Acta Applicandae Mathematicae **82:** 101–117, 2004.
© 2004 *Kluwer Academic Publishers. Printed in the Netherlands.*

Analytic Approximations for a Singularly Perturbed Convection–Diffusion Problem with Discontinuous Data in a Half-Infinite Strip

JOSÉ L. LÓPEZ and ESTER PÉREZ SINUSÍA
Departamento de Matemática e Informática, Universidad Pública de Navarra, 31006-Pamplona, Spain. e-mail: {jl.lopez,ester.perez}@unavarra.es

(Received: 12 August 2003; in final form: 28 October 2003)

Abstract. We consider a singularly perturbed convection–diffusion equation, $-\epsilon\Delta u + \mathbf{v}\cdot\nabla u = 0$, defined on a half-infinite strip, $(x, y) \in (0,\infty)\times(0,1)$ with a discontinuous Dirichlet boundary condition: $u(x,0) = 1$, $u(x,1) = u(0,y) = 0$. Asymptotic expansions of the solution are obtained from an integral representation in two limits: (a) as the singular parameter $\epsilon \to 0^+$ (with fixed distance r to the discontinuity point of the boundary condition) and (b) as that distance $r \to 0^+$ (with fixed ϵ). It is shown that the first term of the expansion at $\epsilon = 0$ contains an error function or a combination of error functions. This term characterizes the effect of discontinuities on the ϵ-behavior of the solution and its derivatives in the boundary or internal layers. On the other hand, near the point of discontinuity of the boundary condition, the solution $u(x, y)$ is approximated by a linear function of the polar angle at the point of discontinuity (0, 0).

Mathematics Subject Classifications (2000): 35C20, 41A60.

Key words: singular perturbation problem, discontinuous boundary data, asymptotic expansions, error function.

1. Introduction

Mathematical models that involve a combination of convective and diffusive processes are quite important in all the science, engineering and other fields where mathematical modeling is required. Very often, a dimensionless parameter that measures the relative strength of diffusion is quite small. This implies that thin boundary and/or interior layers are present in the solution and singular perturbation problems arise. This kind of problem appears, for example, in fluid or gas dynamics (Lu, 1973; Van Dyke, 1964), heat transfer (Aziz and Na, 1984; Bejan, 1984), theory of plates and shells (Knowles and Messic, 1964), or magnetohydrodynamic flow (Gold, 1962; Shercliff, 1953). An extensive selection of singularly perturbed convection–diffusion problems of the physics or engineering may be found in (Morton, 1996): pollutant dispersal in a river estuary, vorticity transport in the incompressible Navier–Stokes equations, atmospheric pollution, groundwater transport, turbulence transport, etc. Besides the small perturbation parameter, an-other source of singular behavior for the solution are the (possible) discontinuities of the boundary data. Think, for example, of the transportation of contaminant in a river with a source of contamination located in a definite portion of the side of water.

Mathematically speaking, a singularly perturbed convection–diffusion problem is a boundary value problem of the second order in which the coefficients of the second-order derivatives are small. In this paper, we focus our attention on two-dimensional linear convection–diffusion (elliptic) problems of the form: find a function $u \in \mathcal{C}(\bar{\Omega}) \cap \mathcal{D}^2(\Omega)$ such that

$$\begin{cases} -\epsilon \Delta u + \mathbf{v} \cdot \nabla u = 0, & x \in \Omega \subset \mathbb{R}^2, \\ u(x)|_{\partial\Omega} = f(\tilde{x}), & \tilde{x} \in \partial\Omega, \end{cases} \qquad (P)$$

where ϵ is a small positive parameter, $\mathbf{v}$ is the convection vector, $\tilde{x}$ is a variable which lives in $\partial\Omega$, and $f(\tilde{x})$ is the Dirichlet data.

The location and shape of the boundary layers of u depend, among other things, on the prescribed velocity field $\mathbf{v}$, on the shape of the boundary $\partial\Omega$, and on the discontinuities of $f(\tilde{x})$. For example, regular boundary layers of size $\mathcal{O}(\epsilon)$ appear on the outflow boundary, whereas parabolic boundary layers of size $\mathcal{O}(\sqrt{\epsilon})$ appear along the characteristic boundaries. For more details on the shape and nature of boundary layers see, for example, (Cook and Ludford, 1973; Eckhaus, 1973; Eckhaus and de Jager, 1966; Hedstrom and Osterheld, 1980; Il'in, 1992) and references therein.

The knowledge of an asymptotic expansion for the solution may help in the development of a suitable numerical method for this kind of problems, because it gives the qualitative behavior of the solution (Vasil'eva *et al.*, 1995, p. 6). There is an extensive literature devoted to the construction of approximated solutions of singular perturbation problems, based on perturbative techniques. The book of Il'in (1992) contains a quite exhaustive and general analysis for different equations and domains. Other important references are, for example, (Eckhaus, 1973; Kevorkian and Cole, 1996) or (O'Malley, 1974).

The standard perturbative approach consists of trying a series representation $u(x, y; \epsilon) = \sum_{k=0}^{\infty} u_k(x, y)\epsilon^k$ for the solution. The functions $u_k(x, y)$ are determined introducing this series into the differential equation, equating the coefficients of like powers of ϵ, and recursively solving a sequence of problems for $u_0, u_1, u_2, \ldots$. But such a perturbative analysis does not work for the discontinuous Dirichlet boundary condition $f(\tilde{x})$ because the coefficients u_k of the expansion contain derivatives of the boundary condition, whereas the solution of the elliptic problem (P) is smooth inside the domain (Temme, 1998). A more sophisticated perturbative approach consists of the matching of asymptotic expansions: the solution is approximated by an *outer approximation* plus several *inner boundary-layer expansions*. Some authors have shown in specific examples that the matched asymptotic expansion method works even with discontinuous boundary data (see, for example, Cook *et al.*, 1972; Grasman, 1974; Kevorkian, 1990, p. 537; Kevorkian and Cole, 1996; Shih, 1996). This technique is quite cumbersome. It requires "a priori" knowledge of the location and nature of the boundary layers in order to choose the correct stretched variable for every inner expansion. In general, this method does not proportionate an expansion uniformly valid in the whole domain, but different expansions for different subdomains of Ω (of course, they match smoothly on overlapping subdomains). Moreover, the calculation of the coefficients of those expansions require to solve a complicated boundary problem for every coefficient.

We propose here a different method to approximate the solution of linear singularly perturbed convection–diffusion problems with discontinuous Dirichlet data, based on the knowledge of an exact representation of the solution. This method (i) does not require a priori knowledge of the location and nature of the boundary layers, (ii) the calculation of the coefficients of the expansion is straightforward, and (iii) produces an expansion uniformly valid in the whole domain (except in the vicinity of discontinuities of the boundary condition).

If the Dirichlet data $f(\tilde{x})$ is continuous except at a finite number of points, where it presents jump discontinuities, then $f(\tilde{x}) = g(\tilde{x}) + \sum_{k=1}^{n} a_k \Theta(\tilde{x} - \tilde{x}_k)$, where $g(\tilde{x})$ is continuous, $a_k \in \mathbb{R}$, and Θ denotes the step function. We can decompose (P) into the problem:

$$\begin{cases} -\epsilon \Delta u + \mathbf{v} \cdot \nabla u = 0, & x \in \Omega \subset \mathbb{R}^2, \\ u(x)|_{\partial\Omega} = g(\tilde{x}), & \tilde{x} \in \partial\Omega, \end{cases} \qquad (P_0)$$

plus n problems of the form:

$$\begin{cases} -\epsilon \Delta u + \mathbf{v} \cdot \nabla u = 0, & x \in \Omega \subset \mathbb{R}^2, \\ u(x)|_{\partial\Omega} = \Theta(\tilde{x} - \tilde{x}_k), & \tilde{x} \in \partial\Omega. \end{cases} \qquad (P_k)$$

An asymptotic expansion for (P_0) may be obtained by the method of matched asymptotic expansions (Il'in, 1992; O'Malley, 1974), whereas an asymptotic expansion for the problems (P_k) may be obtained from an exact representation of the solution.

Some particular problems of the form (P_k) have been already considered in the literature for certain domains Ω. For example, Hedstrom and Osterheld (1980) studied the problem $\epsilon \Delta u - \partial_y u = 0$ on the positive quarter plane with boundary conditions $u(x, 0) = 0$ and $u(0, y) = 1$. They obtained the first two terms of the asymptotic expansion of u for $\epsilon \to 0^+$ from the Fourier integral representation of u. The first term of this expansion is an error function. A more detailed investigation has been developed by Temme (1971): an integral representation for u is obtained from the associated Helmholtz equation and a complete asymptotic expansion of u for $\epsilon \to 0^+$ is derived from this integral representation. The same equation $\epsilon \Delta u - \partial_y u = 0$, but in a generic sector, is considered in (Temme, 1974), where an integral representation for u is obtained from the associated Helmholtz equation. Different asymptotic expansions as $\epsilon \to 0^+$ are obtained depending on the angle of the sector, and again, the error function plays an important role in the analysis. The role of the error function in convection–diffusion problems with discontinuous boundary data has also been pointed out in (Kevorkian, 1990, Section 8.3.3; Kevorkian and Cole, 1996, Section 3.1). In all these problems, the approximation is not valid near the discontinuities of the boundary condition.

The problems mentioned in the above paragraph are all defined over infinite domains. However, for practical purposes, it is more interesting to consider bounded domains (rectangles, circles, ...). As an intermediate step towards the rectangle, Cook and Ludford propose the problem $\epsilon\Delta u - \partial_y u = 0$ on a half-infinite strip (they consider continuous Dirichlet conditions) (Cook and Ludford, 1973). In this paper, we analyze this problem for a more general convection vector, $\epsilon\Delta u - \mathbf{v}\cdot\nabla u = 0$, and a discontinuous Dirichlet condition at a corner. We try to shed light on the influence that the discontinuities of the boundary conditions have on the boundary or interior layers of the solution. We want to investigate if, as in the above mentioned examples, the solution is approximated by an error function. We use similar techniques to those used in (Temme, 1971).

As in the references mentioned, the starting point is an integral representation for the solution. From this integral, we derive asymptotic expansions for the solution, not only in the singular limit $\epsilon \to 0^+$, but also in the limit $r \to 0^+$, where r represents the distance to the discontinuity at the boundary. Then, we approximate the solution on the whole domain, including the neighborhood of the discontinuity point.

In Section 2, we obtain an integral representation for the solution. In Section 3, we derive two asymptotic expansions of the solution as $\epsilon \to 0^+$, whereas in Section 4, we derive an asymptotic expansion as $r \to 0^+$. Some comments are postponed to Section 5.

2. The Problem and an Exact Solution

We consider a singularly perturbed convection–diffusion problem defined on a half-infinite strip parallel to the x-axis with "an infinite source of contamination" located at the whole x-axis:

$$\begin{cases} -\epsilon\Delta U + \mathbf{v}\cdot\nabla U = 0 \quad \text{in } \Omega \equiv (0,\infty)\times(0,1), \\ U(x,0) = 1, \qquad U(x,1) = U(0,y) = 0, \\ \quad U \in \mathcal{C}(\bar{\Omega}\setminus\{(0,0)\}) \cap \mathcal{D}^2(\Omega). \end{cases} \tag{P}$$

The Dirichlet boundary condition has a discontinuity at the point $(0, 0)$. The solution of the same problem, but with a point of discontinuity at $(0, 1)$ is analogous.

After the change of the dependent variable $U(x, y) = F(x, y)\exp(\mathbf{v}\cdot\mathbf{r}/(2\epsilon))$, where $\mathbf{r} \equiv (x, y)$, problem (P) is transformed to the Helmholtz equation for $F(x, y)$:

$$\begin{cases} \Delta F - w^2 F = 0 \quad \text{in } \Omega, \\ F(x,0) = \mathrm{e}^{-wx\sin\beta}, \qquad F(x,1) = F(0,y) = 0, \\ \quad F \in \mathcal{C}(\bar{\Omega}\setminus\{(0,0)\}) \cap \mathcal{D}^2(\Omega), \end{cases} \tag{1}$$

where $w \equiv |\mathbf{v}|/(2\epsilon)$ and $\mathbf{v} \equiv |\mathbf{v}|(\sin\beta, \cos\beta)$.

The solution $U(x, y)$ of problem (P) may not be unique unless we impose a convenient condition upon $U(x, y)$ concerning its growth at infinity. In fact, using the maximum principle of elliptic partial differential equations, we can see that the problem: Find $U \in \mathcal{C}(\bar{\Omega} \setminus \{(0, 0)\}) \cap \mathcal{D}^2(\Omega)$ such that

$$\begin{cases} -\epsilon\Delta U + \mathbf{v}\cdot\nabla U = 0 \quad \text{in } \Omega, \\ U(x,0) = 1, \qquad U(x,1) = U(0,y) = 0, \\ \quad |\lim_{x\to\infty} U(x,y)| < \infty, \end{cases} \tag{P'}$$

has a unique solution for $0 \leqslant \beta \leqslant \pi$.

The exact solution of problem (1), valid for $0 \leqslant \beta \leqslant \pi/2$, may be obtained by taking the sine transform of the differential equation with respect to x:

$$F(x, y) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{t\mathrm{e}^{ixt}}{t^2 + w^2\sin^2\beta}\frac{\sinh((1-y)\sqrt{w^2+t^2})}{\sinh(\sqrt{w^2+t^2})}\,\mathrm{d}t. \tag{2}$$

This integral must be understood as a principal value integral if $\beta = 0$. Then, the function $U_\beta(x, y) \equiv \mathrm{e}^{w(x\sin\beta + y\cos\beta)}F(x, y)$, with $F(x, y)$ defined in (2), is the solution of (P'). We write

$$U_\beta(x,y) = \frac{\mathrm{e}^{w(x\sin\beta+y\cos\beta)}}{\pi i}\int_{-\infty}^{\infty}\left(\mathrm{e}^{-y\sqrt{w^2+t^2}} - \mathrm{e}^{(y-2)\sqrt{w^2+t^2}}\right)\times \\ \times\frac{t\mathrm{e}^{ixt}H(t)\,\mathrm{d}t}{t^2+w^2\sin^2\beta}, \tag{3}$$

where $H(t) \equiv (1 - \mathrm{e}^{-2\sqrt{w^2+t^2}})^{-1}$. Splitting $H(t)$ into a constant term plus an exponentially decaying one:

$$H(t) = 1 + \frac{\mathrm{e}^{-2\sqrt{w^2+t^2}}}{1-\mathrm{e}^{-2\sqrt{w^2+t^2}}}, \tag{4}$$

the right-hand side of (3) becomes the sum of two integrals. After changing the variable $t = w\sinh u$ in the first of these integrals, we obtain

$$U_\beta(x,y) = \mathrm{e}^{w(x\sin\beta+y\cos\beta)}[G(x,y) - G(x,2-y) + R(x,y)], \tag{5}$$

where

$$G(x,Y) \equiv \frac{1}{\pi i}\int_{-\infty}^{\infty}\mathrm{e}^{w[ix\sinh u - Y\cosh u]}\frac{\sinh u\cosh u}{\sinh^2 u + \sin^2\beta}\,\mathrm{d}u$$

and $R(x, y)$ is given by:

$$R(x,y) \equiv \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{t\mathrm{e}^{ixt}}{(t^2+w^2\sin^2\beta)\mathrm{e}^{2\sqrt{w^2+t^2}}}\frac{\sinh((1-y)\sqrt{w^2+t^2})}{\sinh(\sqrt{w^2+t^2})}\,\mathrm{d}t. \tag{6}$$

This integral must also be understood as a principal value integral if $\beta = 0$. Using the polar variables

$$\begin{cases} x = r\sin\phi, & 0 < r < \infty, \\ Y = r\cos\phi, & 0 \leqslant \phi \leqslant \pi/2 \end{cases} \tag{7}$$

adapted to each integral $G(x, Y)$, we have

$$G(x, Y) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \mathrm{e}^{-rw\cosh(u-i\phi)} \frac{\sinh u \cosh u}{\sinh^2 u + \sin^2\beta}\, \mathrm{d}u. \tag{8}$$

The poles of the integrand of $G(x, Y)$ are located at the points $u = \pm i\beta + k\pi i$, $k \in \mathbb{Z}$ and the real part of the exponent reads $-rw\cosh(\Re u)\cos(\phi - \Im u)$. Therefore, we can use the Cauchy Residue Theorem for shifting the integration contour in each integral $G(x, Y)$ to the straight line $\Im u = \phi$. We distinguish two cases:

Case 1: $0 < \beta \leqslant \pi/2$ (see Figure 2(a)). We just apply the Cauchy Residue Theorem to obtain

$$G(x, Y) = I(x, Y) + \mathrm{e}^{-w(x\sin\beta + Y\cos\beta)}\left[\chi_{(0,\pi/2)}(\phi - \beta) + \frac{1}{2}\delta_{\phi,\beta}\right], \tag{9}$$

where $I(x, Y)$ is given by

$$I(x, Y) \equiv \frac{1}{\pi i}\int_{-\infty}^{\infty} \mathrm{e}^{-rw\cosh t} \frac{\sinh(t + i\phi)\cosh(t + i\phi)}{\sinh^2(t + i\phi) + \sin^2\beta}\, \mathrm{d}t. \tag{10}$$

The integral (10) must be understood as a principal value integral if $\phi = \beta$.

Case 2: $\beta = 0$ (see Figure 2(b)). We apply again the Cauchy Residue Theorem, but taking into account that the pole $u = 0$ is now on the integration contour,

$$G(x, Y) = I(x, Y) + \frac{\mathrm{e}^{-wY}}{2}, \tag{11}$$

where $I(x, Y)$ is given in (10). From (5), (9), and (11) we have that

$$\begin{aligned} U_\beta(x, y) = {} & \frac{1}{2}\big[\chi_A(x, y) + \chi_{A_0}(x, y) - \mathrm{e}^{2w(y-1)\cos\beta}(\chi_B(x, y) + \chi_{B_0}(x, y))\big] + \\ & + \mathrm{e}^{w(y\cos\beta + x\sin\beta)}[I(x, y) - I(x, 2 - y) + R(x, y)], \end{aligned} \tag{12}$$

where the functions $\chi_A(x, y)$, $\chi_{A_0}(x, y)$, $\chi_B(x, y)$, and $\chi_{B_0}(x, y)$ are the characteristic functions of the respective regions A, A_0, B, and B_0 depicted in Figure 1:

$$\begin{aligned} A &\equiv \{(x, y) \in \Omega,\ y\tan\beta \leqslant x < \infty\}, \\ B &\equiv \{(x, y) \in \Omega,\ (2 - y)\tan\beta \leqslant x < \infty\}, \end{aligned} \tag{13}$$

and, analogously, for $\chi_{A_0}(x, y)$ and $\chi_{B_0}(x, y)$, where regions A_0 and B_0 are the interior sets of A and B, respectively.

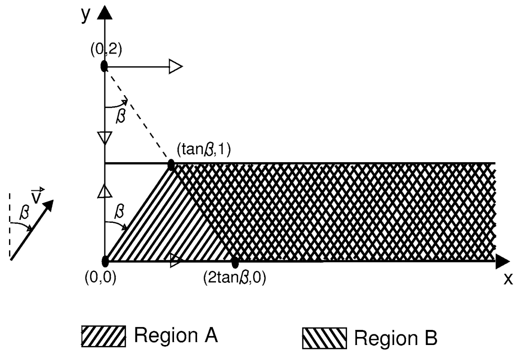
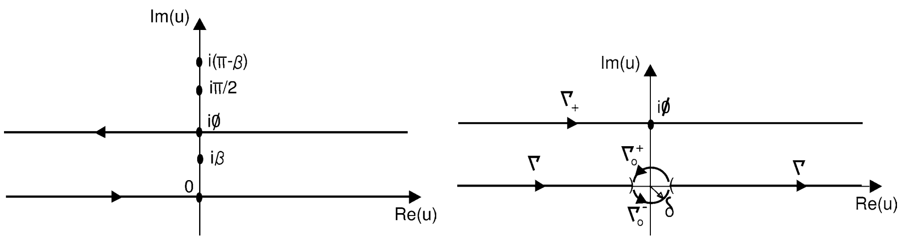


Figure 1. Region A is limited by the straights $y = 0$, $y = 1$, and $x = y\tan\beta$. Region B is limited by the straights $y = 0$, $y = 1$, and $x = (2 - y)\tan\beta$. Every function $G(x, Y)$ in (5) contributes to $U_\beta(x, y)$ with $I(x, Y)$ and the second term in (9) or (11). The first line in (12) (related to the regions A and B) comes from the addition of those second terms. The angle ϕ is different in each one of those terms and is defined by the coordinate system (x, Y) involved in each of them (see Equation (7)): the origin of the two coordinate systems (x, Y) involved in (5) are $(0, 0)$ and $(0, 2)$ and are depicted in the figure.



(a) $0 < \beta \leqslant \pi/2$ (b) $\beta = 0$

Figure 2. (a) The integrand in (8) has just one single pole situated between the straights $u = 0$ and $u = i\pi/2$: $u = i\beta$. (b) The path $\Gamma_0 \equiv \Gamma_0^+ \cup \Gamma_0^-$ is a small circle of radius δ. The path Γ (with $\delta \to 0^+$) is the u-integration contour for the integrand $h(u)$ in $G(x, Y)$. With the shift $u \to u + i\phi$, $\Gamma \cup \Gamma_0^+ \to \Gamma_+$. Therefore, $\int_\Gamma h(u) - \frac{1}{2}\int_{\Gamma_0} h(u) = \int_{\Gamma_+} h(u)$.

If we had not split $H(t)$ in (3) (see Equation (4)), after an analysis similar to the one developed between formulas (3) and (13), we would obtain

$$U_\beta(x, y) = \frac{e^{yw\cos\beta}}{2}\frac{\sinh[(1-y)w\cos\beta]}{\sinh[(w\cos\beta]}[\chi_A(x, y) + \chi_{A_0}(x, y)] + J(x, y), \quad (14)$$

where

$$J(x, y) \equiv \frac{e^{w(y\cos\beta + x\sin\beta)}}{\pi i}\int_{-\infty}^{\infty} e^{wy\cosh(t+i\phi) - rw\cosh t} \times$$
$$\times \frac{\sinh(t+i\phi)\cosh(t+i\phi)}{\sinh^2(t+i\phi) + \sin^2\beta}\frac{\sinh[(1-y)w\cosh(t+i\phi)]}{\sinh[(w\cosh(t+i\phi)]}\,dt. \quad (15)$$

This integral must be understood as a principal value integral if $\phi = \beta$. In this formula $x = r\sin\phi$ and $y = r\cos\phi$.

OBSERVATION 1. The explicit representations given in (12) and (14) are only valid when the angle β between the convection vector **v** and the y-axis is restricted to the interval $[0, \pi/2]$. Nevertheless, an explicit solution $U(x, y)$ of problem (P'), whatever the direction of **v** is, may be obtained as follows:

$$U(x,y) = \begin{cases} U_\beta(x,y), \quad \text{if } 0 \leqslant \beta \leqslant \pi/2, \\ e^{2wy\cos\beta} U_{\pi-\beta}(x,y), \quad \text{if } \pi/2 < \beta \leqslant \pi, \\ e^{2wx\sin\beta} U_{-\beta}(x,y) + \dfrac{(1-e^{2wx\sin\beta})(e^{2w\cos\beta} - e^{2wy\cos\beta})}{e^{2w\cos\beta}-1}, \\ \quad \text{if } -\pi \leqslant \beta \leqslant 0, \beta \neq -\pi/2, \\ e^{-2wx} U_{\frac{\pi}{2}}(x,y) + (1-e^{-2wx})(1-y), \quad \text{if } \beta = -\pi/2, \end{cases}$$

where $U_\beta(x, y)$ is given in (12) or (14). Therefore, in the remainder of the paper, we restrict ourselves to $\beta \in [0, \pi/2]$.

3. Asymptotic Expansion of $U(x, y)$ in the Singular Limit

We are going to show that, as $\epsilon \to 0^+$ and $r \geqslant r_0 > 0$, the solution $U_\beta(x, y)$ of problem (P') may be approximated by a combination of error and elementary functions plus the asymptotic expansion in powers of ϵ.

In this section, we denote by Ω^* the upper half-plane indented at the origin (see Figure 3): $\Omega^* \equiv \Omega \setminus D_{r_0}(0, 0)$.

We will consider first the representation for $U_\beta(x, y)$ given in (12). For large w and fixed r, the asymptotic features of the integral $I(x, Y)$ defined in (10) are: (i) there is a saddle point at $t = 0$. (ii) The pole situated at $t = i(\beta - \phi)$ and the saddle point coalesce as $\phi \to \beta$. A uniform asymptotic expansion of this kind of integrals is obtained by using the error function as the basic approximant (Wong, 1989, Chapter 7). Therefore, we split off the pole of the integrand at $t = i(\beta - \phi)$:

$$\frac{\sinh(t+i\phi)\cosh(t+i\phi)}{\sinh^2(t+i\phi)+\sin^2\beta} = \frac{1+\delta_{\beta,\pi/2}}{4\sinh\frac{1}{2}(t+i(\phi-\beta))} + f(t,\phi,\beta),$$

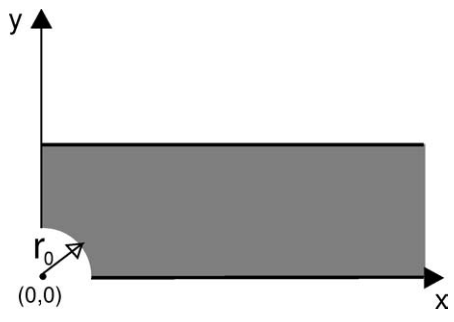


Figure 3. Indented region Ω^* considered in this section.

where obviously, we have just defined

$$f(t,\phi,\beta) \equiv \frac{\sinh(t+i\phi)\cosh(t+i\phi)}{\sinh^2(t+i\phi)+\sin^2\beta} - \frac{1+\delta_{\beta,\pi/2}}{4\sinh\frac{1}{2}(t+i(\phi-\beta))}.$$

Using the complementary error function representation (Temme, 1971)

$$\mathrm{e}^{-r\cos\alpha}\,\mathrm{erfc}\left(\sqrt{2r}\sin\frac{\alpha}{2}\right) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\mathrm{e}^{-r\cosh t}\frac{\mathrm{d}t}{\sinh\frac{1}{2}(t-i\alpha)}, \quad 0<\alpha<2\pi, \tag{16}$$

we obtain that the integral $I(x,Y)$ reads

$$I(x,Y) = \frac{1+\delta_{\beta,\pi/2}}{2}\,\mathrm{sign}\left[\beta - \arctan\left(\frac{x}{Y}\right)\right]\frac{\mathrm{erfc}\sqrt{w\zeta(x,Y)}}{\mathrm{e}^{w(Y\cos\beta+x\sin\beta)}} + \bar{I}(x,Y), \tag{17}$$

where the $\mathrm{sign}(0)=0$ and, using (7),

$$\bar{I}(x,Y) \equiv \frac{1}{\pi i}\int_{-\infty}^{\infty}\mathrm{e}^{-rw\cosh t} f(t,\phi,\beta)\,\mathrm{d}t. \tag{18}$$

Therefore, from (12) we obtain

$$U_\beta(x,y) = U_\beta^0(x,y) + \frac{1}{2\pi\sqrt{2w}}U_\beta^1(x,y), \tag{19}$$

with

$$\begin{aligned} U_\beta^0(x,y) \equiv{}& \frac{1}{2}(1+\delta_{\beta,\pi/2})\left\{\mathrm{sign}\left[\beta-\arctan\left(\frac{x}{y}\right)\right]\mathrm{erfc}\sqrt{w\zeta(x,y)} - \right.\\ &\left. - \mathrm{e}^{2(y-1)w\cos\beta}\,\mathrm{sign}\left[\beta-\arctan\left(\frac{x}{2-y}\right)\right]\mathrm{erfc}\sqrt{w\zeta(x,2-y)}\right\} + \\ &+ \frac{1}{2}\big[\chi_A(x,y)+\chi_{A_0}(x,y) - \mathrm{e}^{2(y-1)w\cos\beta}(\chi_B(x,y)+\chi_{B_0}(x,y))\big], \end{aligned} \tag{20}$$

χ_A, χ_{A_0}, χ_B, and χ_{B_0} defined in (13), the $\mathrm{sign}(0)$ must be understood as zero,

$$\zeta(x,Y) \equiv \sqrt{x^2+Y^2} - x\sin\beta - Y\cos\beta, \tag{21}$$

and

$$U_\beta^1(x,y) = 2\pi\sqrt{2w}\,\mathrm{e}^{w(y\cos\beta+x\sin\beta)}\big[\bar{I}(x,y) - \bar{I}(x,2-y) + R(x,y)\big]. \tag{22}$$

We will show that $U_\beta^1(x,y)$ has an asymptotic expansion in inverse powers of w. We substitute the variable $u=\sinh(t/2)$ in (18):

$$\bar{I}(x,Y) \equiv \frac{\mathrm{e}^{-wr}}{\pi}\int_0^{\infty}\mathrm{e}^{-2rwu^2}g(u^2,\phi,\beta)\,\mathrm{d}u, \tag{23}$$

where

$$g(u,\phi,\beta)=\frac{1}{\sqrt{1+u}}\frac{[\sin^2(\beta)-\sin^2(\phi)-4\cos(2\beta)u(u+1)]\sin(2\phi)}{(2u+1+s_+)(2u+1-s_+)(2u+1+s_-)(2u+1-s_-)}-$$
$$-\frac{(1+\delta_{\beta,\pi/2})\sin(\frac{1}{2}(\beta-\phi))}{2(u+\sin^2\frac{1}{2}(\beta-\phi))}$$

and we have denoted $s_\pm\equiv\cos(\beta\pm\phi)$.

The function $g(u,\phi,\beta)$ has a Taylor expansion at $u=0$ for each $\phi\in[0,\pi/2]$:

$$g(u,\phi,\beta)=\sum_{k=0}^{n-1}\frac{g^{(k)}(0,\phi,\beta)}{k!}u^k+g_n(u,\phi,\beta), \tag{24}$$

where

$$g_n(u,\phi,\beta)\equiv\frac{g^{(n)}(\xi,\phi,\beta)}{n!}u^n$$

for some $\xi\in(0,u)$. The singularities of $g(u,\phi,\beta)$ are away from the positive real axis (let us write d for the distance from the closest one of those singularities to the positive real axis). Therefore, using the Cauchy formula for the derivative $g^{(n)}(\xi,\phi,\beta)$, we see that

$$|g_n(u,\phi,\beta)|\leqslant M\frac{u^n}{d^n}, \tag{25}$$

where M is a bound for $g(w,\phi,\beta)$ on the portion of the complex w-plane surrounding the positive real axis: $\{w\in\mathbb{C},\ |w-u|<d,\ u\in\mathbb{R}^+\}$.

Introducing (24) in (23) and interchanging the sum and integral, we obtain

$$\bar{I}(x,Y)=\frac{\mathrm{e}^{-wr}}{\pi}\left[\sum_{k=0}^{n-1}\frac{g^{(k)}(0,\phi,\beta)}{2k!}\frac{\Gamma(k+1/2)}{(2wr)^{k+1/2}}+\bar{R}_n(x,Y)\right], \tag{26}$$

where, using (7),

$$\bar{R}_n(x,Y)\equiv\int_0^\infty \mathrm{e}^{-2rwu^2}g_n(u^2,\phi,\beta)\,\mathrm{d}u.$$

Therefore, using (25), we have that

$$|\bar{R}_n(x,Y)|\leqslant M\frac{\Gamma(n+1/2)}{(2wd\bar{r})^n},\quad \bar{r}^2\equiv x^2+Y^2. \tag{27}$$

Then we obtain

$$U_\beta^1(x,y)=\sum_{k=0}^{n-1}\frac{T_k(x,y)}{(2w)^k}+R_n(x,y), \tag{28}$$

with

$$T_k(x, y) \equiv \frac{\Gamma(k+\frac{1}{2})}{k!}\left[\frac{g^{(k)}(0,\phi,\beta)}{r^{k+1/2}}\mathrm{e}^{-w\zeta(x,y)} - \frac{g^{(k)}(0,\phi_2,\beta)}{r_2^{k+1/2}}\mathrm{e}^{-w\zeta(x,2-y)}\right], \quad (29)$$

where $r^2 \equiv x^2+y^2$, $r_2^2 \equiv x^2+(2-y)^2$, $\phi \equiv \arctan(x/y)$, $\phi_2 \equiv \arctan(x/(2-y))$ and

$$R_n(x, y) = 2\pi\sqrt{2w}\mathrm{e}^{w(y\cos\beta+x\sin\beta)} \times \times \left[\frac{\mathrm{e}^{-wr}}{\pi}\bar{R}_n(x, y) - \frac{\mathrm{e}^{-wr_2}}{\pi}\bar{R}_n(x, 2-y) + R(x, y)\right]. \quad (30)$$

Finally, we show that $R(x, y)$ in (22) is exponentially small. We perform manipulations on $R(x, y)$ (defined in (6)) similar to those performed on $F(x, y)$ in the previous section: a change of variable $t \to s$ given by $t = w\sinh s$ and a shift in the integration contour $s \to u$ with $u = s + i\phi_2$. Then we are led to

$$R(x, y) = -\mathrm{e}^{-wx\sin\beta-2w\cos\beta}\frac{\sinh[(1-y)w\cos\beta]}{\sinh[w\cos\beta]}\left[\chi_{(0,\pi/2)}(\phi_2-\beta) + \frac{1}{2}\delta_{\phi_2,\beta}\right] + + \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-wr_2\cosh u - y\cosh(u+i\phi_2)}\frac{\sinh(u+i\phi_2)\cosh(u+i\phi_2)}{\sin^2\beta+\sinh^2(u+i\phi_2)} \times \times \frac{\sinh[(1-y)w\cosh(u+i\phi_2)]}{\sinh[w\cosh(u+i\phi_2)]}\,\mathrm{d}u. \quad (31)$$

After the change of variable $u \to t$ given by $t = \sinh(u/2)$ in the last integral, I_R, we can see that this integral has a bound of the form

$$|I_R| \leqslant \bar{M}\mathrm{e}^{-wr_2-w\cos\phi_2}, \quad (32)$$

where $\bar{M}$ is a positive constant independent of x and y. Then, $|R(x, y)| \leqslant \bar{M}\mathrm{e}^{-\alpha w}$ for some $\alpha > 0$ uniformly in Ω. Joining the bounds obtained in (27) for $R_n(x, y)$, $R_n(x, 2-y)$, and the last bound given for $R(x, y)$, we can deduce from (30) that

$$|R_n(x, y)| \leqslant M\frac{\Gamma(n+1/2)}{(2wdr)^n}\mathrm{e}^{-w\zeta(x,y)}. \quad (33)$$

If we consider for $U_\beta(x, y)$ the representation given in (14) instead of the one given in (12), the solution $U(x, y)$ of (P') reads

$$U_\beta(x, y) = \tilde{U}_\beta^0(x, y) + \frac{\mathrm{e}^{-w\zeta(x,y)}}{\sqrt{2wr}}\tilde{U}_\beta^1(x, y), \quad (34)$$

where

$$\tilde{U}_\beta^0(x, y) \equiv \frac{1}{2}\left\{(1+\delta_{\beta,\pi/2})\,\mathrm{sign}\left[\beta - \arctan\left(\frac{x}{y}\right)\right]\mathrm{erfc}\sqrt{w\zeta(x, y)} + + \chi_A(x, y) + \chi_{A_0}(x, y)\right\}\mathrm{e}^{wy\cos\beta}\frac{\sinh[(1-y)w\cos\beta]}{\sinh[w\cos\beta]}. \quad (35)$$

The function $\tilde{U}^1_\beta(x, y)$ has an asymptotic expansion in powers of w^{-1}:

$$\tilde{U}^1_\beta(x, y) = \sum_{k=0}^{n-1} \frac{\tilde{T}_k(x, y)}{(2wr)^k} + \tilde{R}_n(x, y), \tag{36}$$

where empty sums must be understood as zero. The coefficients $\tilde{T}_k(x, y)$ are smooth functions of x and y and $\mathcal{O}(1)$ as $w \to \infty$ uniformly for $(x, y) \in \Omega^*$.

The remainder $\tilde{R}_n(x, y)$ satisfies a bound of the form

$$|\tilde{R}_n(x, y)| \leqslant M \frac{\Gamma(n + 1/2)}{(2wdr)^n}, \tag{37}$$

for some positive constants M and d whose meaning is similar to the meaning of constants M and d in formula (33).

OBSERVATION 2. From (19), (28), and (33) or from (34), (36), and (37) we see that $U_\beta(x, y) = U^0_\beta(x, y) + \mathcal{O}(\sqrt{\epsilon})$ or $U_\beta(x, y) = \tilde{U}^0_\beta(x, y) + \mathcal{O}(\sqrt{\epsilon})$ as $\epsilon \to 0^+$ and away from the point $(0, 0)$ ($\tilde{U}^0_\beta(x, y) - U^0_\beta(x, y) = \mathcal{O}(\sqrt{\epsilon})$). Then, the first-order approximation to the solution of (P') is a combination of one or two error functions and step functions. The error functions in (20) exhibit an interior layer of width $\mathcal{O}(\sqrt{\epsilon})$ and parabolic level lines of the equation $\zeta(x, y) = \text{constant}$. The combination of the error functions and the last line in (20) exhibits a regular boundary layer of width $\mathcal{O}(\epsilon)$ near the piece of the outflow boundary situated between the points $(\tan\beta, 1)$ and $(\infty, 1)$ and a corner layer of area $\mathcal{O}(\sqrt{\epsilon}) \times \mathcal{O}(\epsilon)$ near the point $(\tan\beta, 1)$ (see Figure 4).

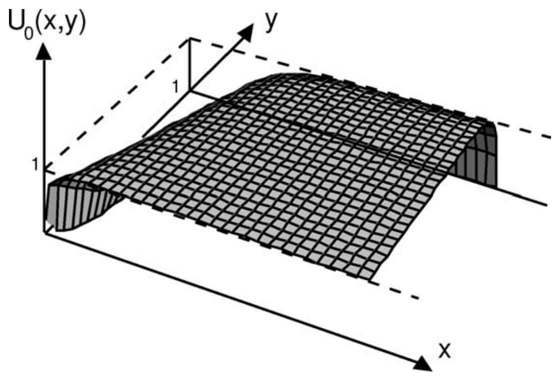


Figure 4. Graph of the first order approximation, $U^0_\beta(x, y)$, to the solution of problem (P') for $\beta = \pi/4$ and $\epsilon = 0.1$. The convection vector $(\tan\beta, 1)$ "drags" the discontinuity of the boundary condition at $(0, 0)$ originating a parabolic layer along the direction of that vector. Moreover, a regular boundary layer occurs at the outflow boundary between $x = \tan\beta$ and $x = \infty$, and a corner layer around the point $(\tan\beta, 1)$ in order to satisfy the boundary condition at $y = 1$.

4. Asymptotic Expansion of $U(x, y)$ Near the Discontinuity

The asymptotic expansions given in (28) and (36) break down as $(x, y) \to (0, 0)$ ($r \to 0^+$ in (29), (33), and (37)). Then, formulas (19) and (34) become useless. The asymptotic approximation of $U_\beta(x, y)$ near the origin requires a completely different analysis than the one developed in Section 3. In this section, we will find the asymptotic approximation of $U_\beta(x, y)$ near the point of discontinuity $(0, 0)$ as $(x, y) \to (0, 0)$ faster than $\epsilon \to 0^+$.

We consider for $U_\beta(x, y)$ the representation given in (12). After the change of variable $\cosh t = u + 1$ in the definition (10) of $I(x, y)$ and using (7), we have

$$I(x, y) = \frac{e^{-wr}}{\pi} \int_0^\infty e^{-rwu} f(u, \phi, \beta)\, du, \tag{38}$$

where

$$\begin{aligned} f(u, \phi, \beta) &= \left[\frac{\sin^2\beta - \sin^2\phi}{u^2} - \cos(2\beta)\left(1 + \frac{2}{u}\right)\right] g(u), \\ g(u) &\equiv \frac{\sin(2\phi)}{u^3\sqrt{1 + \frac{2}{u}}(1 + \frac{1+s_+}{u})(1 + \frac{1-s_+}{u})(1 + \frac{1+s_-}{u})(1 + \frac{1-s_-}{u})} \end{aligned} \tag{39}$$

and $s_\pm \equiv \cos(\beta \pm \phi)$. Splitting $g(u)$ in simple fractions, we obtain the expansion of $f(u, \phi, \beta)$ in inverse powers of u valid for each $\phi \in [0, \pi/2]$:

$$f(u, \phi, \beta) = \sum_{k=0}^{n-1} \frac{V_k(\phi, \beta)}{u^{k+1}} + f_n(u, \phi, \beta), \tag{40}$$

where $f_n(u, \phi, \beta) = \mathcal{O}(u^{-n-1})$ as $u \to \infty$ uniformly for $\phi \in (0, \pi/2)$. The coefficients V_k are just the Taylor coefficients of the expansion of $u^{-1} f(u^{-1}, \phi, \beta)$ at $u = 0$ ($V_0 = 0$).

Using (40), we can apply (Wong, 1989, Chapter 6, Theorem 13(ii)) to the integral on the right-hand side of (38) to obtain

$$I(x, y) = \frac{e^{-wr}}{\pi} \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}[T_k - V_k \log(wr)](wr)^k + R_n(r, \phi, \beta). \tag{41}$$

In this formula, the coefficients T_k are given by (Wong, 1989, Chapter 6, Theorem 13(ii))

$$T_k \equiv V_k \psi(k + 1) + \lim_{s \to k+1} \left\{ M[f; s] + \frac{V_k}{s - k - 1} \right\}, \tag{42}$$

where $M[f; s]$ denotes the Mellin transform of f at s, $\int_0^\infty u^{s-1} f(u, \phi, \beta)\, du$, or its analytic continuation. On the other hand, the remainder $R_n(r, \phi, \beta)$ reads

$$R_n(r, \phi, \beta) \equiv (rw)^n \int_0^\infty f_{n,n}(t) e^{-rwt}\, dt, \tag{43}$$

where

$$f_{n,n}(t) \equiv \frac{(-1)^n}{(n-1)!} \int_t^\infty (u-t)^{n-1} f_n(u,\phi,\beta)\,\mathrm{d}u. \tag{44}$$

In particular, the coefficient T_0, which gives the dominant term of the expansion, reads $T_0 = M[f; 1]$. Splitting $g(u)$ in simple fractions and using (Prudnikov *et al.*, 1990, p. 303, Equation 24) and (Abramowitz and Stegun, 1970, Equation 15.1.6), we obtain

$$T_0 = \begin{cases} 2\phi + \pi, & \text{if } \phi < \beta, \\ 2\phi, & \text{if } \phi = \beta, \\ 2\phi - \pi, & \text{if } \phi > \beta. \end{cases}$$

Introducing (41) in (12) and rearranging the terms, we obtain

$$U_\beta(x,y) = \frac{2\phi}{\pi} + wr\frac{\mathrm{e}^{-w\zeta(x,y)}}{2\pi} U_\beta^2(x,y) + U_\beta^3(x,y), \tag{45}$$

where

$$U_\beta^2(x,y) \equiv \frac{T_0(\phi,\beta)}{rw}[1 - \mathrm{e}^{w\zeta(x,y)}] + \sum_{k=1}^{n-1} \frac{(-1)^k}{k!}[T_k(\phi,\beta) - \\ - V_k(\phi,\beta)\log(rw)](rw)^{k-1} + R_n(x,y), \tag{46}$$

and

$$U_\beta^3(x,y) \equiv \mathrm{e}^{w(y\cos\beta + x\sin\beta)}[-I(x, 2-y) + R(x,y)] - \\ - \frac{1}{2}\mathrm{e}^{2w(y-1)\cos\beta}(\chi_B(x,y) + \chi_{B_0}(x,y)). \tag{47}$$

The functions $I(x, 2-y)$ and $R(x,y)$ verify (17), (26), (27), (31), and (32) with the identification (7). Then, $U_\beta^3(x,y) = \mathcal{O}(\mathrm{e}^{-\alpha/\epsilon})$ as $\epsilon \to 0^+$ for some $\alpha > 0$ uniformly in Ω.

Finally, we obtain an error bound for $R_n(x,y)$ which shows that expansion (46) is convergent. From the Taylor formula for the remainder,

$$f_n(u,\phi,\beta) \equiv \frac{h^{(n)}(\xi)}{n!u^{n+1}}$$

for some $\xi \in (0, u^{-1})$, where $h(s) \equiv s^{-1} f(s^{-1}, \phi, \beta)$. The singularities of $h(s)$ are away from the positive real axis (let us write d for the distance from the closest one of those singularities to the positive real axis). Therefore, using the Cauchy formula for the derivative $h^{(n)}(\xi)$, we see that

$$|f_n(u,\phi,\beta)| \leqslant \frac{M}{d^n u^{n+1}}, \qquad |f_n(u,\phi,\beta)| \leqslant \frac{M}{d^{n-1}u^n}, \tag{48}$$

where M is a bound for $h(w)$ on the portion of the complex w-plane surrounding the positive real axis: $\{w \in \mathbb{C}, |w - s| < d, s \in \mathbb{R}^+\}$.

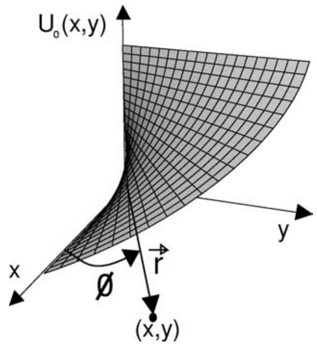


Figure 5. Graph of the first order approximation, $\frac{2\phi}{\pi}$, to the solution of problem (P') near the point of discontinuity (0, 0) of the boundary condition for $\epsilon = 0.1$.

Introducing these bounds in (44), we obtain (López, 2000, Equation (2.23)),

$$|f_{n,n}(t)| \leqslant \frac{M}{d^n(n-1)!}(1 - d\log t) \quad \forall t \in [0,1]$$

and introducing the first bound of (48) in (44), we obtain (López, 2000, Equation (2.24)),

$$|f_{n,n}(t)| \leqslant \frac{M}{n!d^n t} \quad \forall t \in [0,\infty).$$

We divide the integral on the right-hand side of (43) at the point $t = 1$ and use the first bound of $f_{n,n}(t)$ in the interval [0, 1] and the second one in the interval $[1,\infty)$ to finally derive, after simple computations, that

$$|R_n(x,y)| \leqslant M[n(2+d) + |\log(wr)|]\frac{(rw)^{n-1}}{d^n n!}. \tag{49}$$

OBSERVATION 3. From (45), (46), (49), (47), (17), (26), (27), (31), and (32) we see that

$$U_\beta(x,y) = \frac{2\phi}{\pi} + \mathcal{O}\left(\frac{r}{\epsilon}\right) + \mathcal{O}(\mathrm{e}^{-\alpha/\epsilon})$$

as $r \to 0^+$ and $\epsilon \to 0^+$ with $r/\epsilon \to 0^+$ and some $\alpha > 0$. The discontinuities of the inflow boundary condition at the corner are smoothed inside the domain by a linear function of the polar angle ϕ (see Figure 5).

5. Conclusions

The singularly perturbed convection–diffusion problem (P) is defined by means of a discontinuous Dirichlet boundary condition, with a point of discontinuity located at the corner of the boundary. We have derived an integral representation of the solution susceptible to an asymptotic analysis. From this integral, two complementary asymptotic expansions of the solution have been obtained. One expansion is valid in the singular limit $\epsilon \to 0^+$ and away from the points of discontinuity. The other one is valid near the point of discontinuity $(0, 0)$.

These two asymptotic expansions are derived from two quite different asymptotic procedures. Whereas the asymptotic expansion in the singular limit is obtained from a classical uniform method, the asymptotic expansion near the point of discontinuity has been derived by means of a distributional approach. Two quite different asymptotic principles match to the same problem.

The asymptotic expansion in the singular limit shows that the main contribution from the data points of discontinuity to the shape of the solution on the singular layers is contained in a certain combination of error functions. The problem requires the combination of one or two error functions, exponential functions, and step functions to approach the behavior of the solution on the interior layer, the boundary layer, and the corner layer. On the other hand, the asymptotic expansion near the point of discontinuity shows that this discontinuity on the boundary is smoothed inside the domain by means of a simply linear function of the polar angle at the point of discontinuity $(0, 0)$.

We suspect that, as in the problem analyzed here, the error function plays a fundamental role in the approximation of the solution of many singularly perturbed convection–diffusion problems with points of discontinuity in the boundary conditions. In particular, we expect that the approximated solution obtained in this paper may help to approximate the solution of the same problem on a rectangle. This will be the subject of further investigations. Moreover, the asymptotic expansions of the solutions of problem (P) presented here may give a qualitative idea about the behavior of the solutions of more realistic convection–diffusion problems with discontinuous Dirichlet conditions in the inflow boundary. This should help in the development of suitable numerical methods for those problems. For a similar discussion with a parabolic problem see (Hemker and Shishkin, 1993).

Acknowledgements

This work originated from conversations with Nico Temme and Juan C. Jorge. The *Dirección General de Ciencia y Tecnología* (REF. BFM2000-0803) and the *Gobierno de Navarra* (*Res. 92/2002*) are acknowledged for their financial support.

References

Abramowitz, M. and Stegun, I. A. (1970) *Handbook of Mathematical Functions*, Dover, New York.

Aziz, A. and Na, T. Y. (1984) *Perturbation Methods in Heat Transfer*, Hemisphere Publishing Corporation, New York.

Bejan, A. (1984) *Convection Heat Transfer*, Wiley, New York.

Cook, L. P., Ludford, G. S. S. and Walker, J. S. (1972) Corner regions in the asymptotic solution of $\epsilon\nabla^2 u = \partial u/\partial y$ with reference to MHD duct flow, *Proc. Cambridge Philos. Soc.* **72**, 117–121.

Cook, L. P. and Ludford, G. S. S. (1973) The behavior as $\epsilon \to 0^+$ of solutions to $\epsilon\nabla^2 w = \partial w/\partial y$ on the rectangle $0 \leqslant x \leqslant l$, $|y| \leqslant 1$, *SIAM J. Math. Anal.* **4**(1), 161–184.

Eckhaus, W. (1973) *Matched Asymptotic Expansions and Singular Perturbations*, North-Holland, Amsterdam.

Eckhaus, W. and de Jager, E. M. (1966) Asymptotic solutions of singular perturbation problems for linear differential equations of elliptic type, *Arch. Rational Mech. Anal.* **23**, 26–86.

Gold, R. R. (1962) Magnetohydrodynamic pipe flow. Part I, *J. Fluid Mech.* **13**, 505–512.

Grasman, J. (1974) An elliptic singular perturbation problem with almost characteristic boundaries, *J. Math. Anal. Appl.* **46**, 438–446.

Hedstrom, G. W. and Osterheld, A. (1980) The effect of cell Reynolds number on the computation of a boundary layer, *J. Comput. Phys.* **37**, 399–421.

Hemker, P. W. and Shishkin, G. I. (1993) Approximation of parabolic PDEs with a discontinuous initial condition, *East-West J. Numer. Math.* **1**(4), 287–302.

Il'in, A. M. (1992) *Matching of Asymptotic Expansions of Solutions of Boundary Value Problems*, Amer. Math. Soc., Providence.

Kevorkian, J. (1990) *Partial Differential Equations. Analytical Solution Techniques*, Springer-Verlag, New York.

Kevorkian, J. and Cole, J. D. (1996) *Multiple Scale and Singular Perturbation Methods*, Springer-Verlag, New York.

Knowles, J. K. and Messic, R. E. (1964) On a class of singular perturbation problems, *J. Math. Anal. Appl.* **9**, 42–58.

López, J. L. (2000) Asymptotic expansions of symmetric standard elliptic integrals, *SIAM J. Math. Anal.* **31**(4), 754–775.

Lu, P.-C. (1973) *Introduction to the Mechanics of Viscous Fluids*, Holt, Rinehart and Winston, New York.

Morton, K. W. (1996) *Numerical Solution of Convection–Diffusion Problems*, Chapman & Hall, London.

O'Malley, R. E. (1974) *Introduction to Singular Perturbation*, Academic Press, New York.

Prudnikov, A. P., Brychkov, Yu. A. and Marichev, O. I. (1990) *Integrals and Series*, Vol. 1, Gordon and Breach Science Pub.

Shercliff, J. A. (1953) Steady motion of conducting fluids in pipes under transverse magnetic fields, *Proc. Cambridge Philos. Soc.* **49**, 136–144.

Shih, S.-D. (1996) A Novel uniform expansion for a singularly perturbed parabolic problem with corner singularity, *Meth. Appl. Anal.* **3**(2), 203–227.

Temme, N. M. (1971) Analytical methods for a singular perturbation problem. The quarter plane, CWI Report, 125.

Temme, N. M. (1974) Analytical methods for a singular perturbation problem in a sector, *SIAM J. Math. Anal.* **5**(6), 876–887.

Temme, N. M. (1998) Analytical methods for a selection of elliptic singular perturbation problems, In: *Recent Advances in Differential Equations* (Kunming, 1997), Pitman Res. Notes Math. Ser. 386, Longman, Harlow, pp. 131–148.

Van Dyke, M. (1964) *Perturbation Methods in Fluid Dynamics*, Academic Press, New York.

Vasil'eva, A. B., Butuzov, V. F. and Kalachev, L. V. (1995) *The Boundary Function Method for Singular Perturbation Problems*, SIAM, Philadelphia.

Wong, R. (1989) *Asymptotic Approximations of Integrals*, Academic Press, New York.