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Two-point Taylor expansions in the asymptotic approximation of double integrals. Application to the second and fourth Appell functions

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Abstract

The main difficulty in Laplace's method of asymptotic expansions of double integrals is originated by a change of variables. We consider a double integral representation of the second Appell function $F_2(a,b,b',c,c';x,y)$ and illustrate, over this example, a variant of Laplace's method which avoids that change of variables and simplifies the computations. Essentially, the method only requires a Taylor expansion of the integrand at the critical point of the phase function. We obtain in this way an asymptotic expansion of $F_2(a,b,b',c,c';x,y)$ for large b, b', c and c'. We also consider a double integral representation of the fourth Appell function $F_4(a,b,c,d;x,y)$. We show, in this example, that this variant of Laplace's method is uniform when two or more critical points coalesce or a critical point approaches the boundary of the integration domain. We obtain in this way an asymptotic approximation of $F_4(a,b,c,d;x,y)$ for large values of a, b, c and d. In this second example, the method requires a Taylor expansion of the integrand at two points simultaneously. For this purpose, we also investigate in this paper Taylor expansions of two-variable analytic functions with respect to two points, giving Cauchy-type formulas for the coefficients of the expansion and details about the regions of convergence.
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1. Introduction

We consider double integrals of the form

$$F(z)=\iint_D e^{zh(x,y)}f(x,y)\,dx\,dy, \tag{1}$$

where D is a bounded or unbounded domain in $\mathbb{R}^2$, f and h are infinitely differentiable in $\bar{D}$ and z is a large positive parameter. Laplace's method tells us that the main contribution to the asymptotic behaviour of (1) comes from the

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points of the domain D where the phase function $h(x, y)$ attains its largest value [9, Chapter 8]. For instance, if $h(x, y)$ attains its maximum value only at a point $(x_0, y_0) \in D^0$, where $f(x, y)$ and $h(x, y)$ are infinitely differentiable, then $\nabla h(x_0, y_0) = (0, 0)$ and the Hessian matrix of h at that point, $Hh(x_0, y_0)$, is negative definite. Laplace's result is

$$F(z) \sim \frac{\pi f(x_0, y_0)}{z\sqrt{|\mathrm{Det}[Hh(x_0, y_0)]|}} e^{zh(x_0,y_0)}, \quad z \to \infty. \tag{2}$$

The right-hand side above is the first term of a complete asymptotic expansion that can be obtained in the following way [9, Chapter 8, Section 10]. The Hessian matrix $Hh(x_0, y_0)$ can be diagonalized after an orthogonal change of variables. Then, without loss of generality, we may assume that the Taylor expansion of $h(x, y)$ at (x_0, y_0) has the form

$$\begin{aligned} h(x, y) &= h(x_0, y_0) + a(x - x_0)^2 + b(y - y_0)^2 + \cdots \\ &= h(x_0, y_0) + a(x - x_0)^2\big[1 + P(x, y)\big] + b(y - y_0)^2\big[1 + Q(x, y)\big], \end{aligned} \tag{3}$$

where $a, b < 0$, $P(x, y)$ and $Q(x, y)$ are infinitely differentiable at (x_0, y_0) and $P(x_0, y_0) = Q(x_0, y_0) = 0$. Perform in (1) the change of integration variables $(x, y) \to (u, v)$ defined by $u = (x - x_0)\sqrt{1 + P(x, y)}$, $v = (y - y_0)\sqrt{1 + Q(x, y)}$,

$$F(z) = e^{zh(x_0,y_0)} \iint_{D'} e^{z(au^2+bv^2)} g(u, v)\, du\, dv, \tag{4}$$

where D' is the image of D under this change of variables,

$$g(u, v) \equiv f\big(x(u, v), y(u, v)\big)\frac{\partial(x, y)}{\partial(u, v)} \tag{5}$$

and $\partial(x, y)/\partial(u, v)$ is the Jacobian of the transformation $(x, y) \to (u, v)$. If $g(u, v)$ has a Taylor expansion at $(u, v) = (0, 0)$,

$$g(u, v) \sim \sum_{n=0}^{\infty}\sum_{m=0}^{n} c_{m,n-m} u^m v^{n-m}, \tag{6}$$

then we can apply Watson's Lemma to the integral (4): replace (6) in the right-hand side of (4) and interchange sum and integral [9, Chapter 8, Section 10],

$$F(z) \sim e^{zh(x_0,y_0)} \sum_{n=0}^{\infty}\sum_{m=0}^{n} c_{2m,2n-2m} \frac{\Gamma(m + 1/2)\Gamma(n - m + 1/2)}{(-a)^{m+1/2}(-b)^{n-m+1/2} z^{n+1}}, \quad z \to \infty. \tag{7}$$

Therefore, the computation of the coefficients $c_{m,n}$ in the standard Laplace method is very difficult because of the complexity of the above mentioned change of variable (see the example in [9, Chapter 8, Section 10] where the first term of the expansion of a double integral is derived). In general, only a few terms of the expansion are obtained explicitly.

In [2], a modification of Laplace's method for one-dimensional integrals is proposed which avoids the change of variable and simplifies the computation of the coefficients of the expansion. Consider the integral

$$F(x) \equiv \int_a^b e^{xh(t)} f(t)\, dt, \tag{8}$$

where (a, b) is a real interval (finite or infinite), x is a large positive parameter and $f(t)$ and $h(t)$ are infinitely differentiable in (a, b). It is shown in [2] that it is not necessary to use a change of variables in (8) to convert the integral into the Laplace form and then apply the standard Laplace method. It is just necessary to expand $f(t)$ in (8) at the maximum of $h(t)$ in $[a, b]$, say t_0, and interchange sum and integral. If $f(t)$ has a Taylor expansion at $t = t_0$,

$$f(t) \sim \sum_{n=0}^{\infty} f_n (t - t_0)^n, \tag{9}$$

then replace this expansion in (8) and interchange sum and integral. We obtain the asymptotic expansion (see [2] for details)

$$F(x) \sim \sum_{n=0}^{\infty} f_n \int_{-\infty}^{\infty} e^{xh(t)} (t-t_0)^n \, dt, \quad x \to \infty. \tag{10}$$

In [5], it has been shown how this modification of Laplace's method for one-dimensional integrals can be successfully employed to obtain uniform asymptotic expansions of certain orthogonal polynomials.

In this paper we investigate that modification of Laplace's method for double integrals by means of two examples: the second and fourth Appell functions. In Section 2, we show that the second Appell function $F_2(a, b, b', c, c'; u, v)$ can be written in the form (1), where the phase function $h(x, y)$ has only one absolute (and relative) maximum $(x_0, y_0) \in D$. This point (x_0, y_0) can move when the asymptotic parameters b, b', c or c' change their relative size, but it always remains in D. Then, as will be made clear below, we just need to consider the one-point Taylor expansion of the function $f(x, y)$ at that point (x_0, y_0).

We show in Section 3 that the fourth Appell function $F_4(a, b, c, d; u, v)$ can also be written in the form (1) and again, the phase function $h(x, y)$ has only one relative maximum (x_0, y_0). But in this case, x_0 and y_0 depend of u and v and then, the critical point (x_0, y_0) may cross the domain D and leave it for certain values of u and v. Then, as will be made clear below, we need a Taylor expansion of the function $f(x, y)$ at two points simultaneously. For this purpose, in Appendix A we briefly investigate the two-point Taylor expansion of an analytic function of two variables. This problem was already solved for one-variable analytic functions in [8], and more recently considered in [3,4].

A few remarks and comments are given in Section 4.

2. The Appell function $F_2(a, b, b', c, c'; x, y)$ for large b, b', c, c'

The second Appell hypergeometric function is a generalization of the Gauss hypergeometric function ${}_2F_1$ and it is defined by the double series

$$F_2(a, b, b', c, c'; x, y) \equiv \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n m!n!} x^m y^n, \quad |x| + |y| < 1.$$

A double integral representation of the second Appell function is given in [6]:

$$F_2(a, b, b', c, c'; x, y) = \frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(c-b)\Gamma(b')\Gamma(c'-b')} I_2(a, b, b', c, c'; x, y), \tag{11}$$

with

$$I_2(a, b, b', c, c'; x, y) \equiv \int_0^1 ds \int_0^1 dt \, s^{b-1}(1-s)^{c-b-1} t^{b'-1}(1-t)^{c'-b'-1}(1-xs-yt)^{-a}, \tag{12}$$

$x, y \in \mathbb{C}$, $x + y \notin [1, \infty)$, $c > b > 0$ and $c' > b' > 0$. We define the following constants α, β and γ:

$$\alpha \equiv \frac{b'-1}{b-1}, \qquad \beta \equiv \frac{c-b-1}{b-1}, \qquad \gamma \equiv \frac{c'-b'-1}{b-1}.$$

Using these relations, (12) can be written in the form (1)

$$I_2(a, b, b', c, c'; x, y) = \int_0^1 ds \int_0^1 dt \, e^{(b-1)h(s,t)} f(s, t), \tag{13}$$

with

$$\begin{cases} h(s,t) = \log s + \beta \log(1-s) + \alpha \log t + \gamma \log(1-t), \\ f(s,t) = (1-xs-yt)^{-a}. \end{cases}$$

The phase function $h(s,t)$ possesses a relative maximum at

$$(s_0,t_0)=\left(\frac{1}{1+\beta},\frac{\alpha}{\alpha+\gamma}\right).$$

The point (s_0,t_0) belongs to the domain of integration $(0,1)\times(0,1)$ if $\alpha>0$, $\beta>0$ and $\gamma>0$. We will assume throughout this section that b, b', c and c' are large with α, β and γ fixed and positive. Now, instead of using the standard Laplace method to approximate (13) for large b, we just replace the function $f(s,t)$ by its Taylor expansion at the point (s_0,t_0),

$$f(s,t)=\sum_{n=0}^{N-1}\sum_{m=0}^{n}a_{m,n-m}(s_0,t_0)(s-s_0)^m(t-t_0)^{n-m}+r_N(s_0,t_0,s,t), \tag{14}$$

with $r_N(s_0,t_0,s,t)=\mathcal{O}(\|(t-t_0,s-s_0)\|^N)$ as $(s,t)\to(s_0,t_0)$ and

$$a_{m,n-m}(s_0,t_0)=\frac{(a)_n x^m y^{n-m}}{m!(n-m)!}\left[1-\frac{x}{1+\beta}-\frac{\alpha y}{\alpha+\gamma}\right]^{-a-n}.$$

Substitution of the above expansion into (13), followed by reversal of the order of summation and integration, yields the expansion

$$\begin{aligned}F_2(a,b,b',c,c';x,y)={}&\sum_{n=0}^{N-1}\sum_{m=0}^{n}\frac{(a)_n x^m y^{n-m}}{m!(n-m)!}\left[1-\frac{x}{1+\beta}-\frac{\alpha y}{\alpha+\gamma}\right]^{-a-n}\left(-\frac{\alpha}{\alpha+\gamma}\right)^{n-m}\\&\times\left(\frac{-1}{1+\beta}\right)^m{}_2F_1(-m,b;c;1+\beta)\,{}_2F_1\left(m-n,b';c';\frac{\alpha+\gamma}{\alpha}\right)\\&+R_N(a,b,b',c,c';x,y),\end{aligned} \tag{15}$$

with

$$R_N(a,b,b',c,c';x,y)\equiv\frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(c-b)\Gamma(b')\Gamma(c'-b')}\int_0^1 ds\int_0^1 dt\, e^{(b-1)h(s,t)}r_N(s_0,t_0,s,t). \tag{16}$$

The above ${}_2F_1$ functions satisfy the following three-term recurrence relation with respect to the first parameter [1, formula (15.2.10)]:

$${}_2F_1(-m-1,b;c;1+\beta)=\frac{(1+m)(1-\beta)}{m+c}\,{}_2F_1(-m,b;c;1+\beta)+\frac{m\beta}{m+c}\,{}_2F_1(-m+1,b;c;1+\beta).$$

When $b\to\infty$ (and $c\to\infty$ with fixed β),

$${}_2F_1(0,b;c;1+\beta)=1=\mathcal{O}(1)\quad\text{and}\quad {}_2F_1(-1,b;c;1+\beta)=\frac{2b-c}{c(b-1)}=\mathcal{O}\left(\frac{1}{b}\right).$$

From the above recurrence and these asymptotic behaviours, it can be shown by induction that, as $b\to\infty$ (and $c,c',b'\to\infty$ with fixed α,β,γ),

$${}_2F_1(-m,b;c;1/s_0)=\mathcal{O}\big(b^{-\lfloor\frac{m+1}{2}\rfloor}\big)\quad\text{and}\quad {}_2F_1(m-n,b';c';1/t_0)=\mathcal{O}\big(b^{-\lfloor\frac{n-m+1}{2}\rfloor}\big).$$

Hence,

$${}_2F_1(-m,b;c;1/s_0)\,{}_2F_1(m-n,b';c';1/t_0)=\mathcal{O}\big(b^{-\lfloor\frac{n}{2}\rfloor}\big),\quad\text{as } b\to\infty.$$

On the other hand, using $r_N(s_0,t_0,s,t)=\mathcal{O}(\|(s-s_0,t-t_0)\|^N)$, the standard Laplace first order approximation for the double integral in (16) and the Stirling approximation for the quotient of gamma functions in that formula, we find that $R_N(a,b,b',c,c';x,y)=\mathcal{O}(b^{-N/2})$. Then, (15) is an asymptotic expansion for large positive b, b', c and c' with fixed and positive α, β and γ. Table 1 shows a numerical experiment about the accuracy of the approximation (15).

Table 1
Relative errors in the approximation (15) of $F_2(a,b,b',c,c';x,y)$ for $a=1$, $x=1/4$, $y=-1/4$, different b,b',c,c' and truncation index N. The relative error decreases for increasing asymptotic parameters and increasing N

	$b=b'=50$ $c=c'=100$	$b=b'=100$ $c=c'=150$	$b=b'=200$ $c=c'=250$	$b=b'=500$ $c=c'=550$
$N=1$	0.03	0.018	0.0079	0.0018
$N=3$	0.000028	0.00001	1.91e–6	1.06e–7
$N=5$	4.32e–8	9.23e–9	7.72e–10	1.13e–11
$N=7$	7.49e–11	7.21e–12	2.79e–12	1.13e–12

3. The Appell function $F_4(a,b,c,d;x,y)$ for large a,b,c and d

The fourth Appell hypergeometric function $F_4(a,b,c,d;x,y)$ is defined by the double series

$$F_4(a,b,c,d;x,y)\equiv\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_m(d)_n m!n!}x^m y^n,\quad |x|^{1/2}+|y|^{1/2}<1.$$

We consider the following integral representation of F_4 given in [7, p. 566]:

$$F_4\big(a,b,c,d;u(1-v),v(1-u)\big)=\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(d-b)}I_4(a,b,c,d;u,v), \tag{17}$$

with

$$I_4(a,b,c,d;u,v)\equiv\int_0^1 dx\int_0^1 dy\, x^{a-1}y^{b-1}(1-ux)^{a+1-c-d}(1-vy)^{b+1-c-d}(1-x)^{c-a-1} \times(1-y)^{d-b-1}(1-ux-vy)^{c+d-a-b-1}, \tag{18}$$

$c>a>0$, $d>b>0$ and $u,v\in\mathbb{C}$ with $u+v\notin[1,\infty)$. We consider the case that a and c go to infinity at the same speed, as well as b and d,

$$c=a+A,\qquad d=b+B,$$

with A and B fixed positive numbers. Thus, except for an asymptotically irrelevant factor $(1-x)^{A-1}(1-y)^{B-1}$, the integral in (18) can be written in the form (1),

$$I_4(a,b,c,d;u,v)=\int_0^1 dx\int_0^1 dy\, e^{(a-1)h(x,y)}(1-x)^{A-1}(1-y)^{B-1}f(x,y), \tag{19}$$

with

$$\begin{cases} h(x,y)=\log x+\alpha\log y-\beta\log(1-ux)-\gamma\log(1-vy),\\ f(x,y)=(1-ux-vy)^{A+B-1}\end{cases} \tag{20}$$

and $\alpha=\frac{b-1}{a-1}$, $\beta=\frac{b+A+B-1}{a-1}$, $\gamma=\frac{a+A+B-1}{a-1}$ fixed positive numbers. In the remaining of this section we consider $\beta>1$ and $\gamma>\alpha$, which written in the original variables read $c+d>2a$ and $c+d>2b$. Under these conditions, the phase function $h(x,y)$ possesses a relative maximum at the point

$$(x_0,y_0)=\left(\frac{1}{u(1-\beta)},\frac{\alpha}{v(\alpha-\gamma)}\right).$$

This point $(x_0,y_0)\in D=(0,1)\times(0,1)$ if $u(1-\beta)>1$ and $v(\alpha-\gamma)>\alpha$ (in the original variables these conditions read $u(2a-c-d)>b-1$ and $v(2b-c-d)>b-1$, respectively). The point (x_0,y_0) belongs to the boundary of D if $u(1-\beta)=1$ and $v(\alpha-\gamma)\geqslant\alpha$ or $u(1-\beta)\geqslant 1$ and $v(\alpha-\gamma)=\alpha$. And $(x_0,y_0)\notin\bar{D}$ if $u(1-\beta)<1$ or $v(\alpha-\gamma)<\alpha$ (see Fig. 1).

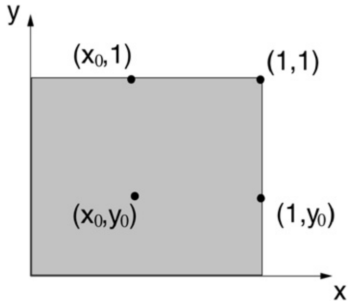

Fig. 1. The location of the absolute maximum P of the phase function $h(x,y)$ in the domain of integration $[0,1]\times[0,1]$ in (19) depends on the relative value of its parameters u, v, α, β and γ: If $u(1-\beta)>1$ and $v(\alpha-\gamma)>\alpha$, then $P=(x_0,y_0)$. If $u(1-\beta)\leqslant 1$ and $v(\alpha-\gamma)>\alpha$, then $P=(1,y_0)$. If $u(1-\beta)>1$ and $v(\alpha-\gamma)\leqslant\alpha$, then $P=(x_0,1)$. If $u(1-\beta)\leqslant 1$ and $v(\alpha-\gamma)\leqslant\alpha$, then $P=(1,1)$.

The standard Laplace method is difficult to apply to the integral (18). As we have shown in the example of the F_2 function, a much simpler method consists of an expansion of the function $f(x,y)$ in (19) at the maximum P of $h(x,y)$ in $\bar{D}$ (one of the points (x_0,y_0), $(1,y_0)$, $(x_0,1)$ or $(1,1)$). But then, the expansion so obtained is not uniform in the parameters $u(1-\beta)$ and $v(1-\gamma/\alpha)$. The purpose of this section is to show that the simplified Laplace method introduced in the previous section for the F_2 function can be straightforwardly converted in a uniform method.

In order to simplify the discussion and show the uniformity of the method, we restrict ourselves to the case $v(\alpha-\gamma)>\alpha$. Then, $0<y_0<1$ and $h(x,y)$ attains its maximum at $(x_0,y_0)\in D$ if $u(1-\beta)>1$ or at $(1,y_0)$ if $u(1-\beta)\leqslant 1$. The method is uniform if we replace the function $f(x,y)$ in (19) by its Taylor expansion, not at the point (x_0,y_0) or at the point $(1,y_0)$, but at both points simultaneously. Because the second components of these two points coincide, we will consider expansion (A.14) with $z_1=x_0$, $z_2=1$, $\omega_1=\omega_2=y_0$ (see Appendix A). For simplicity in the exposition, we consider the case $N=1$ in (A.14) (the general case for arbitrary N is a straightforward generalization)

$$f(x,y)=A_{0,0}+B_{0,0}x+C_{0,0}y+D_{0,0}xy+r_1(x_0,1,y_0,y_0,x,y), \tag{21}$$

with

$$\begin{cases} A_{0,0}=\dfrac{f(x_0,y_0)-x_0f(1,y_0)-y_0f_y(x_0,y_0)+x_0y_0f_y(1,y_0)}{1-x_0},\\ B_{0,0}=\dfrac{f(1,y_0)-f(x_0,y_0)-y_0f_y(1,y_0)+y_0f_y(x_0,y_0)}{1-x_0},\\ C_{0,0}=\dfrac{f_y(x_0,y_0)-x_0f_y(1,y_0)}{1-x_0},\\ D_{0,0}=\dfrac{f_y(1,y_0)-f_y(x_0,y_0)}{1-x_0} \end{cases}$$

and

$$r_1(x_0,1,y_0,y_0,x,y)=\mathcal{O}\big(\big\|(x-x_0,y-y_0)\big\|\,\big\|(x-1,y-y_0)\big\|\big) \tag{22}$$

when $(x,y)\to(x_0,y_0)$ and $(x,y)\to(1,y_0)$. Substitution of the approximation (21) into (19) and this one into (17) yields

$$F_4\big(a,b,c,d;u(1-v),v(1-u)\big)=F(a,b,c,d;u,v)+R_1(a,b,c,d;u,v), \tag{23}$$

with

$$\begin{aligned} &F(a,b,c,d;u,v)\\ &\quad\equiv\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(d-b)}\\ &\qquad\times\big\{A_{0,0}\Phi(a-1,b,A,B,u)\Phi(b-1,a,B,A,v)+B_{0,0}\Phi(a,b,A,B,u)\Phi(b-1,a,B,A,v)\end{aligned}$$

$$+ C_{0,0}\Phi(a-1,b,A,B,u)\Phi(b,a,B,A,v) + D_{0,0}\Phi(a,b,A,B,u)\Phi(b,a,B,A,v)\big\}, \tag{24}$$

$$\Phi(a,b,A,B,u) \equiv \int_0^1 x^a(1-x)^{A-1}(1-ux)^{1-b-A-B}\,dx \tag{25}$$

and

$$R_1(a,b,c,d;u,v) \equiv \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(d-b)} \times \int_0^1 dx \int_0^1 dy\, e^{(a-1)h(x,y)}(1-x)^{A-1}(1-y)^{B-1}r_1(x_0,1,y_0,y_0,x,y). \tag{26}$$

The functions $\Phi(a,b,A,B,u)$ can be evaluated in terms of hypergeometric functions and we obtain

$$\begin{aligned} F(a,b,c,d;u,v) \equiv{}& A_{0,0}\,{}_2F_1(c+d-a-1,a;c;u)\,{}_2F_1(c+d-b-1,b;d;v) \\ &+ B_{0,0}\frac{a}{c}\,{}_2F_1(c+d-a-1,a+1;c+1;u)\,{}_2F_1(c+d-b-1,b;d;v) \\ &+ C_{0,0}\frac{b}{d}\,{}_2F_1(c+d-a-1,a;c;u)\,{}_2F_1(c+d-b-1,b+1;d+1;v) \\ &+ D_{0,0}\frac{ab}{cd}\,{}_2F_1(c+d-a-1,a+1;c+1;u)\,{}_2F_1(c+d-b-1,b+1;d+1;v). \end{aligned} \tag{27}$$

To obtain the asymptotic behaviour of $\Phi(a-s,b,A,B,u)$, and then of $F(a,b,c,d;u,v)$, we rewrite the integral $\Phi(a-s,b,A,B,u)$ for $s=0,1$ given in (25) as a Laplace integral

$$\Phi(a-s,b,A,B,u) = \int_0^1 e^{(a-1)(\log x-\beta\log(1-ux))}(1-x)^{A-1}x^{1-s}\,dx, \quad s=0,1. \tag{28}$$

In this integral, $a\to\infty$ and u,β,A are fixed. Applying the first order approximation given by the standard Laplace method to the integral (28), we deduce that

$$\Phi(a-s,b,A,B,u) = \begin{cases} e^{(a-1)(\log x_0-\beta\log(1-ux_0))}\mathcal{O}(a^{-\frac{1}{2}}) & \text{if } u(1-\beta)>1, \\ e^{-(a-1)\beta\log(1-u)}\mathcal{O}(a^{-\frac{A}{2}}) & \text{if } u(1-\beta)=1, \\ e^{-(a-1)\beta\log(1-u)}\mathcal{O}(a^{-A}) & \text{if } u(1-\beta)<1. \end{cases}$$

On the other hand, as $v(\alpha-\gamma)>\alpha$,

$$\Phi(b-s,a,B,A,v) = e^{(a-1)(\alpha\log y_0-\gamma\log(1-vy_0))}\mathcal{O}\big(a^{-\frac{1}{2}}\big), \quad \text{as } a\to\infty,\ s=0,1.$$

Introducing this in (24) we find that, when $a\to\infty$,

$$F(a,b,c,d,u,v) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)} \times \begin{cases} e^{(a-1)h(x_0,y_0)}\mathcal{O}(a^{-1}) & \text{if } u(1-\beta)>1, \\ e^{(a-1)h(1,y_0)}\mathcal{O}(a^{-\frac{A+1}{2}}) & \text{if } u(1-\beta)=1, \\ e^{(a-1)h(1,y_0)}\mathcal{O}(a^{-A-\frac{1}{2}}) & \text{if } u(1-\beta)<1. \end{cases}$$

In order to find the asymptotic behaviour of the remainder $R_1(a,b,c,d;u,v)$, take into account the bound for the remainder $r_1(x_0,1,y_0,y_0,x,y)$ given in (22) in the integral defining $R_1(a,b,c,d;u,v)$ in (26). Applying the first order approximation given by the standard Laplace method to the integral (26) in the same way we have done for Φ in the integral (28), we find that, when $a\to\infty$,

$$R_1(a,b,c,d;u,v) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)} \times \begin{cases} e^{(a-1)h(x_0,y_0)}\mathcal{O}(a^{-3/2}) & \text{if } u(1-\beta)>1, \\ e^{(a-1)h(1,y_0)}\mathcal{O}(a^{-\frac{A+3}{2}}) & \text{if } u(1-\beta)=1, \\ e^{(a-1)h(1,y_0)}\mathcal{O}(a^{-A-\frac{3}{2}}) & \text{if } u(1-\beta)<1. \end{cases}$$

Then, (23) is an asymptotic approximation of $F_4(a,b,c,d;u(1-v),v(1-u))$ for large positive a, b, c and d with fixed and positive α, β and γ. The approximation is uniform in the parameter $u(1-\beta)$. Table 2 shows a numerical experiment about the accuracy of the approximation (23).

Table 2
Relative errors in the approximation (23) of $F_4(a,b,c,d;u(1-v),v(1-u))$ for different values of a, b, c, d, u and v. For the value of u in the second row, the phase function $h(x,y)$ attains its maximum in D at (x_0,y_0), whereas for the values of u in the third row, $h(x,y)$ attains its maximum in D at $(1,y_0)$

u	$a=10, b=11$ $c=d=12, v=-5.5$	u	$a=50, b=51$ $c=d=52, v=-30$	u	$a=80, b=81$ $c=d=82, v=-60$
-3	0.089083	-25	0.059411	-55	0.032388
-1	0.102995	-10	0.084271	-19	0.078083

4. Concluding remarks

We have considered representations of the second and fourth Appell functions in terms of double integrals. We have studied in these two particular examples of double integrals a simplification of Laplace's method introduced in [2] for simple integrals. We have seen that the method introduced in [2] can be easily generalized to double integrals, obtaining two asymptotic expansions of these two special functions when four of their parameters are large and positive. As it happens for simple integrals, the method is uniform when two critical points of the phase function coalesce or a critical point crosses the boundary of the integration domain. Uniformity requires a Taylor expansion of the integrand at all of the asymptotically relevant points of the phase function (critical points and/or boundary points) simultaneously. To this end, in Appendix A, we have generalized the theory of two-point Taylor expansions of one-variable analytic functions to the case of two-variable analytic functions. We have given details on the regions of convergence (products of Cassini ovals) and on the representations of the coefficients and the remainders of the expansions in terms of Cauchy integrals.

Appendix A. Two-point Taylor expansions of analytic functions of two variables

Theorem A.1. *Let $f(z,\omega)$ be an analytic function on an open set $\Omega_1\times\Omega_2\in\mathbb{C}\times\mathbb{C}$, $z_1,z_2\in\Omega_1$ with $z_1\neq z_2$ and $\omega_1,\omega_2\in\Omega_2$ with $\omega_1\neq\omega_2$. Then, $f(z,\omega)$ admits the two-point Taylor expansion*

$$\begin{aligned}f(z,\omega)={}&\sum_{n=0}^{N-1}\sum_{m=0}^{n}\big[a_{m,n-m}(z_1,z_2,\omega_1,\omega_2)(z-z_1)(\omega-\omega_1)\\&+a_{m,n-m}(z_1,z_2,\omega_2,\omega_1)(z-z_1)(\omega-\omega_2)+a_{m,n-m}(z_2,z_1,\omega_1,\omega_2)(z-z_2)(\omega-\omega_1)\\&+a_{m,n-m}(z_2,z_1,\omega_2,\omega_1)(z-z_2)(\omega-\omega_2)\big](z-z_1)^m(z-z_2)^m(\omega-\omega_1)^{n-m}(\omega-\omega_2)^{n-m}\\&+r_N(z_1,z_2,\omega_1,\omega_2;z,\omega),\end{aligned}\tag{A.1}$$

where $r_N(z_1,z_2,\omega_1,\omega_2;z,\omega)=\mathcal{O}(\|(z-z_1,\omega-\omega_1)\|^N\|(z-z_2,\omega-\omega_2)\|^N)$ when $(z,\omega)\to(z_1,\omega_1)$ or $(z,\omega)\to(z_2,\omega_2)$ and the coefficients $a_{m,n}$ of the expansion are given by the double Cauchy integral

$$a_{m,n}(z_1,z_2,\omega_1,\omega_2)\equiv\frac{1}{(2\pi i)^2(z_2-z_1)(\omega_2-\omega_1)}\int_{\mathcal{C}_1}\frac{du}{(u-z_1)^m(u-z_2)^{m+1}}\int_{\mathcal{C}_2}\frac{f(u,v)\,dv}{(v-\omega_1)^n(v-\omega_2)^{n+1}}.\tag{A.2}$$

The contours of integration $\mathcal{C}_1$ and $\mathcal{C}_2$ are simple closed loops that encircle the points z_1 and z_2 and ω_1 and ω_2, respectively in the counterclockwise direction and also the point z and are contained in Ω_1 and Ω_2, respectively (see Fig. 2).

The expansion (A.1) is convergent for $(z,\omega)\in O^1_{z_1,z_2}\times O^2_{\omega_1,\omega_2}$, where $O^1_{z_1,z_2}$ and $O^2_{\omega_1,\omega_2}$ are the Cassini ovals

$$O^1_{z_1,z_2}\equiv\big\{z\in\Omega_1,\ \big|(z-z_1)(z-z_2)\big|<r_1\big\},\qquad O^2_{\omega_1,\omega_2}\equiv\big\{\omega\in\Omega_2,\ \big|(\omega-\omega_1)(\omega-\omega_2)\big|<r_2\big\},\tag{A.3}$$

with $r_1\equiv\inf_{u\in\mathbb{C}\setminus\Omega_1}\{|(u-z_1)(u-z_2)|\}$ and $r_2\equiv\inf_{v\in\mathbb{C}\setminus\Omega_2}\{|(v-\omega_1)(v-\omega_2)|\}$.

Proof. By Cauchy's theorem

$$f(z,\omega)=\frac{1}{(2\pi i)^2}\int_{\mathcal{C}_1}\frac{du}{u-z}\int_{\mathcal{C}_2}\frac{f(u,v)\,dv}{v-\omega},\tag{A.4}$$

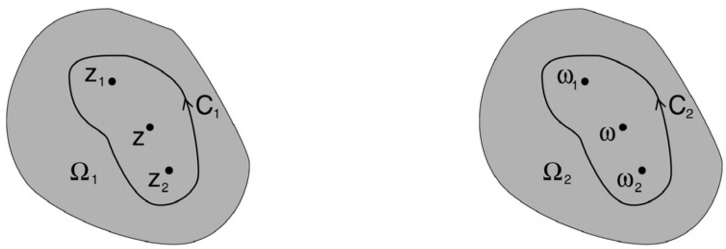

Fig. 2. Contours $\mathcal{C}_1$ and $\mathcal{C}_2$ in Theorem A.1.

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are the contours defined above. We write

$$\frac{1}{u-z}=\frac{z+u-z_1-z_2}{(u-z_1)(u-z_2)}\frac{1}{1-\frac{(z-z_1)(z-z_2)}{(u-z_1)(u-z_2)}} \tag{A.5}$$

and

$$\frac{1}{v-\omega}=\frac{\omega+v-\omega_1-\omega_2}{(v-\omega_1)(v-\omega_2)}\frac{1}{1-\frac{(\omega-\omega_1)(\omega-\omega_2)}{(v-\omega_1)(v-\omega_2)}}. \tag{A.6}$$

We introduce the expansion

$$\frac{1}{1-s}=\sum_{k=0}^{K-1}s^k+\frac{s^K}{1-s},$$

in (A.5) and (A.6) for $s=\frac{(z-z_1)(z-z_2)}{(u-z_1)(u-z_2)}$ and $s=\frac{(\omega-\omega_1)(\omega-\omega_2)}{(v-\omega_1)(v-\omega_2)}$, respectively, and these in (A.4). After straightforward calculations we obtain (A.1)–(A.2) with

$$\begin{aligned}
r_N(z_1,z_2,\omega_1,\omega_2;z,\omega)\equiv{}&\sum_{n=0}^{N-1}\frac{1}{(2\pi i)^2}\int_{\mathcal{C}_1}\frac{z+u-z_1-z_2}{(u-z_1)(u-z_2)}\frac{(z-z_1)^n(z-z_2)^n}{(u-z_1)^n(u-z_2)^n}\,du\\
&\times\int_{\mathcal{C}_2}\frac{f(u,v)\,dv}{(v-\omega_1)^N(v-\omega_2)^N(v-\omega)}(\omega-\omega_1)^N(\omega-\omega_2)^N\\
&+\sum_{m=0}^{N-1}\frac{1}{(2\pi i)^2}\int_{\mathcal{C}_2}\frac{\omega+v-\omega_1-\omega_2}{(v-\omega_1)(v-\omega_2)}\frac{(\omega-\omega_1)^m(\omega-\omega_2)^m}{(v-\omega_1)^m(v-\omega_2)^m}\,dv\\
&\times\int_{\mathcal{C}_1}\frac{f(u,v)\,du}{(u-z_1)^N(u-z_2)^N(u-z)}(z-z_1)^N(z-z_2)^N\\
&+\frac{1}{(2\pi i)^2}\int_{\mathcal{C}_1}\frac{du}{(u-z_1)^N(u-z_2)^N(u-z)}(z-z_1)^N(z-z_2)^N\\
&\times\int_{\mathcal{C}_2}\frac{f(u,v)\,dv}{(v-\omega_1)^N(v-\omega_2)^N(v-\omega)}(\omega-\omega_1)^N(\omega-\omega_2)^N\\
&-\sum_{n=1}^{N-1}\sum_{m=n}^{N-1}\big[a_{m,N-1+n-m}(z_1,z_2,\omega_1,\omega_2)(z-z_1)(\omega-\omega_1)\\
&+a_{m,N-1+n-m}(z_1,z_2,\omega_2,\omega_1)(z-z_1)(\omega-\omega_2)\\
&+a_{m,N-1+n-m}(z_2,z_1,\omega_1,\omega_2)(z-z_2)(\omega-\omega_1)\\
&+a_{m,N-1+n-m}(z_2,z_1,\omega_2,\omega_1)(z-z_2)(\omega-\omega_2)\big]\\
&\times(z-z_1)^m(z-z_2)^m(\omega-\omega_1)^{N-1+n-m}(\omega-\omega_2)^{N-1+n-m}.
\end{aligned} \tag{A.7}$$

For any $(z,\omega)\in O^1_{z_1,z_2}\times O^2_{\omega_1,\omega_2}$, we can take contours $\mathcal{C}_1\in\Omega_1$ and $\mathcal{C}_2\in\Omega_2$ such that $|(z-z_1)(z-z_2)|<|(u-z_1)(u-z_2)|$, $\forall u\in\mathcal{C}_1$ and $|(\omega-\omega_1)(\omega-\omega_2)|<|(v-\omega_1)(v-\omega_2)|$, $\forall v\in\mathcal{C}_2$. In these contours, $|f(u,v)|$ is bounded by some constant C: $|f(u,v)|\leqslant C$. Introducing these bounds in (A.7) we see that $|r_N(z_1,z_2,\omega_1,\omega_2;z,\omega)|\leqslant C_N(\|(z-z_1,\omega-\omega_1)\|^N\|(z-z_2,\omega-\omega_2)\|^N)$, with C_N independent of z and ω, and also that $\lim_{N\to\infty}r_N(z_1,z_2,\omega_1,\omega_2;z,\omega)=0$ and the proof follows. □

A.1. A more explicit form of the coefficients

Formula (A.2) is not appropriate for numerical computations. A more practical formula to compute the coefficients of the above two-point Taylor expansion is given in the following proposition.

Proposition A.1. *Coefficients $a_{m,n}(z_1,z_2,\omega_1,\omega_2)$ in the expansion* (A.1) *are also given by the formulas*

$$a_{0,0}(z_1,z_2,\omega_1,\omega_2)=\frac{f(z_2,\omega_2)}{(z_2-z_1)(\omega_2-\omega_1)}. \tag{A.8}$$

For $m=1,2,3,\ldots,$

$$a_{m,0}(z_1,z_2,\omega_1,\omega_2)=\frac{1}{\omega_2-\omega_1}\sum_{k=0}^{m}\frac{(m+k-1)!}{k!(m-k)!}\frac{(-1)^{m+1}mf^{(m-k,0)}(z_2,\omega_2)+(-1)^k kf^{(m-k,0)}(z_1,\omega_2)}{m!(z_1-z_2)^{m+k+1}}. \tag{A.9}$$

For $n=1,2,3,\ldots,$

$$a_{0,n}(z_1,z_2,\omega_1,\omega_2)=\frac{1}{z_2-z_1}\sum_{j=0}^{n}\frac{(n+j-1)!}{j!(n-j)!}\frac{(-1)^{n+1}nf^{(0,n-j)}(z_2,\omega_2)+(-1)^j jf^{(0,n-j)}(z_2,\omega_1)}{n!(\omega_1-\omega_2)^{n+j+1}}. \tag{A.10}$$

For $m,n=1,2,3,\ldots,$

$$\begin{aligned}a_{m,n}(z_1,z_2,\omega_1,\omega_2)={}&\sum_{k=0}^{m}\sum_{j=0}^{n}\frac{(m+k-1)!}{k!(m-k)!m!(z_1-z_2)^{m+k+1}}\frac{(n+j-1)!}{j!(n-j)!n!(\omega_1-\omega_2)^{n+j+1}}\\&\times\big[(-1)^{m+n}mnf^{(m-k,n-j)}(z_2,\omega_2)+(-1)^{m+j+1}mjf^{(m-k,n-j)}(z_2,\omega_1)\\&+(-1)^{n+k+1}knf^{(m-k,n-j)}(z_1,\omega_2)+(-1)^{k+j}kjf^{(m-k,n-j)}(z_1,\omega_1)\big].\end{aligned} \tag{A.11}$$

Proof. We deform the contour of integration $\mathcal{C}_1$ in Eq. (A.2) to any contour of the form $\mathcal{C}_1^1\cup\mathcal{C}_1^2$ also contained in Ω_1, where $\mathcal{C}_1^1$ ($\mathcal{C}_1^2$) is a simple closed loop which encircles the point z_1 (z_2) in the counterclockwise direction and does not contain the point z_2 (z_1) inside. Analogously, we deform the contour of integration $\mathcal{C}_2$ in Eq. (A.2) to any contour of the form $\mathcal{C}_2^1\cup\mathcal{C}_2^2$ also contained in Ω_2, where $\mathcal{C}_2^1$ ($\mathcal{C}_2^2$) is a simple closed loop which encircles the point ω_1 (ω_2) in the counterclockwise direction and does not contain the point ω_2 (ω_1) inside. Then,

$$\begin{aligned}a_{m,n}(z_1,z_2,\omega_1,\omega_2)={}&\frac{1}{(2\pi i)^2(z_2-z_1)(\omega_2-\omega_1)}\\&\times\sum_{i=1}^{2}\sum_{j=1}^{2}\int_{\mathcal{C}_1^i}\frac{du}{(u-z_1)^m(u-z_2)^{m+1}}\int_{\mathcal{C}_2^j}\frac{f(u,v)\,dv}{(v-\omega_1)^n(v-\omega_2)^{n+1}}.\end{aligned} \tag{A.12}$$

Taking into account considerations pointed out in Fig. 3 and the Cauchy formula for the derivatives,

$$\begin{aligned}a_{m,n}(z_1,z_2,\omega_1,\omega_2)={}&\frac{1}{(z_2-z_1)(\omega_2-\omega_1)}\left\{\frac{1}{(m-1)!(n-1)!}\frac{\partial^{m-1}}{\partial u^{m-1}}\frac{\partial^{n-1}}{\partial v^{n-1}}\left(\frac{f(u,v)}{(u-z_2)^{m+1}(v-\omega_2)^{n+1}}\right)\bigg|_{\substack{u=z_1\\v=\omega_1}}\right.\\&+\frac{1}{(m-1)!n!}\frac{\partial^{m-1}}{\partial u^{m-1}}\frac{\partial^{n}}{\partial v^{n}}\left(\frac{f(u,v)}{(u-z_2)^{m+1}(v-\omega_1)^{n}}\right)\bigg|_{\substack{u=z_1\\v=\omega_2}}\end{aligned}$$

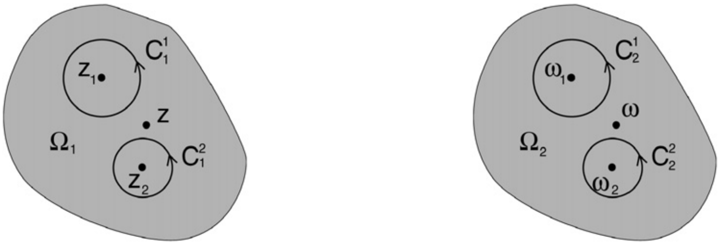

Fig. 3. The following analyticity properties are satisfied by the functions appearing in the integrands of (A.12): (a) The function $(u-z_2)^{-m-1}\times(v-\omega_2)^{-n-1}f(u,v)$ is analytic inside $\mathcal{C}_1^1\times\mathcal{C}_2^1$. (b) The function $(u-z_2)^{-m-1}(v-\omega_1)^{-n}f(u,v)$ is analytic inside $\mathcal{C}_1^1\times\mathcal{C}_2^2$. (c) The function $(u-z_1)^{-m}(v-\omega_2)^{-n-1}f(u,v)$ is analytic inside $\mathcal{C}_1^2\times\mathcal{C}_2^1$. (d) The function $(u-z_1)^{-m}(v-\omega_1)^{-n}f(u,v)$ is analytic inside $\mathcal{C}_1^2\times\mathcal{C}_2^2$.

$$+\frac{1}{m!(n-1)!}\frac{\partial^m}{\partial u^m}\frac{\partial^{n-1}}{\partial v^{n-1}}\left(\frac{f(u,v)}{(u-z_1)^m(v-\omega_2)^{n+1}}\right)\Big|_{\substack{u=z_2\\ v=\omega_1}}$$
$$+\frac{1}{m!n!}\frac{\partial^m}{\partial u^m}\frac{\partial^{n}}{\partial v^{n}}\left(\frac{f(u,v)}{(u-z_1)^m(v-\omega_1)^{n}}\right)\Big|_{\substack{u=z_2\\ v=\omega_2}}\Bigg\}. \tag{A.13}$$

From here, Eqs. (A.8)–(A.11) follow after straightforward computations. □

A.2. An alternative form of the expansion

The expansion (A.1) is not valid when $z_1=z_2$ and/or $\omega_1=\omega_2$. In this case, the coefficients $a_{m,n}(z_1,z_2,\omega_1,\omega_2)$ diverge as $z_1\to z_2$ and/or $\omega_1\to\omega_2$, although the remainder $r_N(z_1,z_2,\omega_1,\omega_2;z,\omega)$ remains well defined. In order to avoid this inconvenience, we consider the following alternative representation:

$$f(z,\omega)=\sum_{n=0}^{N-1}\sum_{m=0}^{n}[A_{m,n-m}+B_{m,n-m}z+C_{m,n-m}\omega+D_{m,n-m}z\omega]$$
$$\times(z-z_1)^m(z-z_2)^m(\omega-\omega_1)^{n-m}(\omega-\omega_2)^{n-m}+r_N(z_1,z_2,\omega_1,\omega_2;z,\omega), \tag{A.14}$$

with

$$A_{m,n}=z_1\omega_1a_{m,n}(z_1,z_2,\omega_1,\omega_2)+z_1\omega_2a_{m,n}(z_1,z_2,\omega_2,\omega_1)+z_2\omega_1a_{m,n}(z_2,z_1,\omega_1,\omega_2)$$
$$+z_2\omega_2a_{m,n}(z_2,z_1,\omega_2,\omega_1),$$
$$B_{m,n}=-\omega_1\big[a_{m,n}(z_1,z_2,\omega_1,\omega_2)+a_{m,n}(z_2,z_1,\omega_1,\omega_2)\big]-\omega_2\big[a_{m,n}(z_1,z_2,\omega_2,\omega_1)+a_{m,n}(z_2,z_1,\omega_2,\omega_1)\big],$$
$$C_{m,n}=-z_1\big[a_{m,n}(z_1,z_2,\omega_1,\omega_2)+a_{m,n}(z_1,z_2,\omega_2,\omega_1)\big]-z_2\big[a_{m,n}(z_2,z_1,\omega_1,\omega_2)+a_{m,n}(z_2,z_1,\omega_2,\omega_1)\big],$$

and

$$D_{m,n}=a_{m,n}(z_1,z_2,\omega_1,\omega_2)+a_{m,n}(z_1,z_2,\omega_2,\omega_1)+a_{m,n}(z_2,z_1,\omega_1,\omega_2)+a_{m,n}(z_2,z_1,\omega_2,\omega_1), \tag{A.15}$$

which are regular when $z_1\to z_2$ and/or $\omega_1\to\omega_2$. In fact, we have

$$A_{m,n}\equiv\frac{1}{(2\pi i)^2}\int_{\mathcal{C}_1}\frac{(u-z_1-z_2)\,du}{[(u-z_1)(u-z_2)]^{m+1}}\int_{\mathcal{C}_2}\frac{(v-\omega_1-\omega_2)f(u,v)\,dv}{[(v-\omega_1)(v-\omega_2)]^{n+1}},$$
$$B_{m,n}\equiv\frac{1}{(2\pi i)^2}\int_{\mathcal{C}_1}\frac{du}{[(u-z_1)(u-z_2)]^{m+1}}\int_{\mathcal{C}_2}\frac{(v-\omega_1-\omega_2)f(u,v)\,dv}{[(v-\omega_1)(v-\omega_2)]^{n+1}},$$
$$C_{m,n}\equiv\frac{1}{(2\pi i)^2}\int_{\mathcal{C}_1}\frac{(u-z_1-z_2)\,du}{[(u-z_1)(u-z_2)]^{m+1}}\int_{\mathcal{C}_2}\frac{f(u,v)\,dv}{[(v-\omega_1)(v-\omega_2)]^{n+1}}$$

and

$$D_{m,n} \equiv \frac{1}{(2\pi i)^2} \int_{\mathcal{C}_1} \frac{du}{[(u-z_1)(u-z_2)]^{m+1}} \int_{\mathcal{C}_2} \frac{f(u,v)\,dv}{[(v-\omega_1)(v-\omega_2)]^{n+1}}.$$

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