\Re w-1\\(n+\Re w+1)/2\end{array}\right|\sin^2\left(\frac{\mathrm{Arg}(z)}{2}\right)\right)$$

and the remainder $R_{n,s}(w_1,w_2;z)$ *in the expansion (19) satisfies*

$$|R_{n,s}(w_1,w_2;z)|\le \frac{c_n\pi(|w_1|+|w_2|)_n\Gamma(n+\Re(w_1+w_2+s)-1)h(xz,w_1)h(yz,w_2)}{\Gamma(n+\Re s)\Gamma(n+\Re w_1+\Re w_2)|\sin(\pi\Re s)||vz|^{n+\Re(w_1+w_2+s)-1}}\times F\left(\left.\begin{array}{c}1-\Re s, n+\Re(s+w_1+w_2)-1\\(n+\Re w_1+\Re w_2+1)/2\end{array}\right|\frac{1}{2}\left(1-\frac{r}{|vz|}\right)\right),$$

where

$$v\equiv \mathrm{Min}\{|x|,|y|\}, \qquad r\equiv \mathrm{Min}\{\Re(xz),\Re(yz)\} \tag{22}$$

and

$$h(z,w)\equiv\begin{cases}1 & \text{if}\quad \mathrm{Arg}(z)\Im w\ge 0\\ e^{|\mathrm{Arg}(z)\Im w|} & \text{if}\quad \mathrm{Arg}(z)\Im w<0.\end{cases} \tag{23}$$

Proof. Introducing the bound $|f_n(t)| \le c_n t^{-n-\Re s}$ in the definition (14) of $f_{n,n}(t)$ we obtain

$$|f_{n,n}(t)| \le \frac{c_n\Gamma(\Re s)}{\Gamma(n+\Re s)t^{\Re s}} \qquad \forall \; t \in [0,\infty).$$

Introducing this bound in the definition (17) of $R_{n,s}(w;z)$ and using the duplication formula of the gamma function and [[22], p. 309, eq. 7] we obtain the first bound. The second bound is obtained using the inequalities $|t+xz|^2$, $|t+yz|^2 \ge t^2+2rt+|vz|^2$ in the definition (20) of $R_{n,s}(w_1,w_2;z)$, formula [[22], p. 309, eq. 7] and the equality

$$\sum_{k=0}^{n}\binom{n}{k}(|w_1|)_k(|w_2|)_{n-k} = (|w_1|+|w_2|)_n. \tag{24}$$

□

Proposition 2. *If, for $\Re s = 1$, each remainder $f_n(t)$ in the expansion (9) of the function $f(t)$ satisfies the bound $|f_n(t)| \le c_n t^{-n-1}$ $\forall$ $t \in [0,\infty)$ for some positive constant c_n then, the remainder $R_{n,s}(w;z)$ in the expansions (16) and (18) satisfies*

$$\begin{aligned}|R_{n,s}(w;z)| \le & \frac{\bar{c}_n\pi(|w|)_n\Gamma(n+\Re w-1/2)h(z,w)}{\Gamma(n+1/2)\Gamma(n+\Re w)|z|^{n+\Re w-1/2}}\times \\ & F\left(\begin{array}{c}1/2, n+\Re w-1/2\\ (n+\Re w+1)/2\end{array}\middle| \sin^2\left(\frac{\mathrm{Arg}(z)}{2}\right)\right) \equiv \mathbf{R}_n^{(1)}(w;z),\end{aligned} \tag{25}$$

where $\bar{c}_n \equiv$Max$\{c_n, c_{n-1}+|a_{n-1}|\}$ and

$$\begin{aligned}|R_{n,s}(w;z)| \le \frac{(|w|)_n}{|z|^{n+\Re w}}\Bigg\{&\frac{\epsilon(c_{n-1}+|a_{n-1}|)+c_n}{(n-1)!\Theta(z,\epsilon)^{n+\Re w}} + \frac{c_n}{n!}\left|1+\frac{\epsilon}{z}\right|^{-n-\Re w}\Bigg[\log|z| + \\ &\frac{(n+\Re w)[(2\epsilon+\Re z+|\Re z|)(|z|^{-1}-1)+(|\Re z|-\Re z)\log|z|]}{2(n+\Re w+1)|z+\epsilon|}H_1 + \\ &\frac{4\epsilon+\Re z+|\Re z|-2\epsilon|z|}{2\epsilon(n+\Re w+1)|z|}H_0 + \frac{2|\epsilon+z|H_{-1}}{\epsilon((n+\Re w)^2-1)|z|}\Bigg]\Bigg\}h(z,w) \equiv \mathbf{R}_n^{(2)}(w;z),\end{aligned} \tag{26}$$

where ϵ is an arbitrary positive number,

$$H_m \equiv F\left(\begin{array}{c}2-m, n+\Re w+m\\ (n+\Re w+3)/2\end{array}\middle| \sin^2\left(\frac{\mathrm{Arg}(z+\epsilon)}{2}\right)\right) \tag{27}$$

and

$$\Theta(z,\epsilon) \equiv \left\{\begin{array}{ccc} 1 & \text{if} & \Re z \ge 0 \\ |\sin(\mathrm{Arg}(z)| & \text{if} & \epsilon \ge -\Re z > 0. \\ |1+\epsilon/z| & \text{if} & -\Re z > \epsilon > 0. \end{array}\right. \tag{28}$$

For large z and fixed n, the optimum value for ϵ is given approximately by

$$\epsilon^2 = \frac{c_n}{n(c_{n-1}+|a_{n-1}|)}\left[\frac{2H_{-1}}{(n+\Re w)^2-1} + \frac{(\Re z+|\Re z|)H_0}{2(n+\Re w+1)|z|}\right]. \tag{29}$$

The remainder $R_{n,s}(w_1,w_2;z)$ in expansions (19) and (21) satisfies

$$|R_{n,s}(w_1,w_2;z)|\le \mathbf{R}_n^{(i)}(w_1+w_2;xz)+\mathbf{R}_n^{(i)}(w_1+w_2;yz), \tag{30}$$

With either $i=1$ or $i=2$. If x, y, z, w, w_1 and w_2 are positive real numbers, then

$$|R_{n,s}(w;z)|\le \left[n\epsilon(c_{n-1}+|a_{n-1}|)+c_n\big(S_n(z,\epsilon,w)+T_n(z,\epsilon,w)\big)\right]\frac{(w)_n}{n!z^{n+w}}, \tag{31}$$

where ϵ is again an arbitrary positive number,

$$S_n(z,\epsilon,w)\equiv \text{Min}\left\{\frac{nz\left[(\epsilon+z)^{n+w-1}-z^{n+w-1}\right]}{\epsilon(n+w-1)(\epsilon+z)^{n+w-1}},\psi(n+1)+\gamma\right\}$$

and

$$T_n(z,\epsilon,w)\equiv\frac{z^{n+w}}{(n+w)(\epsilon+z)^{n+w}}F\left(n+w,1;n+w+1;\frac{z}{\epsilon+z}\right).$$

For large z and fixed n, the optimum value for ϵ is given by

$$\epsilon=\frac{c_n}{n(c_{n-1}+|a_{n-1}|)}. \tag{32}$$

The remainder $R_{n,s}(w_1,w_2;z)$ in expansions (19) and (21) satisfies the bound (31) replacing w by w_1+w_2 and z by vz, with $v\equiv Min\{|x|,|y|\}$.

Proof. From $|f_{n-1}(t)|\le c_{n-1}t^{-n}$ and $f_n(t)=f_{n-1}(t)-a_{n-1}t^{-n}$ $\forall\ t\in[0,\infty)$, $n\in\mathbb{N}$, we obtain $|f_n(t)|\le(c_{n-1}+|a_{n-1}|)t^{-n}$ $\forall\ t\in[0,\infty)$. In order to obtain the bound (26) we divide the integral defining $f_{n,n}(t)$ in (14) by a fixed point $u=\epsilon\ge t$ and use the bound $|f_n(t)|\le(c_{n-1}+|a_{n-1}|)t^{-n}$ in the integral over $[t,\epsilon]$ and the bound $|f_n(t)|\le c_nt^{-n-1}$ in the integral over $[\epsilon,\infty)$. Using $u-t\le u$ in the integral over $[t,\epsilon]$ we obtain

$$|f_{n,n}(t)|\le\frac{1}{(n-1)!}\left[(c_{n-1}+|a_{n-1}|)\log\left(\frac{\epsilon}{t}\right)+\frac{c_n}{\epsilon}\right]\qquad\forall\ \ t\in[0,\epsilon],\qquad \epsilon>0. \tag{33}$$

On the other hand, $\forall\ t\in[0,\infty)$ we introduce the bound $|f_n(t)|\le c_nt^{-n-1}$ in the integral definition of $f_{n,n}(t)$ and perform the change of variable $u=tv$. We obtain

$$|f_{n,n}(t)|\le\frac{c_n}{n!}\frac{1}{t}\qquad\forall\qquad t\in[0,\infty). \tag{34}$$

We divide the integral in the right hand side of (17) at the point $t=\epsilon$ and use the bound (34) in the integral over $[\epsilon,\infty)$ and the bound (33) in the integral over $[0,\epsilon]$. We obtain

$$|R_{n,s}(w;z)|\le\frac{(|w|)_n}{n!}\left[nc_n\int_0^1\frac{dt}{|\epsilon t+z|^{n+\Re w}}+c_n\int_1^\infty\frac{dt}{t|\epsilon t+z|^{n+\Re w}}+\right.$$
$$\left.n\epsilon(c_{n-1}+|a_{n-1}|)\int_0^1\frac{\log(t^{-1})dt}{|\epsilon t+z|^{n+\Re w}}\right]h(z,w). \tag{35}$$

Removing a factor $|z|^{n+\Re w}$ from the denominator in the integrand of the first and third integrals in the right hand side of (35) and using the bound $|\epsilon t/z+1|\geq\Theta(z,\epsilon)$ we easily obtain that those two integrals are bounded by $(|z|\Theta(z,\epsilon))^{-n-\Re w}$. On the other hand, we perform the change of variable $t\to|z|t$ in the second integral in the right hand side of (35) and integrate by parts to obtain

$$|z|^{n+\Re w}\int_1^\infty\frac{dt}{t|\epsilon t+z|^{n+\Re w}}=\frac{\log|z|}{|1+\epsilon/z|^{n+\Re w}}+$$
$$\epsilon(n+\Re w)\int_{|z|^{-1}}^\infty\frac{(\epsilon t+\cos(\text{Arg}(z)))\log t dt}{[(\epsilon t+\cos(\text{Arg}(z)))^2+\sin^2(\text{Arg}(z))]^{(n+\Re w)/2+1}}.$$

Now, with the change of variable $t\to t/\epsilon+|z|^{-1}$ and using $-\log|z|\leq\log(t/\epsilon+|z|^{-1})\leq t/\epsilon+|z|^{-1}-1\ \forall\ t\in[0,\infty)$ and [[22], p. 309, eq. 7] we obtain (26).

To obtain (25) we use $|f_n(t)|\leq c_nt^{-n-1}$ and $|f_n(t)|\leq(c_{n-1}+|a_{n-1}|)t^{-n}$. Then, we have $f_n(t)\leq c_nt^{-n-1/2}$ if $t\geq1$ and $f_n(t)\leq(c_{n-1}+|a_{n-1}|)t^{-n-1/2}$ if $t\leq1$. Therefore, $f_n(t)\leq\bar{c}_nt^{-n-1/2}\ \forall\ t\in[0,\infty)$. Then, $f_n(t)$ satisfies the bound required in proposition 1 with $\Re s=1/2$ and c_n replaced by $\bar{c}_n$. Repeating now the calculations of the proof of proposition 1 we obtain (25).

If we get rid of irrelevant terms for large z, the right hand side of (26), as function of ϵ, has a minimum for ϵ given in (29).

Bounds (30) are obtained using the inequality $|t+xz|^{-\Re w_1}|t+yz|^{-\Re w_2}\leq|t+xz|^{-\Re w_1-\Re w_2}+|t+yz|^{-\Re w_1-\Re w_2}$ in the definition (20) of $R_{n,s}(w_1,w_2;z)$ and formulas (24), (25) and (26).

Bounds (31)-(32) and the bound for $R_{n,1}(w_1,w_2;z)$ for real positive x, y, w, w_1, w_2 and z have been obtained in [[14], propositions 2 and 4]. □

The following two lemmas introduce two families of functions $f(t)$ which verify the bound $|f_n(t)|\leq c_nt^{-n-\Re s}\ \forall\ t\in[0,\infty)$. Moreover, for these functions $f(t)$, the constants c_n can be easily obtained from $f(t)$.

Lemma 5. *Suppose $f(t)$ verifies (9) with $\Re s>0$ and consider the function $g(u)\equiv u^{-s-K}f(u^{-1})$. If $g(z)$ is a bounded analytic function in the region W of the complex $z-$plane comprised by the points situated at a distance $<\sigma$ from the positive real axis (see fig. 1), then,*

$$|f_n(t)|\leq Cr^{-n}t^{-n-\Re s},$$

where C is a bound of $|g(z)|$ in W and $0<r<\sigma$.

Proof. From the asymptotic expansion (9) and the Lagrange formula for the remainder in the Taylor expansion of $g(u)$ at $u=0$, we have

$$g(u)=\sum_{k=0}^{n-1}a_ku^k+R_n(u),$$

where

$$R_n(u)=\frac{1}{n!}\left.\frac{d^ng(u)}{du^n}\right|_{u=\xi}u^n,\qquad\qquad \xi\in(0,u).$$

Using the Cauchy formula for the derivative of an analytic function,

$$\frac{d^n g(u)}{du^n} = \frac{n!}{2\pi i}\int_{\mathcal{C}} \frac{g(z)}{(z-\xi)^{n+1}}dz,$$

where $\mathcal{C}$ is a circle of radius r around ξ contained into the region W. Then, for fixed ξ and r, performing the change of variable $z = \xi + re^{i\theta}$, and using $|g(\xi + re^{i\theta})| \le C$ for $\theta \in [0, 2\pi)$ with C independent of θ, r and ξ, we obtain the wished result. $\square$

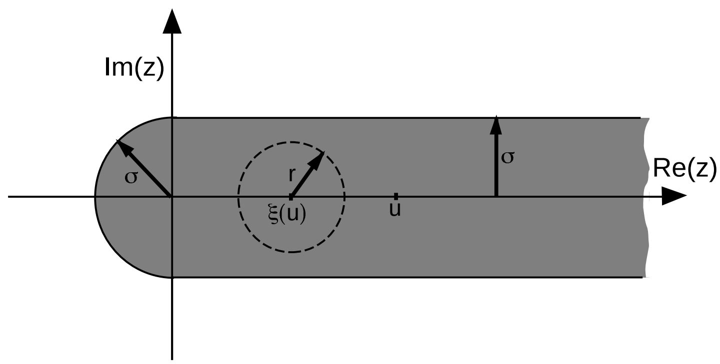

Figure 1. *Analyticity region W for the function $g(z)$ considered in lemma 5. The integration variable u in (14) is real and unbounded and therefore, the analyticity region for $g(z)$ must contain the positive real axis. The circle of radius r centered at $\xi(u)$, with $0 < \xi(u) < u$, used in the proof of lemma 5 must be contained in this region and therefore, $r < \sigma$.*

Lemma 6. *If the expansion (9) verifies the error test, then*

$$|f_n(t)| \le |a_n| t^{-n-\Re s} \qquad \text{and} \qquad |f_n(t)| \le |a_{n-1}| t^{-(n-1)-\Re s}.$$

Proof. A proof of the first inequality can be found in [[20], p. 68]. The second inequality follows from the first one, from $\text{sign}(f_n(t)) \neq \text{sign}(f_{n-1}(t))$ and

$$f_n(t) = f_{n-1}(t) - \frac{a_{n-1}}{t^{n-1+s}}.$$

$\square$

Corollary 1. *If $f(t)$ verifies the hypotheses of lemma 5, then $R_{n,s}(w;z)$ and $R_{n,s}(w_1, w_2; z)$ satisfy the bounds given in propositions 1 and 2 with $c_n = Cr^{-n}$. Moreover, the expansions given in theorems 1 and 2 are convergent when the parameter $|z|$ is longer than the inverse of the width of the region considered in lemma 5 (see figure 1): when $\sigma|z| \ge 1$ if $\Re w < 1$ or $\sigma|z| > 1$ if $\Re w \ge 1$. The expansions given in theorems 3 and 4 are convergent when the parameter $|vz|$, with $v \equiv Min\{|x|, |y|\}$, is longer than the inverse of the width of that region: when $\sigma|vz| \ge 1$ if $\Re w_1 + \Re w_2 < 1$ or $\sigma|vz| > 1$ if $\Re w_1 + \Re w_2 \ge 1$.*

For $\Re s = 1$, *the convergence of these expansions requires also that* $\lim_{n\to\infty} n^{w-1} a_n z^{-n} = 0$ *or* $\lim_{n\to\infty} n^{w_1+w_2-1} a_n (vz)^{-n} = 0$ *respectively.*

Corollary 2. *If the expansion (9) of* $f(t)$ *verifies the error test, then* $R_n(w;z)$ *and* $R_{n,s}(w_1, w_2; z)$ *satisfy the bounds given in propositions 1 and 2 replacing* c_n *by* $|a_n|$ *and* c_{n-1} *by* 0. *Moreover, the expansions given in theorems 1 and 2 are convergent when the coefficients* a_n *in the asymptotic expansion (9) verify* $\lim_{n\to\infty} n^{w-1} a_n z^{-n} = 0$. *The expansions given in theorems 3 and 4 are convergent when* $\lim_{n\to\infty} n^{w_1+w_2-1} a_n (vz)^{-n} = 0$, $v \equiv Min\{|x|, |y|\}$.

3. Asymptotic expansions of the Appell's function F_1

In order to obtain asymptotic expansions of $F_1(a, b, c, d; x, y)$ for $|x| \to \infty$ and/or $|y| \to \infty$ we just apply theorems 1-4 to the integrals (2) or (3). Error bounds for the remainders are obtained from corollaries 1 and 2.

Corollary 3. *For* $\Re a > 0$, $\Re(d-a) > 0$, $1+a-b \notin \mathbb{Z}$, $|Arg(z)| < \pi$ *and* $y-1 \notin \mathbb{R}^+ \cup \{0\}$ *if* $\Re c \geq 1$,

$$F_1(a, b, c, d; 1-z, y) = \frac{\Gamma(d)}{\Gamma(d-a)\Gamma(a)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma(b+k)}{\Gamma(b)} \frac{(-1)^k B_k}{k! z^{k+b}} + \frac{\pi(-1)^K}{\Gamma(b)\sin(\pi s)} \sum_{k=0}^{n-K-1} \frac{(-1)^k \Gamma(k+a)}{\Gamma(k-b+a+1)} \frac{B_k^F}{z^{k+a}} + R_n(a, b, c, d; z, y) \right\}, \tag{36}$$

where $K \equiv \mathrm{Int}(\Re(1-b+a))$ *(*$Int(x)$ *means the integer part of* x*) and the coefficients* B_k^F *are defined in (8). Coefficients* B_k *are given by*

$$B_k \equiv \frac{\Gamma(k+d-a)\Gamma(a-k-b)}{\Gamma(d-b)(1-y)^c} F\left(\begin{matrix} c, k+d-a \\ d-b \end{matrix} \middle| \frac{y}{y-1} \right). \tag{37}$$

If $\Re(1+a-b) \notin \mathbb{Z}$ *and* $n \geq 0$, *a bound for the remainder is given by*

$$|R_n(a, b, c, d; z, y)| \leq \frac{c_n \pi (|b|)_n \Gamma(n + \Re(b+s) - 1) h(z, b)}{\Gamma(n+\Re s)\Gamma(n+\Re b)|\sin(\pi \Re s)||z|^{n+\Re(b+s)-1}} \times F\left(\begin{matrix} 1-\Re s, n+\Re(s+b)-1 \\ (n+\Re b+1)/2 \end{matrix} \middle| \sin^2\left(\frac{\mathrm{Arg}(z)}{2}\right) \right), \tag{38}$$

where $s \equiv 1+a-b-K$ *and* $h(z, b)$ *was defined in (23). We can take* $c_n = |B_{n-K}^F|$ *if the following conditions over the parameters hold:*

$$a, b, c, d \in \mathbb{R}, \quad b+c-d \leq 0, \quad c \geq 0, \quad \Re(1-y) > 0. \tag{39}$$

In any case, we can take $c_n = C_y r^{-n}$, *where*

$$C_y \geq \mathrm{Sup}_{u\in W} \left| (1+u)^{b+c-d} (1+(1-y)u)^{-c} \right|, \tag{40}$$

W is the region considered in lemma 5 for $g(u)=u^{b-a-1}f_y^F(u^{-1})$ with

$$0<r<\text{Min}\left\{1,|1-y|^{-1}\xi(c)\right\}, \qquad \xi(c)\equiv\begin{cases}1 & \text{if} \quad c\notin \mathbb{Z}^-\cup\{0\}\\ +\infty & \text{if} \quad c\in \mathbb{Z}^-\cup\{0\}.\end{cases} \tag{41}$$

On the other hand, if $\Re(1+a-b)\in\mathbb{Z}$ and $n\geq 0$, $n\in\mathbb{N}$, two bounds for the remainder are given by

$$\begin{aligned}|R_n(a,b,c,d;z,y)|\leq\frac{\bar{c}_n\pi(|b|)_n\Gamma(n+\Re b-1/2)h(z,b)}{\Gamma(n+1/2)\Gamma(n+\Re b)|z|^{n+\Re b-1/2}}\times\\ F\left(\begin{array}{c}1/2,n+\Re b-1/2\\(n+\Re b+1)/2\end{array}\middle|\sin^2\left(\frac{\text{Arg}(z)}{2}\right)\right)\equiv\mathbf{R}_n^{(1)}(B_n^F,b;z)\end{aligned} \tag{42}$$

and

$$\begin{aligned}|R_n(a,b,c,d;z,y)|\leq\frac{(|b|)_n}{|z|^{n+\Re b}}\left\{\frac{c_n}{n!}\left|1+\frac{\epsilon}{z}\right|^{-n-\Re b}\left[\log|z|+\right.\right.\\ \frac{(n+\Re b)[(2\epsilon+\Re z+|\Re z|)(|z|^{-1}-1)+(|\Re z|-\Re z)\log|z|]}{2(n+\Re b+1)|z+\epsilon|}H_1+\\ \left.\frac{4\epsilon+\Re z+|\Re z|-2\epsilon|z|}{2\epsilon(n+\Re b+1)|z|}H_0+\frac{2|\epsilon+z|}{\epsilon((n+\Re b)^2-1)|z|}H_{-1}\right]+\\ \left.\frac{\epsilon(c_{n-1}+|B_{n-K-1}^F|)+c_n}{(n-1)!\Theta(z,\epsilon)^{n+\Re b}}\right\}h(z,b)\equiv\mathbf{R}_n^{(2)}(C_yr^{-n},B_n^F,b;z).\end{aligned} \tag{43}$$

In these formulas, $\bar{c}_n$ =Max$\{|B_{n-K}^F|,|B_{n-K-1}^F|\}$ with $c_n=|B_{n-K}^F|$ and $c_{n-1}=0$ if conditions (39) hold. In any case, we can take $\bar{c}_n$ =Max$\{c_n,c_{n-1}+|B_{n-K-1}^F|\}$, with $c_n=C_yr^{-n}$ given above. In (43), ϵ is an arbitrary positive number, $\Theta(z,\epsilon)$ is given in (28) and H_k is given in (27) setting $w=b$. For large z and fixed n, the optimum value for ϵ is given approximately by (29) setting $w=b$. Moreover, the expansion (36) is convergent when Max$\{|1-y|\xi(c)^{-1},1\}<|z|$.

Proof. To obtain the expansion (36), just apply theorem 1 to the integral (2) with $f(t)=f_y^F(t)$ given in (4), $a_k=B_{k-K}^F$ given in (8), $w=b$ and s and K given above. After the change of variable $t=u(1-u)^{-1}$, the mellin transform of $f_y^F(t)$ reads

$$M[f_y^F;k+1]=(1-y)^{-c}\int_0^1\frac{u^{k+d-a-1}}{(1-u)^{k+b-a+1}}\left(1+\frac{y}{1-y}u\right)^{-c}du.$$

Then, the first term in (37) follows from [[22], p. 306, eq. 5].

If (39) holds, then, by [[13], lemmas 3 and 4], the function $f_y^F(t)$ verifies the error test. Therefore, by corollary 2, the remainder in the expansion (36) verifies the bounds given in propositions 1 and 2 with $c_n=|B_{n-K}^F|$, $c_{n-1}=0$. In any case, by lemma 5 and corollary 1, the remainder in the expansion (36) verifies the bounds given in propositions 1 and 2 with $c_n=C_yr^{-n}$, C_y and r verifying (40) and (41) respectively. Therefore, the bounds (38), (42) and (43) hold.

Using (41) and introducing in (38) and (42) the bound

$$|B_n^F| \leq M \left| \begin{pmatrix} b-d \\ n \end{pmatrix} \right| \left[\text{Max}\left\{1, |1-y|\xi(c)^{-1}\right\}\right]^n,$$

where M is a constant independent of n, we obtain that $\lim_{n\to\infty} R_n(a,b,c,d;z,y) = 0$ if $\text{Max}\{|1-y|\xi(c)^{-1}, 1\} < |z|$. □

Corollary 4. *For $\Re a$, $\Re(d-a) > 0$, $y-1 \notin \mathbb{R}^+ \cup \{0\}$ if $\Re c \geq 1$, $1+a-b \in \mathbb{Z}$ and $|\text{Arg}(z)| < \pi$,*

$$\begin{aligned} F_1(a,b,c,d;1-z,y) = \frac{\Gamma(d)}{\Gamma(d-a)\Gamma(a)} \left\{ \sum_{k=0}^{b-a-1} B_k^F \frac{\Gamma(k+a)\Gamma(b-k-a)}{\Gamma(b)z^{k+a}} + \right. \\ \left. \sum_{k=0}^{n-1} \frac{(-1)^k (b)_k}{k! z^{k+b}} \left[B_{k+b-a}^F \big(\log(z) - \gamma - \psi(k+b)\big) + B_k \right] + R_n(a,b,c,d;z,y) \right\}, \end{aligned} \tag{44}$$

where the coefficients B_k^F are given in (8) and the coefficients B_k are given by

$$\begin{aligned} B_k \equiv & B_{k+b-a}^F (\gamma + \psi(k+1)) + \frac{(-1)^{k+b-a-1}\Gamma(k+d-a)}{\Gamma(d-b)(k+b-a)!(1-y)^c} \times \\ & \left\{ [\psi(k+d-a) - \psi(k+b-a+1)]\, F\left(\begin{matrix} c, k+d-a \\ d-b \end{matrix} \middle| \frac{y}{y-1} \right) + \right. \\ & \left. F'\left(\begin{matrix} c, k+d-a \\ d-b \end{matrix} \middle| \frac{y}{y-1} \right) \right\}. \end{aligned} \tag{45}$$

For $n \in \mathbb{N}$, two bounds for the remainder are given by (42) and (43) in corollary 3 replacing B_{n-K}^F by B_{n-a+b}^F. And again, the expansion (44) is convergent if $\text{Max}\{|1-y|\xi(c)^{-1}, 1\} < |z|$, where $\xi(c)$ is defined in (41).

Proof. To obtain the expansion (44), just apply theorem 2 to the integral (2) with $f(t) = f_y^F(t)$ given in (4), $s = 1$, $K = a-b$, $a_k = B_{k+b-a}^F$ and $w = b$.
On the other hand, the coefficients B_k^F in the expansion (7) of $f_y^F(t)$ may be written

$$B_k^F = \frac{1}{k!} \frac{d^k}{dt^k} \left[t^{b-a-1} f_y^F(t^{-1}) \right]_{t=0}.$$

Using the Cauchy formula for the derivative of an analytic function, we obtain

$$B_{k+b-a}^F = \frac{d^{k+b-a}}{dt^{k+b-a}} \left[\frac{t^k (1-t)^{b-a-1}}{(k+b-a)!} f_y^F \left(\frac{t}{1-t} \right) \right]_{t=1}. \tag{46}$$

The coefficient B_n in (44) is just b_{n+1} given by (15)(b) with $a_n = B_{n+b-a}^F$. The Mellin transform in formula (15)(b) is given by (37). When $z \to n$, there are two singular terms in this limit: $B^F_{n+b-a}/(z-n)$ and $B(z+d-a,a-z-b)$. Setting $z=n+\eta$, expanding these terms at $\eta=0$ and using (46) we obtain (45).

The bounds (42) and (43) are obtained as in corollary 3 (but using only proposition 2). □

Corollary 5. *For $\Re a>0$, $\Re(d-a)>0$, $1+a-b-c\notin \mathbb{Z}$, $|Arg(xz)|<\pi$ and $|Arg(yz)|<\pi$,*

$$\begin{aligned} F_1(a,b,c,d;1-xz,1-yz) = \frac{\Gamma(d)}{\Gamma(d-a)\Gamma(a)} \left\{ \sum_{k=0}^{n-1} \frac{(-1)^k B_k}{k! z^{k+b+c}} + \right. \\ \sum_{k=0}^{n-K-1} \frac{\Gamma(b+c-a-k)\Gamma(a+k)}{\Gamma(b+c) x^{a+k-c} y^c z^{k+a}} A_k^F F\left(\begin{matrix} b+c-a-k, c \\ b+c \end{matrix} \middle| 1-\frac{x}{y}\right) + \\ \left. R_n(a,b,c,d;xz,yz) \right\}, \end{aligned} \tag{47}$$

where $K\equiv \mathrm{Int}(\Re(1+a-b-c))$ and the coefficients A_k^F are defined in (6). Coefficients B_k are given by

$$B_k \equiv \frac{\Gamma(k+d-a)\Gamma(a-k-b-c)}{\Gamma(d-b-c)} \sum_{j=0}^{k} \binom{k}{j} \frac{(b)_j (c)_{k-j}}{x^{b+j} y^{k+c-j}}. \tag{48}$$

If $\Re(1+a-b-c)\notin \mathbb{Z}$ and $n\ge 0$, a bound for the remainder is given by

$$\begin{aligned} |R_n(a,b,c,d;xz,yz)| \le & \frac{c_n \pi (|b|+|c|)_n \Gamma(n+\Re(b+c+s)-1) h(xz,b) h(yz,c)}{\Gamma(n+\Re s)\Gamma(n+\Re(b+c))|\sin(\pi \Re s)| |vz|^{n+\Re(b+c+s)-1}} \times \\ & F\left(\begin{matrix} 1-\Re s, n+\Re(s+b+c)-1 \\ (n+\Re(b+c)+1)/2 \end{matrix} \middle| \frac{1}{2}\left(1-\frac{r}{|vz|}\right)\right), \end{aligned} \tag{49}$$

where $s\equiv 1+a-b-c-K$, $h(z,w)$ was defined in (23) and r and v where defined in (22).

In formula (49) we can take $c_n=|A^F_{n-K}|$ if $a,b,c,d\in \mathbb{R}$ and $b+c-d\le 0$. In any case, we can take $c_n=C$, where

$$C > \mathrm{Sup}_{u\in W} \left|(1+u)^{b+c-d}\right| \tag{50}$$

and W is the region considered in lemma 5 for $g(u)=u^{b+c-a-1} f^F(u^{-1})$.

On the other hand, if $\Re(1+a-b-c)\in \mathbb{Z}$ and $n\ge 0$, the remainder in the expansion (47) satisfies

$$|R_n(a,b,c,d;xz,yz)| \le \mathbf{R}_n^{(1)}(A_n^F, b+c; 1-xz) + \mathbf{R}_n^{(1)}(A_n^F, b+c; 1-yz) \tag{51}$$

or

$$|R_n(a,b,c,d;xz,yz)| \le \mathbf{R}_n^{(2)}(C, A_n^F, b+c; 1-xz) + \mathbf{R}_n^{(2)}(C, A_n^F, b+c; 1-yz) \tag{52}$$

where $\mathbf{R}_n^{(1)}$ *and* $\mathbf{R}_n^{(2)}$ *are given in (42) and (43). Moreover, the expansion (47) is convergent if* $v|z|>1$.

Proof. To obtain the expansion (47), just apply theorem 3 to the integral (3) with $f(t)=f^F(t)$ given in (4), $a_k=A_{k-K}^F$ given in (6), $w_1=b$, $w_2=c$ and s and K given above. The calculation of coefficient C_k in formula (19) of theorem 3 requires the calculation of the Mellin transform $M[f^F;k+1]$. After trivial manipulations, and using the integral representation of the Beta function, we obtain $M[f^F;k+1]=B(k+d-a,a-k-b-c)$ and (48) follows.

If $a,b,c,d\in\mathbb{R}$ and $b+c-d\le 0$ holds, then, by [[13], lemmas 3 and 4], the function $f^F(t)$ verifies the error test. Therefore, by corollary 2, the remainder in the expansion (47) verifies the bounds given in propositions 1 and 2 with $c_n=|A_{n-K}^F|$, $c_{n-1}=0$. In any case, by lemma 5 and corollary 1, the remainder in the expansion (47) verifies the bounds given in formula (30) of proposition 2 with $w_1=b$, $w_2=c$, $r=1$ and $c_n=C$, C verifying (50). Therefore, the bound (51) holds.

Finally, using the same argument that we used at the end of the proof of the corollary 3, we obtain that, if $|xz|>1$ and $|yz|>1$, then $\lim_{n\to\infty}R_n^{(i)}=0$ and therefore, $\lim_{n\to\infty}R_n(a,b,c,d;xz,yz)=0$. □

Corollary 6. *For* $\Re a>0$, $\Re(d-a)>0$, $1+a-b-c\in\mathbb{Z}$, $|Arg(xz)|<\pi$ *and* $|Arg(yz)|<\pi$,

$$\begin{aligned}F_1(a,b,c,d;1-xz,1-yz)=&\frac{\Gamma(d)}{\Gamma(d-a)\Gamma(a)}\left\{\sum_{k=0}^{n-1}\frac{(-1)^kC_{k+1}}{k!z^{k+b+c}}+\right.\\&\sum_{k=0}^{b+c-a-1}\frac{\Gamma(b+c-a-k)\Gamma(a+k)A_k^F}{\Gamma(b+c)x^{k+a-c}y^cz^{a+k}}F\left(\left.\begin{array}{c}b+c-a-k,c\\b+c\end{array}\right|1-\frac{x}{y}\right)+\\&\sum_{k=0}^{n-1}\frac{(-1)^kA_{k+b+c-a}^F}{k!(k+b+c)z^{k+b+c}}\left[B_k\big(\log(xz)-\gamma-\psi(k+b+c)\big)+B_k'\right]+\\&\left.R_n(a,b,c,d;xz,yz)\right\},\end{aligned}\qquad(53)$$

where coefficients A_k^F *are given in (6) and coefficients* B_k, B_k' *and* C_{k+1} *are given by*

$$B_k\equiv\sum_{j=0}^{k+1}\binom{k+1}{j}\frac{(b)_j(c)_{k+1-j}}{x^{b+j-1}y^{c+k+1-j}}F\left(\left.\begin{array}{c}1,k+1+c-j\\k+1+b+c\end{array}\right|1-\frac{x}{y}\right),$$

$$B_k'\equiv\sum_{j=0}^{k+1}\binom{k+1}{j}\frac{(b)_j(c)_{k+1-j}}{x^{b+j-1}y^{c+k+1-j}}F'\left(\left.\begin{array}{c}1,k+1+c-j\\k+1+b+c\end{array}\right|1-\frac{x}{y}\right)$$

and

$$\begin{aligned}C_{k+1}\equiv&\left\{\frac{\Gamma(k+d-a)(-1)^{k+b+c-a-1}}{(k+b+c-a)!\Gamma(d-b-c)}\left(\psi(k+d-a)-\psi(k+b+c-a+1)\right)+\right.\\&\left.A_{k+b+c-a}^F\left(\gamma+\psi(k+1)\right)\right\}\times\sum_{j=0}^{k}\binom{k}{j}\frac{(b)_j(c)_{k-j}}{x^{b+j}y^{k+c-j}}.\end{aligned}\qquad(54)$$

respectively.

The remainder in the expansion (53) satisfies the bounds (51) and (52). Moreover, the expansion (53) is convergent if $|xz| > 1$ *and* $|yz| > 1$.

Proof. To obtain the expansion (53), just apply theorem 4 to the integral (3) with $f(t) = f^F(t)$ given in (4), $s = 1$, $K = a - b - c$, $a_k = A^F_{k+b+c-a}$, $w_1 = b$ and $w_2 = c$.

On the other hand, the coefficients A^F_k in the expansion (5) of $f^F(t)$ may be written

$$A^F_k = \frac{1}{k!}\frac{d^k}{dt^k}\left[t^{b+c-a-1}f^F(t^{-1})\right]_{t=0}.$$

Using the Cauchy formula for the derivative of an analytic function, we obtain

$$A^F_{k+b+c-a} = \frac{d^{k+b+c-a}}{dt^{k+b+c-a}}\left[\frac{t^k(1-t)^{b+c-a-1}}{(k+b+c-a)!}f^F\left(\frac{t}{1-t}\right)\right]_{t=1}. \tag{55}$$

Coefficients C_{n+1} in (53) are given by b_{n+1} in (15)(b) with $a_n = A^F_{n+b+c-a}$. The Mellin transform in formula (15)(b) is given by (48). When $z \to n$, there are two singular terms in the limit in (15)(b): $A^F_{n+b+c-a}/(z-n)$ and $B(z+d-a, a-z-b-c)$. Setting $z = n + \eta$, expanding these terms at $\eta = 0$ and using (55) we obtain (54).

The bound (51) are obtained as in corollary 5. □

3.1. Numerical experiments

The following tables show numerical experiments performed with the program *mathematica* about the approximation and the accuracy of the error bounds supplied by corollaries 3-6. In these tables, the second column represents the integral $F_1(a,b,c,d;x,y)$. The third and sixth columns represent the approximation given by corollaries 3, 4, 5 or 6 for $n = 2$ and $n = 3$ respectively. Fourth and seventh columns represent the respective relative errors, and fifth and last columns are the respective relative error bounds given in those corollaries. The *c.p.u.* time used by mathematica to compute a "correct" value of F_1 (by an undisclosed method) is of the order of 1 second, whereas the time used by mathematica to compute an approximation, including its error bound, is of the order of 10^{-2} seconds.

Parameter values: $a = 1.5,\ b = 2.05,\ c = 1,\ d = 3.25,\ y = -0.9$

x	F_1	Second or. approx.	Relative error	Relative er. bound	Third or. approx.	Relative error	Relative er. bound
-10	0.0192501	0.01395848	0.275	0.39	0.01790552	0.07	0.097
-20	0.00870325	0.00810254	0.069	0.09	0.00862363	0.0091	0.0117
-50	0.00274596	0.00271654	0.0107	0.012	0.00274436	5.8e-4	6.84e-4
-100	0.00108523	0.00108241	0.0026	0.0029	0.00108515	7.13e-5	8.e-5
-200	0.000414904	0.000414641	6.35e-4	6.93e-4	0.0004149	8.73e-6	9.46e-6

Table 1: Approximation supplied by (36) and error bounds given by (38).

Parameter values: $a = 1.5,\ b = 4.05,\ c = 1,\ d = 1.75,\ y = -0.9, \mathrm{Arg}(x) = -3\pi/4$

$\lvert x\rvert$	F_1	First or. approx.	Relative error	Relative er. bound	Second or. approx.	Relative error	Relative er. bound
20	0.00080787 - 0.0019668i	0.000831 - 0.0019555i	0.012	0.068	0.00081234 - 0.001968i	0.0022	0.007
50	0.00020672 - 0.000498i	0.0002077 - 0.0004976i	0.002	0.029	0.00020679 - 0.000498i	1.4e-4	0.001
100	7.31557e-5 - 1.76271e-4i	7.324337e-5 - 1.76234e-4i	5.e-4	0.01	7.315886e-5 - 1.762723e-4i	1.78e-5	3.2e-4
200	2.58616e-5 - 6.23597e-5i	2.586926e-5 - 6.235645e-5i	1.2e-4	0.007	2.58616e-5 - 6.23597e-5i	1.e-6	8.e-5
300	1.40754e-5 - 3.39519e-5i	1.407729e-5 - 3.395107e-5i	5.5e-5	0.005	1.40754e-5 - 3.395186e-5i	3.3e-7	3.7e-5

Table 2: Approximation supplied by (36) and error bounds given by (38).

Parameter values: $a = 1 - 0.5i,\ b = 3 + i,\ c = 3,\ d = 3,\ y = -0.5$

x	F_1	Second or. approx.	Relative error	Relative er. bound	Third or. approx.	Relative error	Relative er. bound
-10	0.00395015 + 0.0241468i	0.00393933 + 0.024349i	0.0083	0.083	0.00392435 + 0.0241975i	0.0023	0.0236
-20	-0.00150212 + 0.012645i	-0.0015046 + 0.012659i	0.0011	0.0087	-0.00150328 + 0.0126466i	1.6e-4	1.29e-3
-50	-0.00263747 + 0.00448026i	-0.0026376 + 0.004486i	6.97e-5	3.95e-4	-0.00263748 + 0.00448027i	4.e-6	2.41e-5
-100	-0.00198698 + 0.00169895i	-0.00198699 + 0.00169896i	8.5e-6	3.6e-5	-0.00198698 + 0.00169895i	2.48e-7	1.12e-6
-200	-0.00122261 + 4.71985e-4i	-0.00122261 + 4.71986e-4i	1.06e-6	3.27e-6	-0.0012226 + 4.71985e-4i	1.58e-8	5.1e-8

Table 3: Approximation supplied by (36) and error bounds given by Min{(42),(43)}.

Parameter values: $a = 2,\ b = 3,\ c = 3,\ d = 2.5,\ y = -0.9$

x	F_1	Second or. approx.	Relative error	Relative er. bound	Third or. approx.	Relative error	Relative er. bound
-10	0.00289672	0.00376548	0.3	0.885	0.00324391	0.12	0.3
-20	8.73402e-4	9.17957e-4	0.051	0.16	8.826362e-4	0.01	0.029
-50	1.64142e-4	1.6486e-4	0.0044	0.016	1.64202e-4	0.0004	0.0012
-100	4.42497e-5	4.42786e-5	6.52e-4	0.0027	4.42509e-5	2.76e-5	1.e-4
-200	1.1611e-5	1.1612e-5	9.5e-5	4.6e-4	1.1610998e-5	2.01e-6	8.7e-6

Table 4: Approximation supplied by (44) and error bounds given by Min{(42),(43)}.

4. Conclusions

Asymptotic expansions of generalized Stieltjes transforms for complex values of the parameters have been derived in section 2, including error bounds. They extend to the complex case the known methods given in [27], [[28], chap. 6], [13], [14] for real parameters. Using these methods we have derived four expansions of the first Appell's hypergeometric function F_1 in corollaries 3-6, including error bounds for the remainder. Moreover, these expansions are convergent when the asymptotic variable is large enough.

Parameter values: $a = 2,\ b = 3,\ c = 3,\ d = 2.5,\ y = -0.9, \mathrm{Arg}(x) = -\pi/2$

$\|x\|$	F_1	Second or. approx.	Relative error	Relative er. bound	Third or. approx.	Relative error	Relative er. bound
10	-0.00313761 - 0.00155835i	-0.00182859 - 0.00215644i	0.4	1.8	-0.00335547 - 0.00214796i	0.18	0.7
20	-9.57456e-4 - 3.17093e-4i	-9.127204e-4 - 3.58223e-4i	0.06	0.29	-9.66098e-4 - 3.270276i	0.013	0.058
50	-1.77437e-4 - 3.31918e-5i	-1.76966e-4 - 3.39021e-5i	0.0047	0.027	-1.77498e-4 - 3.323225e-5i	4.e-4	0.0022
100	-4.69382e-4 - 5.51933e-6i	-4.692343e-4 - 5.548144e-6i	6.8e-4	0.0047	-4.69394e-4 - 5.519946e-6i	2.9e-5	1.9e-4
200	-1.20975e-5 - 8.70185e-7i	-1.20971e-5 - 8.712899e-7i	9.9e-5	8.12e-4	-1.20976e-5 - 8.701947e-7i	2.08e-6	1.6e-5

Table 5: Approximation supplied by (44) and error bounds given by Min{(42),(43)}.

Parameter values: $a = 1.5,\ b = 0.05,\ c = 2,\ d = 4$

$x,\ y$	F_1	Second or. approx.	Relative error	Relative er. bound	Third or. approx.	Relative error	Relative er. bound
-10,-25	0.00669952	0.00646809	0.034	0.78	0.00668157	0.0027	0.13
-20,-40	0.00370763	0.00365988	0.013	0.15	0.00370534	0.0006	0.013
-50,-65	0.00194611	0.00193718	0.0046	0.012	0.00194585	1.3e-4	4.6e-4
-100,-200	4.232485e-4	4.230478e-4	4.7e-4	0.005	4.232466e-4	4.62e-6	9.8e-5
-200,-350	1.908408e-4	1.908122e-4	1.5e-4	0.001	1.908406e-4	8.26e-7	9.86e-6

Table 6: Approximation supplied by (47) and error bounds given by (49).

Parameter values: $a = 2.5,\ b = 4.05,\ c = 1,\ d = 3, \mathrm{Arg}(x) = -3\pi/4,\ \mathrm{Arg}(y) = 0$

$\|x\|,\ y$	F_1	Second or. approx.	Relative error	Relative er. bound	Third or. approx.	Relative error	Relative er. bound
10, -15	-1.1346201e-6 - 0.000204246i	-1.322963e-6 - 0.00020423i	9.25e-4	1.8e-3	-1.1455236e-6 - 0.000204237i	7.e-5	1.5e-4
20, -20	-1.42374e-6 - 4.316887e-5i	-1.43107e-6 - 4.316743e-5i	1.7e-4	2.4e-4	-1.42395e-6 - 4.316865e-5i	7.2e-6	1.e-5
50, -55	-6.266874e-9 - 4.072453e-4i	-6.313751e-9 - 4.07244e-4i	1.2e-5	1.8e-5	-6.267419e-9 - 4.072453e-4i	2.e-7	3.4e-7
100, -110	2.791197e-9 - 7.151052e-7i	2.790133e-9 - 7.151048e-7i	1.56e-6	2.45e-6	2.79119e-9 - 7.151051e-7i	1.46e-8	2.23e-8
150, -200	7.346265e-9 - 2.337076e-7i	7.346172e-9 - 2.3374e-7i	4.23e-7	8.e-7	7.346265e-9 - 2.3374e-7i	2.8e-9	5.e-9

Table 7: Approximation supplied by (47) and error bounds given by (49).

When the parameters defining the function $f(t)$ or $f_y(t)$ in the integral representation of the Appell's function F_1 (equations (2) and (3) respectively) verify the conditions given in (39) then, $f(t)$ or $f_y(t)$ belong to a special kind of functions: the remainder term in their asymptotic expansion in inverse powers of t satisfies the error test. This fundamental property allows us to use corollary 2 to derive a more accurate error bound for the remainder in the asymptotic expansions of F_1 given in corollaries 3-6. These

Parameter values: $a = 2 + i,\ b = 2 - 1.5i,\ c = 1,\ d = 3$

$x,\ y$	F_1	Second or. approx.	Relative error	Relative er. bound	Third or. approx.	Relative error	Relative er. bound
-10, -15	0.0001248 - 0.0003i	0.0001224 - 0.000298i	0.009	0.2	0.0001246 - 0.0002999i	0.0008	0.006
-20, -40	-3.04309e-5 - 7.e-5i	-3.04379e-5 - 7.e-5i	0.002	0.04	-3.04308e-5 - 6.9838e-5i	9.8e-5	7.e-4
-50, -65	-1.357482e-5 - 7.414746e-6i	-1.357084e-5 - 7.408461e-6i	4.8e-4	0.005	-1.357474e-5 - 7.414624e-6i	9.5e-6	3.16e-5
-100, -130	-3.78267e-6 + 7.694256e-7i	-3.78222e-6 + 7.69585e-7i	1.2e-4	9.3e-4	-3.78266e-6 + 7.694369e-7i	1.2e-6	2.9e-6
-200, -250	-6.362044e-7 + 7.622805e-7i	-6.361753e-7 + 7.6227e-7i	3.e-5	1.7e-4	-6.362043e-7 + 7.622805e-7i	1.6e-7	2.7e-7

Table 8: Approximation supplied by (47) and error bounds given by Min{(51), (52)}.

Parameter values: $a = 2 + i,\ b = 3 - 1.5i,\ c = 1,\ d = 4, \mathrm{Arg}(x) = -\pi/2, \mathrm{Arg}(y) = \pi$

$\|x\|,\ \|y\|$	F_1	Second or. approx.	Relative error	Relative er. bound	Third or. approx.	Relative error	Relative er. bound
10, 15	-0.0004909 + 0.00079287i	-0.0004903 + 0.00079364i	0.001	0.3	-0.0004908 + 0.00079279i	0.0001	0.06
20, 40	3.774639e-5 + 2.095e-4i	3.7774e-5 + 2.094998e-4i	0.0001	0.03	3.774621e-5 + 2.094987-4i	7.4e-6	0.003
50, 65	3.78677223-5 + 2.0116287e-5i	3.78679583-5 + 2.0115959e-5i	9.4e-6	1.e-3	3.78677137-5 + 2.0116282e-5i	2.12e-7	5.e-5
100, 130	1.058444e-5 - 2.2504918e-6i	1.058444e-5 - 2.250504e-6i	1.18e-6	1.e-4	1.058444e-5 - 2.2504917e-6i	1.3e-8	2.2e-6
200, 250	1.7122688e-6 - 2.169836e-6i	1.7122686e-6 - 2.169836e-6i	1.5e-7	9.4e-6	1.7122688e-6 - 2.169836e-6i	9.e-10	9.89e-8

Table 9: Approximation supplied by (47) and error bounds given by Min{(51), (52)}.

Parameter values: $a = 2,\ b = 1,\ c = 1,\ d = 3.2$

$x,\ y$	F_1	Second or. approx.	Relative error	Relative er. bound	Third or. approx.	Relative error	Relative er. bound
-50, -70	0.0055576	0.005437717	0.02	0.1	0.00554541	0.002	0.004
-100, -200	0.0001827	0.000182648	2.7e-4	0.0017	0.0001827	3.2e-6	6.e-6
-500, -650	1.5569377e-5	1.55691513e-5	1.4e-5	9.7e-5	1.55693768e-5	3.7e-8	6.6e-8
-1000, -1100	5.1555639e-6	5.1555424e-6	4.1e-6	3.2e-5	5.1555638e-6	5.8e-9	1.1e-8
-1500, -2000	2.0549834e-6	2.05498025e-6	1.5e-6	1.5e-5	2.0549834e-6	1.3e-9	3.5e-9

Table 10: Approximation supplied by (53) and error bounds given by Min{(51),(52)}.

bounds have been obtained from the error test and, as numerical computations show (see tables 1-11), they exhibit a remarkable accuracy.

Parameter values: $a = 2,\ b = 1,\ c = 1,\ d = 3.3, \mathrm{Arg}(x) = -4\pi/5, \mathrm{Arg}(y) = 0$

$\|x\|,\ \|y\|$	F_1	Second or. approx.	Relative error	Relative er. bound	Third or. approx.	Relative error	Relative er. bound
100, 150	1.998863e-4 - 1.16859e-4i	1.99866499e-4 - 1.167804e-4i	3.5e-4	0.0018	1.998865e-4 - 1.1685818e-4i	4.e-6	6.26e-6
200, 250	6.9458999e-5 - 4.2285549e-5i	6.94566577e-5 - 4.227787e-5i	9-87e-5	5.3e-4	6.9459004e-5 - 4.22855e-5i	5.99e-7	9.3e-7
500, 565	1.473406e-5 - 9.28714e-6i	1.4733969e-5 - 9.28686e-6i	1.68e-5	1.e-4	1.473406e-5 - 9.28714e-6i	4.2e-8	8.e-8
1000, 1060	4.4257489e-6 - 2.8438858e-6i	4.42574127e-6 - 2.843864e-6i	4.4e-6	4.e-5	4.4257489e-6 - 2.8438858e-6i	5.65e-9	1.34e-8
1500, 1550	2.153354e-6 - 1.396248e-6i	2.1533524e-6 - 1.39624e-6i	2.e-6	1.9e-5	2.153354e-6 - 1.3962485e-6i	1.7e-9	4.67e-9

Table 11: Approximation supplied by (53) and error bounds given by Min{(51),(52)}.

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