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Asymptotic expansions of Mellin convolutions by means of analytic continuation

José L. López

Departamento de Matemática e Informática, Universidad Pública de Navarra, 31006-Pamplona, Spain

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Abstract

We present a method for deriving asymptotic expansions of integrals of the form $\int_0^\infty f(t)h(xt)\,dt$ for small x based on analytic continuation. The expansion is given in terms of two asymptotic sequences, the coefficients of both sequences being Mellin transforms of h and f. Many known and unknown asymptotic expansions of important integral transforms are derived trivially from the approach presented here. This paper reconsiders earlier work of McClure and Wong [Explicit error terms for asymptotic expansions of Stieltjes transforms, J. Inst. Math. Appl. 22 (1978) 129–145; Exact remainders for asymptotic expansions of fractional integrals, J. Inst. Math. Appl. 24 (1979) 139–147] and [Asymptotic approximations of integrals, Academic Press, New York, 1989. Chaps. 5 and 6], where elements of distribution theory are used, and Wong [Explicit error terms for asymptotic expansions of Mellin convolutions, J. Math. Anal. Appl. 72(2) (1979) 740–756], where, as in the present paper, the asymptotic expansions are obtained without the use of distributions. In this paper we re-derive the expansions given in Wong [Explicit error terms for asymptotic expansions of Mellin convolutions, J. Math. Anal. Appl. 72(2) (1979) 740–756] by using a different approach and we obtain new results which are not present in Wong [Explicit error terms for asymptotic expansions of Mellin convolutions, J. Math. Anal. Appl. 72(2) (1979) 740–756]: a proof of the asymptotic character of the expansions and accurate error bounds.

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1. Introduction

We consider integrals of the form:

$$I(x) \equiv \int_0^\infty f(t)h(xt)\,dt. \tag{1}$$

We are interested in finding asymptotic expansions of $I(x)$ for large or small x under mild conditions on f and h. The ideas developed in this paper may be generalized to complex x but for the sake of clearness, we restrict ourselves to positive values of x. Without lost of generality we can think of x as a small parameter. (If x is large, perform the change of variable $t \to t/x$ and replace the roles of f and h in (1).) Many integral transforms can be put in form (1): Laplace, Fourier, Stieltjes, Hankel, Poisson, Glasser, Lambert,…[7].

E-mail address: jl.lopez@unavarra.es.

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A formal asymptotic expansion of (1) may be obtained by replacing an expansion of $h(xt)$ in increasing powers of xt in (1) and interchanging summation and integration (when $h(t) = e^{-t}$ this is Watson's Lemma). But this easy procedure does not work if the positive moments of $f(t)$ do not exist. (Important examples of failure are elliptic integrals, Epstein–Hubbel integrals or the Appell's function F_1 among others.)

McClure and Wong (in the following M&W) solved this problem for certain kernels h using elements of the theory of distributions and requiring an asymptotic expansion of f in decreasing powers of t [2,3,6].

In [1] we proposed a technique different from that of M&W inspired by the ideas introduced in [5]: we used analytic continuation instead of distributions. In this paper we generalize the method introduced in [1]: we only require for h to have an asymptotic expansion at $t = 0$ instead of the more stringent condition $h \in \mathscr{C}^{(\infty)}[0, \infty)$ required in [1]. Moreover, we offer here much easier proofs than those given in [1] and more accurate error bounds. This technique results in a quite simple and very general method of asymptotic expansions of integrals $I(x)$. (Essentially, this technique only requires for $h(t)$ to have an asymptotic expansion at $t = 0$ and for $f(t)$ to have an asymptotic expansion at $t = \infty$.)

The asymptotic expansions obtained here were already obtained by Wong in [5] using also analytic continuation techniques but the proofs in this paper are different from those of [5]. From these new proofs we can show the asymptotic character of the expansions without requiring additional hypotheses (the proof of the asymptotic character of the expansion in [5] requires additional hypotheses).

In the next section we present the method for obtaining asymptotic expansions of $I(x)$ and establish the fundamental results of the paper. In Section 3 we show the asymptotic character of the expansions and give precise error bounds for the remainders under additional hypotheses on f and h. In Section 4 we present some conclusions and a few comments.

2. The method

We need some definitions and two technical lemmas to formulate the concepts mentioned in the introduction accurately.

Definition 2.1. We denote by $\mathscr{H}_{a,\beta}$ the set of functions $h \in L^1_{\text{Loc}}(0, \infty)$ verifying:

(i) h has an asymptotic expansion at $t = 0$:

$$h(t) = \sum_{k=0}^{n-1} A_k t^{k-a} + h_n(t), \quad n = 1, 2, 3, \ldots, \quad a \in \mathbb{R}, \tag{2}$$

where $\{A_k\}$ is a sequence of complex numbers and $h_n(t) = \mathcal{O}(t^{n-a})$ when $t \to 0^+$.

(ii) $h(t) = \mathcal{O}(t^{-\beta})$ when $t \to \infty$ for some $\beta \in \mathbb{R}$.

Definition 2.2. We denote by $\mathscr{F}_{b,\alpha}$ the set of functions $f \in L^1_{\text{Loc}}(0, \infty)$ verifying:

(i) f has an asymptotic expansion at infinity:

$$f(t) = \sum_{k=0}^{n-1} \frac{B_k}{t^{k+b}} + f_n(t), \quad n = 1, 2, 3, \ldots, \quad b \in \mathbb{R}, \tag{3}$$

where $\{B_k\}$ is a sequence of complex numbers and $f_n(t) = \mathcal{O}(t^{-n-b})$ when $t \to \infty$.

(ii) $f(t) = \mathcal{O}(t^{-\alpha})$ when $t \to 0^+$ for some $\alpha \in \mathbb{R}$.

In the remaining of the paper we require for the parameters a, b, α and β to satisfy the following conditions:

Condition I: $a + \alpha < 1 < b + \beta$.

Condition II: $\alpha < b$ and $a < \beta$.

Condition I assures the integrability of $h(xt)f(t)$ in (1). Condition II does not suppose any lost of generality because of (a) and (b) below.

(a) If $\alpha \geqslant b$ (then $\beta > a + \alpha - b \geqslant a$ from condition I) we write:

$$I(x) = \int_0^\infty \tilde{f}(t)h(xt)\,dt + x^{-s}\int_0^\infty \tilde{f}(t)\tilde{h}(xt)\,dt = I_1(x) + x^{-s}I_2(x),$$

with

$$\tilde{f}(t) \equiv t^s f(t), \quad \tilde{h}(t) \equiv \frac{h(t)}{1+t^s}$$

and s being any natural number verifying $s \geqslant \gamma \equiv (\alpha + \beta - a - b)/2$ (observe that $\gamma > 0$). Then $h \in \mathscr{H}_{a,\beta}$, $\tilde{f} \in \mathscr{F}_{\tilde{b},\alpha}$ and $\tilde{h} \in \mathscr{H}_{\tilde{a},\tilde{\beta}}$ with $\tilde{b} \equiv b + s$, $\tilde{\beta} \equiv \beta - s$ and $\tilde{a} \equiv a - s$. In $I_1(x)$ we have $a + \alpha < 1 < \tilde{b} + \beta$, $a < \beta$ and $\alpha < \tilde{b}$. In $I_2(x)$ we have $\tilde{a} + \alpha < 1 < \tilde{b} + \tilde{\beta}$, $\tilde{a} < \tilde{\beta}$ and $\alpha < \tilde{b}$.

(b) If $a \geqslant \beta$ (then $b > \alpha + a - \beta \geqslant \alpha$ from condition I) we write:

$$I(x) = \int_0^\infty f(t)\tilde{h}(xt)\,dt + x^{-s}\int_0^\infty \tilde{f}(t)\tilde{h}(xt)\,dt = I_1(x) + x^{-s}I_2(x),$$

with

$$\tilde{f}(t) \equiv t^{-s} f(t), \quad \tilde{h}(t) \equiv \frac{h(t)}{1+t^{-s}}$$

and s any integer verifying $s \leqslant \gamma$ (observe that now $\gamma < 0$). Then $\tilde{h} \in \mathscr{H}_{a,\tilde{\beta}}$, $f \in \mathscr{F}_{b,\alpha}$ and $\tilde{f} \in \mathscr{F}_{\tilde{b},\tilde{\alpha}}$ with $\tilde{\beta} \equiv \beta - s$, $\tilde{b} \equiv b + s$ and $\tilde{\alpha} \equiv \alpha + s$. In $I_1(x)$ we have $a + \alpha < 1 < b + \tilde{\beta}$, $a < \tilde{\beta}$ and $\alpha < b$. In $I_2(x)$ we have $a + \tilde{\alpha} < 1 < \tilde{b} + \tilde{\beta}$, $a < \tilde{\beta}$ and $\tilde{\alpha} < \tilde{b}$.

Definition 2.3. Let $g \in L^1_{\mathrm{Loc}}(0,\infty)$. We denote by $M[g;z]$ the Mellin transform of g, $\int_0^\infty t^{z-1}g(t)\,dt$ (when this integral exists), or its analytic continuation as a function of z.

The proofs of the following two lemmas are based on the results of [6, Chapter 3].

Lemma 2.4. *The Mellin transform $M[f;z]$ of every function $f \in \mathscr{F}_{b,\alpha}$ exists and defines a meromorphic function of z in the half plane $\Re z > \alpha$. More precisely,*

(i) *In the strip $\alpha < \Re z < b$,*

$$M[f;z] = \int_0^\infty t^{z-1} f(t)\,dt. \tag{4}$$

(ii) *For $n \in \mathbb{N}$, in the strip $\alpha < \Re z < n + b$,*

$$M[f;z] = \int_0^1 t^{z-1} f(t)\,dt - \sum_{k=0}^{n-1} \frac{B_k}{z-k-b} + \int_1^\infty t^{z-1} f_n(t)\,dt. \tag{5}$$

$M[f;z]$ has simple poles at the points $z = k + b$, $k = 0, 1, 2, \ldots$ with residues $-B_k$.

(iii) *For $n \in \mathbb{N}$, in the strip $n + b - 1 < \Re z < n + b$,*

$$M[f;z] = \int_0^\infty t^{z-1} f_n(t)\,dt.$$

Proof. Thesis (i) follows immediately from the definition of $\mathscr{F}_{b,\alpha}$ and $\alpha < b$. The analyticity of the integral in the right hand side of (4) in the strip $\alpha < \Re z < b$ follows from [4, pp. 240–241], (we will use repeatedly without mention this well-known result).

Then, in the strip $\alpha < \Re z < b$:

$$M[f; z] = \int_0^\infty t^{z-1} f(t)\,\mathrm{d}t = \int_0^1 t^{z-1} f(t)\,\mathrm{d}t + \int_1^\infty t^{z-1} f(t)\,\mathrm{d}t.$$

Introducing expansion (3) in the last integral above we obtain equality (5) in the strip $\alpha < \Re z < b$. The first integral in the right hand side of (5) defines an analytic function of z for $\Re z > \alpha$, whereas the last integral in (5) defines an analytic function of z for $\Re z < n + b$. Therefore, the right hand side of (5) defines the analytic continuation of $\int_0^\infty t^{z-1} f(t)\,\mathrm{d}t$ to the strip $\alpha < \Re z < n + b$. From here, thesis (ii) is evident. Thesis (iii) follows from (ii). Multiplying (3) by t^{z-1} and integrating we have:

$$\int_0^1 t^{z-1} f(t)\,\mathrm{d}t = \sum_{k=0}^{n-1} \frac{B_k}{z-k-b} + \int_0^1 t^{z-1} f_n(t)\,\mathrm{d}t, \quad \Re z > n + b - 1. \tag{6}$$

Thesis (iii) follows from (5) by using (6). □

Lemma 2.5. *The Mellin transform $M[h; z]$ of every function $h \in \mathscr{H}_{a,\beta}$ exists and defines a meromorphic function of z in the half plane $\Re z < \beta$. More precisely,*

(i) *In the strip $a < \Re z < \beta$,*

$$M[h; z] = \int_0^\infty t^{z-1} h(t)\,\mathrm{d}t.$$

(ii) *For $m \in \mathbb{N}$, in the strip $a - m < \Re z < \beta$,*

$$M[h; z] = \int_0^1 t^{z-1} h_m(t)\,\mathrm{d}t + \sum_{k=0}^{m-1} \frac{A_k}{z+k-a} + \int_1^\infty t^{z-1} h(t)\,\mathrm{d}t. \tag{7}$$

$M[h; z]$ has simple poles at the points $z = a - k$, $k = 0, 1, 2, \ldots$ with residues A_k.

(iii) *For $m \in \mathbb{N}$, in the strip $a - m < \Re z < a + 1 - m$,*

$$M[h; z] = \int_0^\infty t^{z-1} h_m(t)\,\mathrm{d}t.$$

Proof. Define $\tilde{h}(t) \equiv h(t^{-1})$. Observe that $M[h; z] = M[\tilde{h}; -z]$ and that $\tilde{h} \in \mathscr{F}_{-a,-\beta}$. Apply Lemma 2.4 to $M[\tilde{h}; -z]$ with b replaced by $-a$ and α replaced by $-\beta$. □

Now we can formulate the main results of the paper.

Theorem 2.6. *Let $f \in \mathscr{F}_{b,\alpha}$ and $h \in \mathscr{H}_{a,\beta}$ with $a + b \notin \mathbb{Z}$. Let conditions I and II hold. Then, for any $n = 1, 2, 3, \ldots$ and $m = n + \lfloor a + b \rfloor$,*

$$\int_0^\infty h(xt) f(t)\,\mathrm{d}t = \sum_{k=0}^{n-1} B_k M[h; 1-k-b] x^{k+b-1} + \sum_{k=0}^{m-1} A_k M[f; k+1-a] x^{k-a} + \int_0^\infty f_n(t) h_m(xt)\,\mathrm{d}t. \tag{8}$$

Proof. Denote $f_0(t) \equiv f(t)$ and

$$F_k(z) \equiv \int_0^\infty t^z f_k(t) h(t)\,\mathrm{d}t. \tag{9}$$

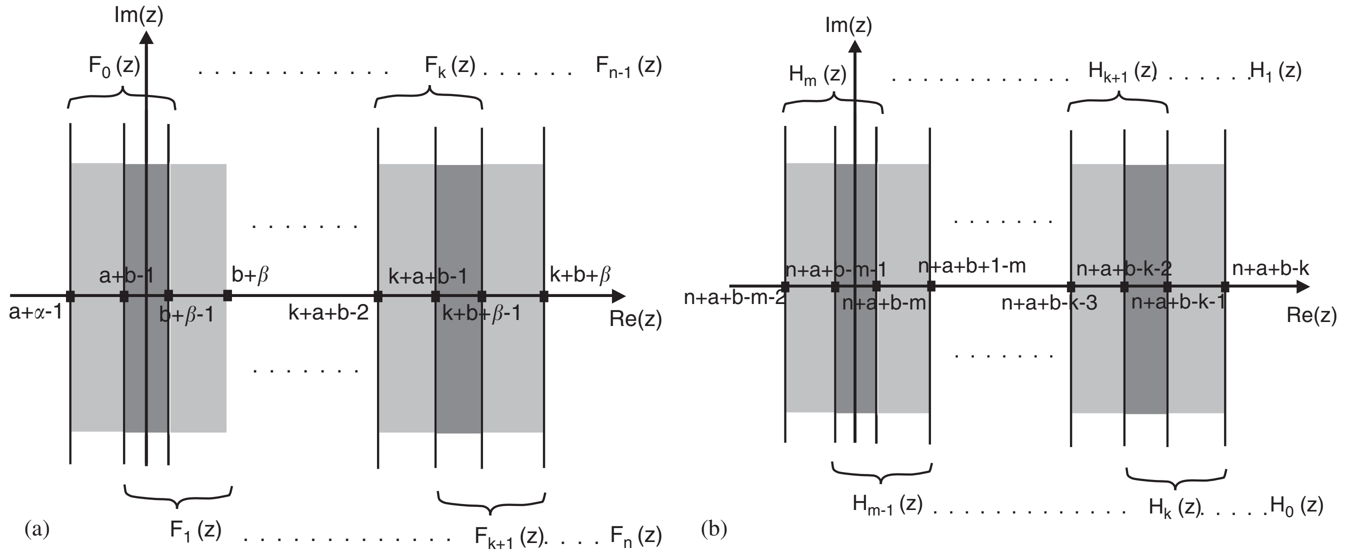

Fig. 1. The strips of analyticity $\Re z \in (k+a+b-2, k+b+\beta-1)$ of $F_k(z)$, $k>0$, are shown in (a). The strip of analyticity of $F_0(z)$ is $\Re z \in (\alpha+a-1, b+\beta-1)$. Their width is $\beta-a+1$ for $k>0$ and $b+\beta-a-\alpha$ for $k=0$. Then, for $k \geqslant 0$, $F_{k+1}(z)$ and $F_k(z)$ are both analytic in the strip $\Re z \in (k+a+b-1, k+b+\beta-1)$ of width $\beta-a$. In that strip, $F_{k+1}(z)=F_k(z)-B_k M[h; z+1-k-b]$. On the other hand, $M[h; z+1-k-b]$ is a meromorphic function of z in the half plane $\Re z < k+b+\beta-1$. Letting k run from $k=n-1$ up to $k=0$, an analytic continuation process of $F_n(z)$ is obtained from the strip $(n+a+b-2, n+b+\beta-1)$ to the strip $(\alpha+a-1, b+\beta-1)$. The strips of analyticity $\Re z \in (n+b+a-k-2, n+b+a-k)$ of $H_k(z)$, $k>0$, are shown in (b). The strip of analyticity of $H_0(z)$ is $\Re z \in (n+b+a-2, n+b+\beta-1)$. As in (a), letting k run from $k=0$ up to $k=m-1$, an analytic continuation process of $H_0(z)$ is obtained from the strip $(n+b+a-2, n+b+\beta-1)$ to the strip $(n+b+a-m-2, n+b+a-m)$.

For $k=1,2,3,\ldots,$ $F_k(z)$ is an analytic function of z in the strip $k+b+a-2<\Re z<k+b+\beta-1$. $F_0(z)$ is an analytic function of z in the strip $\alpha+a-1<\Re z<b+\beta-1$. From (3) we have $f_{k+1}(t)=f_k(t)-B_k t^{-k-b}$. Using this in (9) and (i) of Lemma 2.5 we have, for any $k=0,1,2,\ldots,$ that the equality

$$F_{k+1}(z)=F_k(z)-B_k M[h; z+1-k-b] \tag{10}$$

holds in the strip $k+b+a-1<\Re z<k+b+\beta-1$. $F_{k+1}(z)$ is an analytic function of z in the strip $k+b+a-1<\Re z<k+b+\beta$ and, from Lemma 2.5, $M[h; z+1-k-b]$ is a meromorphic function of z in the half plane $\Re z<k+b+\beta-1$. Hence analytic continuation of $F_{k+1}(z)$ is obtained from strip $k+b+a-1<\Re z<k+b+\beta$ to strip $k+b+a-2<\Re z<k+b+\beta-1$ if $k>0$. If $k=0$, analytic continuation of $F_1(z)$ is obtained from strip $b+a-1<\Re z<b+\beta$ to strip $\alpha+a-1<\Re z<b+\beta-1$. Repeating this process from $k=n-1$ up to $k=0$ we obtain that

$$F_0(z)-\sum_{k=0}^{n-1} B_k M[h; z+1-k-b] \tag{11}$$

defines the analytic continuation of $F_n(z)$ from the strip $n+b+a-2<\Re z<n+b+\beta-1$ to the strip $\alpha+a-1<\Re z<b+\beta-1$ (see Fig. 1(a)).

On the other hand, denote $h_0(t)\equiv h(t)$ and

$$H_k(z)\equiv \int_0^\infty t^z f_n(t) h_k(t)\,\mathrm{d}t. \tag{12}$$

For $k=1,2,3,\ldots,$ $H_k(z)$ is an analytic function of z in the strip $n+b+a-k-2<\Re z<n+b+a-k$. $H_0(z)$ is an analytic function of z in the strip $n+b+a-2<\Re z<n+b+\beta-1$. From (2) we have $h_k(t)=h_{k+1}(t)+A_k t^{k-a}$. Using this in (12) and (i) of Lemma 2.4 with f replaced by $f_n \in \mathscr{H}_{n+b,n+b-1}$ we have, for any $k=0,1,2,\ldots,$

that the equality

$$H_k(z) = H_{k+1}(z) + A_k M[f_n; z+1+k-a] \tag{13}$$

holds in the strip $n+b+a-k-2 < \Re z < n+b+a-k-1$. $H_{k+1}(z)$ is an analytic function of z in the strip $n+b+a-k-3 < \Re z < n+b+a-k-1$ and, from (iii) of Lemma 2.4, $M[f_n; z+1+k-a] = M[f; z+1+k-a]$ in the strip $n+b+a-k-2 < \Re z < n+b+a-k-1$. Moreover, from Lemma 2.4, $M[f; z+1+k-a]$ is a meromorphic function of z for $\Re z > a+\alpha-k-1$. Hence, for $k=1, 2, 3, \ldots$, analytic continuation of $H_k(z)$ is obtained from strip $n+b+a-k-2 < \Re z < n+b+a-k$ to strip $\mathrm{Max}\{n+b-2, \alpha\}+a-k-1 < \Re z < n+b+a-k-1$. For $k=0$, analytic continuation of $H_0(z)$ is obtained from strip $n+b+a-2 < \Re z < n+b+\beta-1$ to strip $\mathrm{Max}\{n+b-2, \alpha\}+a-1 < \Re z < n+b+a-1$. Repeating this process from $k=0$ up to $k=m-1$ we obtain that

$$H_m(z) + \sum_{k=0}^{m-1} A_k M[f; z+k+1-a] \tag{14}$$

defines the analytic continuation of $H_0(z)$ from the strip $n+b+a-2 < \Re z < n+b+\beta-1$ to the strip $\mathrm{Max}\{n+b-2, \alpha\}+a-m < \Re z < n+b+a-m$ (see Fig. 1(b)).

Observe that $F_n(z) = H_0(z)$ in the strip $n+b+a-2 < \Re z < n+b+\beta-1$. Then, both functions, (11) and (14) represent the analytic continuation of $F_n(z) = H_0(z)$ from the strip $n+a+b-2 < \Re z < n+b+\beta-1$ to the strip $\mathrm{Max}\{\alpha, n+b-m-1\}+a-1 < \Re z < b+\mathrm{Min}\{\beta-1, n+a-m\}$. The analytic continuation is unique and then, functions (11) and (14) coincide in the strip $\mathrm{Max}\{\alpha, n+b-m-1\}+a-1 < \Re z < b+\mathrm{Min}\{\beta-1, n+a-m\}$:

$$\int_0^\infty t^z f(t)h(t)\,\mathrm{d}t = \sum_{k=0}^{n-1} B_k M[h; z+1-k-b] + \sum_{k=0}^{m-1} A_k M[f; z+k+1-a] + \int_0^\infty t^z f_n(t)h_m(t)\,\mathrm{d}t. \tag{15}$$

The point $z=0$ belongs to the strip of validity of the above formula. The desired result is this identity at $z=0$ with $h(t)$ replaced by $h(xt)$. □

Theorem 2.7. *Let $f \in \mathscr{F}_{b,\alpha}$ and $h \in \mathscr{H}_{a,\beta}$ with $a+b \in \mathbb{N}$. Let conditions I and II hold. Then, for any $n=1, 2, 3, \ldots$ and $m=n+a+b-1$,*

$$\begin{aligned}\int_0^\infty h(xt)f(t)\,\mathrm{d}t = &\sum_{k=0}^{a+b-2} A_k M[f; k+1-a]x^{k-a} + \sum_{k=0}^{n-1} x^{k+b-1} \\ &\times \left\{-B_k A_{k+a+b-1}\log x + \lim_{z\to 0}\,[B_k M[h; z+1-k-b] + A_{k+a+b-1} M[f; z+k+b]]\right\} \\ &+ \int_0^\infty f_n(t)h_m(xt)\,\mathrm{d}t.\end{aligned} \tag{16}$$

Proof. The proof is similar to the proof of the above theorem. In that proof we have not used $a+b \notin \mathbb{Z}$ to derive formula (15). In fact, that formula still holds for $a+b \in \mathbb{Z}$. But then both, $M[h; z+1-k-b]$ and $M[f; z+k+1-a]$ have a simple pole at $z=0$ and we cannot simply take $z=0$ in that formula. But we can take the limit $z \to 0$: from (ii) of Lemma 2.5 we have that, for $k=0, 1, 2, \ldots$, $M[h; z+1-k-b]$ has a simple pole at $z=0$ with residue $A_{a+b+k-1}$. From (ii) of Lemma 2.4 we have that, for $k=a+b-1, a+b-2, a+b-3, \ldots$, $M[f; z+k+1-a]$ has simple poles at $z=0$ with residue $-B_{k+1-a-b}$. Therefore, the first $a+b-1$ terms of the second sum in the right hand side of (15) have a finite limit at $z=0$. The poles of the remaining n terms of that sum (from $k=a+b-1$ up to $k=m=n+a+b-1$) cancel one by one with the poles of the n terms of the first sum in the right hand side of (15) and then, the limit $z \to 0$ exist. Therefore, taking the limit $z \to 0$ we obtain

$$\int_0^\infty h(t)f(t)\,dt = \sum_{k=0}^{a+b-2} A_k M[f;k+1-a] + \lim_{z\to 0}\sum_{k=0}^{n-1}\{B_k M[h;z+1-k-b] + A_{k+a+b-1}M[f;z+k+b]\} + \int_0^\infty f_n(t)h_m(t)\,dt. \tag{17}$$

Now, replace $h(t)$ by $h(xt)$ in the above equation and use

$$\lim_{z\to 0}\{x^{k+b-1}[B_k x^{-z}M[h;1+z-k-b] + A_{k+a+b-1}M[f;z+k+b]]\}$$
$$= x^{k+b-1}\left\{\lim_{z\to 0}[B_k M[h;1+z-k-b] + A_{k+a+b-1}M[f;z+k+b]] - A_{k+a+b-1}B_k\log x\right\}. \tag{18}$$

To derive this equality we have used $x^{-z} = 1 - z\log x + \mathcal{O}(z^2)$ and

$$M[h;z+1-b-k] = \frac{A_{k+a+b-1}}{z} + \mathcal{O}(1),$$

when $z \to 0$. This last identity follows from (ii) of Lemma 2.5. □

Theorem 2.8. *Let $f \in \mathcal{F}_{b,\alpha}$ and $h \in \mathcal{H}_{a,\beta}$ and $1-a-b \in \mathbb{N}$. Let conditions I and II hold. Then, for any $m=1,2,3,\ldots$ and $n = m+1-a-b$,*

$$\int_0^\infty h(xt)f(t)\,dt = \sum_{k=0}^{-a-b} B_k M[h;1-k-b]x^{k+b-1} + \sum_{k=0}^{m-1} x^{k-a} \times\left\{-A_k B_{k+1-a-b}\log x + \lim_{z\to 0}[B_{k+1-a-b}M[h;z+a-k] + A_k M[f;z+k+1-a]]\right\} + \int_0^\infty f_n(t)h_m(xt)\,dt. \tag{19}$$

Proof. It is similar to the proof of the above theorem using

$$\lim_{z\to 0}\{x^{k-a}[B_{k+1-a-b}x^{-z}M[h;z+a-k] + A_k M[f;z+k+1-a]]\}$$
$$= x^{k-a}\left\{\lim_{z\to 0}[B_{k+1-a-b}M[h;z+a-k] + A_k M[f;z+k+1-a]] - A_k B_{k+1-a-b}\log x\right\}$$

instead of (18). □

3. Asymptotic properties and error bounds

Theorem 3.1. *Within the hypotheses of Theorems* 2.6–2.8, *expansions* (8), (16) *and* (19) *are asymptotic expansions for small x*:

$$\int_0^\infty f_n(t)h_m(xt)\,dt = \mathcal{O}(x^{n+b-1}) \quad \text{when } x\to 0 \quad \text{and} \quad a+b\notin\mathbb{Z}, \tag{20}$$

that is, when $m = n + \lfloor a+b \rfloor$, and

$$\int_0^\infty f_n(t)h_m(xt)\,dt = \mathcal{O}(x^{m-a}\log x) \quad \text{when } x\to 0 \quad \text{and} \quad a+b\in\mathbb{Z}, \tag{21}$$

that is, when $m = n+a+b-1$.

Proof. On the one hand, from (i) of Definition 2.2, there is a $c_n^1>0$ and a t_0^1 such that $|f_n(t)|\leqslant c_n^1 t^{-n-b}$ for $t\geqslant t_0^1$. From (i) of Definition 2.1, there is a $c_m^2>0$ and a t_0^2 such that $|h_m(xt)|\leqslant c_m^2(xt)^{m-a}$ for $xt\leqslant t_0^2$. For small enough x we have that $t_0^1<t_0^2/x$ and we can take a $t_0\in[t_0^1,t_0^2/x]$. Then,

$$\int_0^\infty f_n(t)h_m(xt)\,\mathrm{d}t=\int_0^{t_0} f_n(t)h_m(xt)\,\mathrm{d}t+\int_{t_0}^\infty f_n(t)h_m(xt)\,\mathrm{d}t.$$

$$\left|\int_0^\infty f_n(t)h_m(xt)\,\mathrm{d}t\right|\leqslant c_m^2x^{m-a}\int_0^{t_0}|f_n(t)|t^{m-a}\,\mathrm{d}t+c_n^1x^{n+b-1}\int_{xt_0}^\infty|h_m(t)|t^{-n-b}\,\mathrm{d}t$$

$$\leqslant(x^{m-a}+x^{n+b-1})\left[c_m^2\int_0^{t_0}|f_n(t)|t^{m-a}\,\mathrm{d}t+c_n^1\int_0^\infty|h_m(t)|t^{-n-b}\,\mathrm{d}t\right]. \tag{22}$$

The integrals between brackets in the last line of (22) are finite for $0<n+a+b-m<1$ (when $a+b\notin\mathbb{Z}$ and $m=n+\lfloor a+b\rfloor$). Then, (20) follows from (22).

If $n+b=m-a+1$ (when $a+b\in\mathbb{Z}$), the second integral in the last line of (22) is divergent and inequality (22) is true but useless. In this case, using $h_m(t)=h_{m+1}(t)+A_mt^{m-a}$ in the second line of (22) we have:

$$\int_{xt_0}^\infty|h_m(t)|t^{-n-b}\,\mathrm{d}t\leqslant\int_1^\infty|h_m(t)|t^{-n-b}\,\mathrm{d}t+\int_{xt_0}^1(|h_{m+1}(t)|+|A_mt^{m-a}|)t^{-n-b}\,\mathrm{d}t\leqslant\int_1^\infty|h_m(t)|t^{-n-b}\,\mathrm{d}t$$

$$+\int_0^1|h_{m+1}(t)|t^{-n-b}\,\mathrm{d}t+|A_m\log(xt_0)|. \tag{23}$$

The last line above is finite for $0<n+a+b-m<2$ (in particular when $m=n+a+b-1$). Using (23) in the second line of (22), (21) follows. □

Remark 3.2. In [6, Chapter 6], the additional hypothesis $\mathrm{Sup}_{t\in(0,\infty)}\{|t^{n+\alpha}f_n(t)|\}<\infty$ is required to show the asymptotic character of the expansions. We see in Theorem 3.1 above that this extra hypothesis is not necessary.

Theorem 3.1 does not offer precise bounds for the remainder. The following proposition shows that a precise bound may be obtained if the bound $|f_n(t)|\leqslant c_n^1t^{-n-b}$ used in the proof of Theorem 3.1 holds $\forall t\in(0,\infty)$ and not only for $t\in[t_0^1,\infty)$ and the bound $|h_m(t)|\leqslant c_m^2t^{m-a}$ holds $\forall t\in(0,\infty)$ and not only for $t\in(0,t_0^2]$.

Proposition 3.3. *Let the remainder $f_n(t)$ in expansion* (3) *satisfy bound $|f_n(t)|\leqslant F_nt^{-n-b}\forall t\in(0,\infty)$ and let the remainder $h_m(t)$ in* (2) *satisfy bound $|h_m(t)|\leqslant H_mt^{m-a}\forall t\in(0,\infty)$ for some positive constants F_n and H_m. Then, the remainder in Theorems* 2.6–2.8 *satisfies the bound*

$$\left|\int_0^\infty f_n(t)h_m(xt)\,\mathrm{d}t\right|\leqslant\begin{cases}C_{n,m}^1x^{n+b-1} & \text{if } a+b\notin\mathbb{Z},\\ [C_{n,m}^2+F_nH_m|\log x|]x^{m-a} & \text{if } a+b\in\mathbb{Z},\end{cases}$$

with

$$C_{n,m}^1\equiv F_n\left[\frac{H_m}{1+\lfloor a+b\rfloor-a-b}+\frac{|A_{m-1}|+H_{m-1}}{b+a-\lfloor a+b\rfloor}\right]$$

and

$$C_{n,m}^2\equiv F_n(|A_{m-1}|+H_{m-1})+H_m(|B_{n-1}|+F_{n-1}).$$

Proof. Consider first the case $a+b\notin\mathbb{Z}$ ($m=n+\lfloor a+b\rfloor$). Write

$$\int_0^\infty f_n(t)h_m(xt)\,\mathrm{d}t=\frac{1}{x}\int_0^1 f_n\left(\frac{t}{x}\right)h_m(t)\,\mathrm{d}t+\frac{1}{x}\int_1^\infty f_n\left(\frac{t}{x}\right)h_m(t)\,\mathrm{d}t.$$

Introduce the decomposition $h_m(t)=h_{m-1}(t)-A_{m-1}t^{m-a-1}$ into the last integral in the right hand side above. The proposition follows after using $|f_n(t)|\leqslant F_nt^{-n-b}$, $|h_m(t)|\leqslant H_mt^{m-a}$ and $m=n+\lfloor a+b\rfloor$.

Consider now the case $a+b\in\mathbb{Z}$ ($m=n+a+b-1$). Write

$$\int_0^\infty f_n(t)h_m(xt)\,dt=\int_0^1 f_n(t)h_m(xt)\,dt+\int_1^{1/x} f_n(t)h_m(xt)\,dt+\int_{1/x}^\infty f_n(t)h_m(xt)\,dt.$$

Perform the change of variable $t\to t/x$ in the last integral, introduce the decomposition $f_n(t)=f_{n-1}(t)-B_{n-1}t^{1-n-b}$ into the first integral in the right hand side and the decomposition $h_m(t)=h_{m-1}(t)-A_{m-1}t^{m-a-1}$ into the last integral. The proposition follows using $|f_n(t)|\leqslant F_n t^{-n-b}$, $|h_m(t)|\leqslant H_m t^{m-a}$ and $m=n+a+b-1$. $\square$

4. Conclusions

When the positive moments of $f(t)$ do not exist, asymptotic expansions of integral transforms $\int_0^\infty f(t)h(xt)\,dt$ for small x where first obtained by M&W using the theory of distributions [6, Chapters 5 and 6] or of analytic continuation [5]. In this paper we have generalized the method used by M&W in [6, Chapter 6] in such a way that it may be applied not only to the functions h given there, but to any function $h\in\mathscr{H}_{a,\beta}$ (see Definition 2.1). We have new proofs for the expansions given in [5], using analytic continuation techniques in a different way.

The asymptotic character of the expansions in M&W's theory [6,5] is shown when the remainder in the expansion of the function $f(t)$ satisfies the additional hypothesis $|f_n(t)|\leqslant c_n t^{-n-b}\forall t\in(0,\infty)$. In Theorem 3.1 above we have shown that this extra hypothesis is not necessary, although it is quite convenient to have practical error bounds (see Proposition 3.2).

The method presented in this paper unifies classical procedures and distributional ones in a unique formulation (Theorems 2.6–2.8). Classical methods require (among other hypotheses) for $h(t)$ to have an expansion at $t=0$. The distributional M&W's technique requires (among other hypotheses) for $f(t)$ to have an expansion at $t=\infty$. The technique presented here requires both, an expansion of $h(t)$ at $t=0$ and an expansion of $f(t)$ at $t=\infty$ (and no more essential hypotheses). It is an exercise to check, for example, that the asymptotic expansions given in [6, Chapter 6] for Stieltjes transforms for large argument, Laplace transforms for small argument, Watsons lemma (Laplace transforms for large argument), fractional integrals for large arguments or Fourier transforms for small argument are trivial corollaries of Theorems 2.6–2.8 given above. Asymptotic expansions of Stieltjes transforms for small argument, Lambert transforms for small or large argument and Poisson transform for small or large argument may be easily derived from Theorems 2.6–2.8. Also, many asymptotic expansions of important special functions such as Elliptic integrals, Appell functions, Lauricela functions or Epstein–Hubbel integrals, previously obtained by using the distributional method, may be re-derived easily from those theorems. Moreover, accurate error bounds for these expansions follow from Proposition 3.2.

M&W's theory deals also with the possibility of $f(t)$ having an oscillatory asymptotic expansion at infinity of the form $f(t)=e^{ict}\sum_{k=0}^{n-1}B_k t^{-k-b}+f_n(t)$ instead of (3). Asymptotic expansions of Stieltjes transforms of these kind of functions are derived in [6, Chapter 6, Section 3]. Theorems 2.6–2.8, may be perhaps generalized to this kind of functions. Moreover, they may be generalized to complex values of the parameters and x. This is subject of further investigations.

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