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Asymptotic approximations for a singularly perturbed convection–diffusion problem with discontinuous data in a sector

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Abstract

We consider a singularly perturbed convection–diffusion equation, $-\varepsilon\Delta u + \vec{v}\cdot\vec{\nabla}u = 0$ on an arbitrary sector shaped domain, $\Omega \equiv \{(r,\phi)|r>0, 0<\phi<\alpha\}$ being r and ϕ polar coordinates and $0<\alpha<2\pi$. We consider for this problem discontinuous Dirichlet boundary conditions at the corner of the sector: $u(r,0)=0$, $u(r,\alpha)=1$. An asymptotic expansion of the solution is obtained from an integral representation in two limits: (a) when the singular parameter $\varepsilon \to 0^+$ (with fixed distance r to the discontinuity point of the boundary condition) and (b) when that distance $r \to 0^+$ (with fixed ε). It is shown that the first term of the expansion at $\varepsilon = 0$ contains an error function. This term characterizes the effect of the discontinuity on the ε-behaviour of the solution and its derivatives in the boundary or internal layers. On the other hand, near discontinuity of the boundary condition $r=0$, the solution $u(r,\phi)$ of the problem is approximated by a linear function of the polar angle ϕ.

MSC: 35C20; 41A60

Keywords: Singular perturbation problem; Discontinuous boundary data; Asymptotic expansions; Error function

1. Introduction

Mathematical models that involve a combination of convective and diffusive processes are quite important in all of science, engineering and other fields where mathematical modeling is required. Very often, the dimensionless parameter that measures the relative strength of the diffusion is very small. This

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implies that thin boundary and/or interior layers are present in the solution and singular perturbation problems arise. This kind of problem appears, for example, in fluid or gas dynamics [15,23], heat transfer [2,3], theory of plates and shells [13], or magnetohydrodynamic flow [8,19]. An extensive selection of singularly perturbed convection–diffusion problems of the physics or engineering may be found in [17]: pollutant dispersal in a river estuary, vorticity transport in the incompressible Navier–Stokes equations, atmospheric pollution, groundwater transport, turbulence transport, etc. Besides the small perturbation parameter, another source of singular behaviour for the solution are the (possible) discontinuities of the boundary data: think for example in transportation of contaminant in a river with a source of contamination located on a finite portion of the side of the water.

Mathematically speaking, a singularly perturbed convection–diffusion problem is a boundary value problem of the second order in which the coefficients of the second-order derivatives are small. In this paper, we focus our attention on two-dimensional linear convection–diffusion (elliptic) problems of the form: find a function $u \in \mathscr{C}(\bar{\Omega}) \cap \mathscr{D}^2(\Omega)$ such that

$$\begin{cases} -\varepsilon\Delta u + \overrightarrow{v} \cdot \overrightarrow{\nabla} u = h(x), & x \in \Omega \subset \mathbb{R}^2, \\ u(x)|_{\partial\Omega} = f(\tilde{x}), & \tilde{x} \in \partial\Omega, \end{cases} \tag{1}$$

where ε is a small positive parameter, $\overrightarrow{v}$ is the convection vector, $h(x)$ is a nonhomogeneous term, $\tilde{x}$ is a variable which lives in $\partial\Omega$, $f(\tilde{x})$ is the Dirichlet datum and $\mathscr{D}^2(\Omega)$ is the set of functions with partial derivatives up to order two defined in all points of Ω.

The location and shape of the boundary layers of u depend, among other things, on the prescribed velocity field $\overrightarrow{v}$, on the shape of the boundary $\partial\Omega$ and on the existence of discontinuities in $f(\tilde{x})$. For example, regular boundary layers of size $\mathcal{O}(\varepsilon)$ appear on the outflow boundary, whereas parabolic boundary layers of size $\mathcal{O}(\sqrt{\varepsilon})$ appear along the characteristic boundaries. For more details on the shape and nature of boundary layers see for example [5–7,9,10] and references therein.

The knowledge of an asymptotic expansion for the solution may help in the development of a suitable numerical method for these kind of problems because it gives the qualitative behaviour of the solution [24, p. 6]. An ε-uniformly convergent method requires the analysis of uniform convergence and then, accurate error bounds for the local error. The accuracy of these error bounds depends on the precision in the approximation given by the first terms of the asymptotic expansion. The design of the numerical technique is based on the exact integration of the first terms of the asymptotic expansion or of functions which have a similar behaviour in the singular layer. Along this line, some references which propose exponential fitting techniques or special meshes based on asymptotic expansions are [4] or [11]. A classical reference is [16].

To get the exact solution of a boundary problem in terms of elementary functions is, in general, an impossible mission. Then, an approximation of the solution adapted to the singular character of this kind of problems (an asymptotic expansion) is of interest. There is an extensive literature devoted to the construction of approximated solutions of singular perturbation problems based on matching of asymptotic expansions. The book of Il'in [10] contains a quite exhaustive and general analysis for different equations and domains. Other important references are for example [6,12] or [18]. However, a perturbative analysis based on an expansion of the solution in powers of the perturbation parameter does not always work for discontinuous Dirichlet boundary conditions [22]. This is so, because the coefficients of the expansion contain derivatives of the boundary condition, whereas the solution of the elliptic problem (1) is smooth inside the domain.

If the Dirichlet datum $f(\tilde{x})$ is continuous except at a finite number of points, where it presents jump discontinuities, then $f(\tilde{x}) = g(\tilde{x}) + \sum_{k=1}^{n} a_k \Theta(\tilde{x} - \tilde{x}_k)$, where $g(\tilde{x})$ is continuous, $a_k \in \mathbb{R}$ and Θ denotes the step function. We can decompose (1) into the problem:

$$\begin{cases} -\varepsilon\Delta u + \overrightarrow{v} \cdot \overrightarrow{\nabla} u = h(x), & x \in \Omega \subset \mathbb{R}^2, \\ u(x)|_{\partial\Omega} = g(\tilde{x}), & \tilde{x} \in \partial\Omega, \end{cases} \tag{P_0}$$

plus n problems of the form:

$$\begin{cases} -\varepsilon\Delta u + \overrightarrow{v} \cdot \overrightarrow{\nabla} u = 0, & x \in \Omega \subset \mathbb{R}^2, \\ u(x)|_{\partial\Omega} = \Theta(\tilde{x} - \tilde{x}_k), & \tilde{x} \in \partial\Omega. \end{cases} \tag{P_k}$$

An asymptotic expansion for (P_0) may be obtained by the method of matching asymptotic expansions [10]. An asymptotic expansion for the problems (P_k) may be obtained from an exact representation of the solution.

Some particular problems of form (P_k) have been already considered in the literature. For example, Hedstrom and Osterheld [9] studied the problem $\varepsilon\Delta u - \partial_y u = 0$ on the positive quarter plane with boundary conditions $u(x, 0) = 0$ and $u(0, y) = 1$. They obtained the first two terms of the asymptotic expansion of u for $\varepsilon \to 0^+$ from a Fourier integral representation of u. The first term of this expansion is an error function. A more detailed investigation has been developed in [20]: an integral representation for u is obtained from the associated Helmholtz equation and a complete asymptotic expansion of u for $\varepsilon \to 0^+$ is derived from this integral representation. The same equation $\varepsilon\Delta u - \partial_y u = 0$, but in a generic sector, is considered in [21], where an integral representation for u is obtained from the associated Helmholtz equation. Different asymptotic expansions as $\varepsilon \to 0^+$ are obtained depending on the angle of the sector and again the error function plays an important role in the analysis. A similar problem defined in the interior of a circle is analyzed in [22]. In all these problems, the approximation is not valid near the discontinuities of the boundary condition.

In this paper we try to shed light on the influence that the discontinuities in the boundary conditions on corner points of the domain have on the boundary or interior layers of the solution. We want to investigate if, as in the examples mentioned, the solution is approximated by an error function. For this purpose we analyze the problem $-\varepsilon\Delta u + \overrightarrow{v} \cdot \overrightarrow{\nabla} u = 0$ on a general sector with a discontinuous boundary condition at the corner of the sector. We consider a general convection vector $\overrightarrow{v}$ (in [21], $\overrightarrow{v} = (0, 1)$) because the location and size of the singular layers depend also on the relative direction of $\overrightarrow{v}$ with respect to the sector: boundary layers appear if $\overrightarrow{v}$ points out of the sector, whereas interior layers are present if $\overrightarrow{v}$ points into the sector. As in the references mentioned in the paragraph above, the starting point to analyze the problem is an integral representation of the solution. We approximate the solution by deriving asymptotic expansions from this integral, not only in the singular limit $\varepsilon \to 0^+$, but also in the limit $r \to 0^+$, where r represents the distance to the discontinuities. Then, we approximate the solution on the whole domain, including the neighbourhood of the discontinuity point $r = 0$. For the approximation in the singular limit we use similar techniques to those used in [20].

In Section 2 we obtain an integral representation for the solution. In Section 3 we derive an asymptotic expansion of the solution for $\varepsilon \to 0^+$ whereas in Section 4 we derive an asymptotic expansion for $r \to 0^+$. Some comments and conclusions are postponed to Section 5.

2. The problem and its exact solution

We use polar coordinates to describe an infinite sector $\bar{\Omega}$ of amplitude α in the plane with its corner point removed: $x = r\cos\phi$, $y = r\sin\phi$, $(r,\phi) \in \bar{\Omega} \equiv (0,\infty) \times [0,\alpha]$. Its interior set is $\Omega \equiv (0,\infty)\times(0,\alpha)$ (see Fig. 1 (a)).

We consider a singularly perturbed convection–diffusion problem defined on this sector with an "infinite source of contamination" located at one side of the sector:

$$\begin{cases} U \in \mathscr{C}(\bar{\Omega}) \cap \mathscr{D}^2(\Omega) \text{ and} & U \text{ bounded at } r=0,\\ -\varepsilon\Delta U + \vec{v}\cdot\vec{\nabla} U = 0 & \text{in } \Omega,\\ U(r,0)=0 \text{ and } U(r,\alpha)=1 & \text{for } r>0, \end{cases} \tag{2}$$

where $\vec{v} \equiv (\cos\beta, \sin\beta)$ is a constant vector, $\varepsilon > 0$ is a small parameter, $0 \leqslant \beta < 2\pi$ and $0 < \alpha < 2\pi$. (Observe the discontinuous Dirichlet condition at the corner $r = 0$.)

After the change of the unknown $U(r,\phi) = 1 - F(r,\phi)\exp(\vec{v}\cdot\vec{r}/(2\varepsilon))$, where $\vec{r} \equiv (r\cos\phi, r\sin\phi)$, problem (2) is transformed into the Yukawa equation for $F(r,\phi)$:

$$\begin{cases} F \in \mathscr{C}(\bar{\Omega}) \cap \mathscr{D}^2(\Omega) \text{ and} & F \text{ bounded at } r=0,\\ \Delta F - w^2 F = 0 & \text{in } \Omega\\ F(r,0) = \mathrm{e}^{-wr\cos\beta} \text{ and } F(r,\alpha)=0 & \text{for } r>0, \end{cases} \tag{3}$$

where $w \equiv 1/(2\varepsilon)$.

We will obtain a solution of problem (3) and therefore of problem (2) in Proposition 2, but this solution may not be unique unless we impose a convenient condition upon $U(r,\phi)$ (or upon $F(r,\phi)$) concerning its growth at infinity. Then, we add a radiation condition to (2) and consider the following problem:

$$\begin{cases} U \in \mathscr{C}(\bar{\Omega}) \cap \mathscr{D}^2(\Omega) \text{ and} & U \text{ bounded at } r=0,\\ -\varepsilon\Delta U + \vec{v}\cdot\vec{\nabla} U = 0 & \text{in } \Omega,\\ U(r,0)=0 \text{ and } U(r,\alpha)=1 & \text{for } r>0,\\ U(r,\phi) = \delta_{\phi,\pi+\beta} + o\left(\dfrac{\mathrm{e}^{wr(1+\cos(\beta-\phi))}}{\sqrt{wr}}\right) & \text{as } r\to\infty \text{ and } \phi\in(0,\alpha). \end{cases} \tag{P}$$

In what follows, $\chi_A(x)$ represents the characteristic function of the set A and $\delta_{a,b}$ is the Kronecker delta:

$$\chi_A(x) \equiv \begin{cases} 1 & \text{if } x\in A,\\ 0 & \text{if } x\notin A, \end{cases} \qquad \delta_{a,b} \equiv \begin{cases} 1 & \text{if } a=b,\\ 0 & \text{if } a\neq b. \end{cases}$$

We have the following uniqueness result:

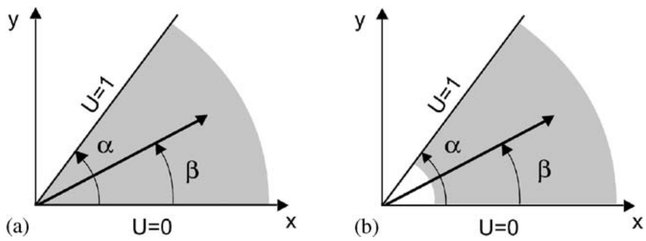

Fig. 1. (a) Domain $\bar{\Omega} \equiv (0,\infty)\times[0,\alpha]$ of problem (P). (b) Indented region $\Omega^* \equiv (r_0,\infty)\times(0,\alpha)$ in Theorem 1.

Proposition 1. *Problem (P) has at most one solution.*

Proof. Suppose that U_1 and U_2 are two solutions of (P). Then, the function $G(r,\phi) \equiv (U_1(r,\phi) - U_2(r,\phi))\mathrm{e}^{-wr\,\cos(\beta-\phi)}$ verifies:

$$\begin{cases} G \in \mathscr{C}(\bar{\Omega}) \cap \mathscr{D}^2(\Omega) \text{ and} & G \text{ bounded at } r=0, \\ \Delta G - w^2 G = 0 & \text{in } \Omega \\ G(r,0)=0 \text{ and } G(r,\alpha)=0 & \text{for } r>0, \\ G(r,\phi) = o\left(\dfrac{\mathrm{e}^{wr}}{\sqrt{wr}}\right) \text{ as } r\to\infty & \text{and } \phi\in(0,\alpha). \end{cases} \tag{4}$$

Consider the following auxiliary function defined on $\bar{\Omega}$ and at $r=0$:

$$V_a(r,\phi) \equiv \begin{cases} \dfrac{G(r,\phi)}{H_a(wr)} & \text{if } r\neq 0, \\ 0 & \text{if } r=0, \end{cases} \qquad H_a(wr) \equiv K_0(wr) + I_0(wr) + a,$$

where K_0 and I_0 are modified Bessel functions of order zero and a is a positive constant. The function $H_a(wr)$ is positive for $wr>0$, of the order $\mathscr{O}(\mathrm{e}^{wr}/\sqrt{wr})$ as $wr\to\infty$ and $\mathscr{O}(\log(wr))$ as $wr\to 0$ [1, Eqs. (9.7.1) and (9.6.13)]. Moreover, $H_a(wr)\in\mathscr{C}(\bar{\Omega})\cap\mathscr{D}^2(\Omega)$ and satisfies the equation: $\Delta H_a - w^2 H_a + aw^2 = 0$ in Ω [1, Eq. (9.6.1)]. Therefore, the auxiliary function V_a is continuous on $\bar{\Omega}$ and at $r=0$ and verifies:

$$\begin{cases} \Delta V_a + \dfrac{2}{H_a}\vec{\nabla} H_a \cdot \vec{\nabla} V_a = \dfrac{aw^2}{H_a} V_a & \text{in } \Omega, \\ V_a(r,0) = V_a(r,\alpha) = 0 & \forall\, r>0, \\ \lim_{r\to 0} V_a(r,\phi) = \lim_{r\to\infty} V_a(r,\phi) = 0 & \forall\,\phi\in[0,\alpha]. \end{cases}$$

Consider the open finite sector of radius R: $\Omega_R \equiv (0,R)\times(0,\alpha)$. At points $(r,\phi)\in\Omega_R$ where $\vec{\nabla} V_a = 0$ and $V_a\neq 0$, we have that $V_a\cdot\Delta V_a > 0$. Therefore, V_a has not positive relative maximums neither negative relative minimums in Ω_R. Then $\mathrm{Sup}_{\Omega_R}|V_a| \leqslant \mathrm{Sup}_{\partial\Omega_R}|V_a|$.

Using that $\lim_{r\to\infty} V_a(r,\phi) = 0$ we have that, $\forall\delta>0$, there is a $R>0$ such that $|V_a(R,\phi)|\leqslant\delta$. On the other hand, $V_a(r,0) = V_a(r,\alpha) = 0$ $\forall r>0$ and $\lim_{r\to 0} V_a(r,\phi) = 0$ $\forall\phi\in[0,\alpha]$. Therefore, $|V_a(r,\phi)|\leqslant\delta$ $\forall\delta>0$ and every $(r,\phi)\in\Omega_R$. Taking the limit $\delta\to 0$ $(R\to\infty)$ we have that $V_a=0$ on $\bar{\Omega}$. Therefore, $G=0$ and $U_1=U_2$ on $\bar{\Omega}$. $\square$

Remark 1. Only for $\phi=\beta\pm\pi$, the radiation condition given in the last line of (P) specifies a precise limit of U at $r=\infty$: $\lim_{r\to\infty} U(r,\beta+\pi)=1$ and $\lim_{r\to\infty} U(r,\beta-\pi)=0$. For the remaining values of $\phi\in(0,\alpha)$, that radiation condition allows an exponential growing of U at $r=\infty$. That is, to have uniqueness in problem (2), we require to specify a precise radiation condition in the direction $\phi=\beta+\pi$ or $\phi=\beta-\pi$ only if one of these directions is contained in the sector. In other directions of the sector we only require a not too wild growing of U. A geometrical interpretation of this is the following: uniqueness requires for U to have a defined value at the inflow boundary. Uniqueness requires the convection vector $\vec{v}$ to drag a concrete boundary value inside Ω. When the inflow boundary has not a defined boundary condition, the vector $\vec{v}$ does not drag any specific value inside Ω and uniqueness is not assured (see Fig. 2).

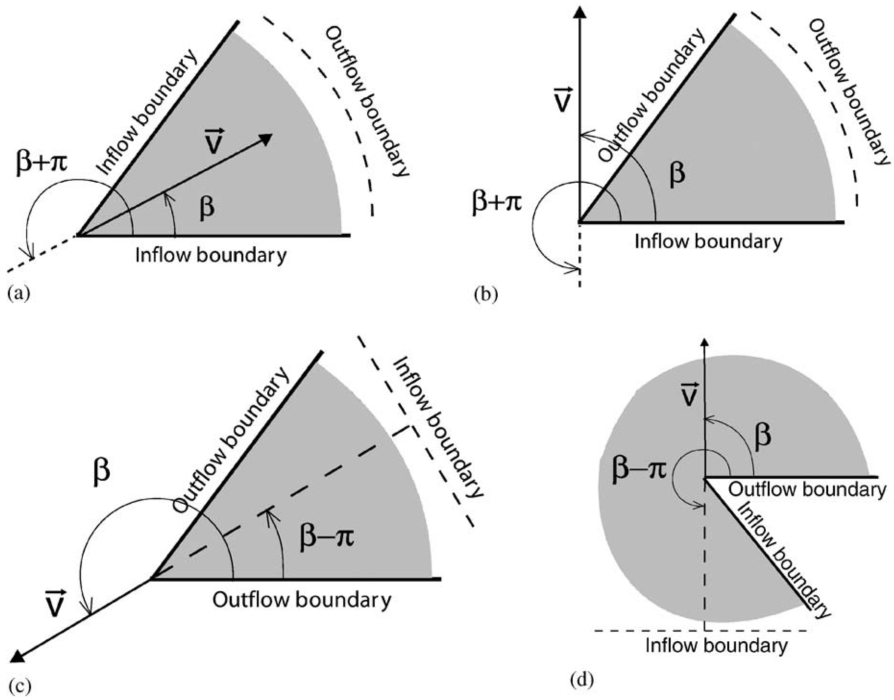

Fig. 2. (a)–(b) The inflow boundary is contained in the union of the boundary lines $\phi = 0$ and $\phi = \alpha$ if $\beta \pm \pi \notin (0, \alpha)$. (c)–(d) The inflow boundary is not contained in the union of those boundary lines and "contains a portion of arc at the infinity" if $\beta + \pi \in (0, \alpha)$ or $\beta - \pi \in (0, \alpha)$.

In order to construct the unique solution of problem (P) we will need the following function defined by means of an integral:

Definition 1. We define the function

$$I_{\alpha,\beta}(r, \phi) \equiv \frac{e^{wr\cos(\beta-\phi)}}{2\alpha} \int_{-\infty}^{\infty} e^{-wr\cosh t} \frac{\sin(\mu\phi)}{\cosh[\mu(t - i\beta)] - \cos(\mu\phi)}\, dt, \tag{5}$$

where $\mu \equiv \pi/\alpha$. It must be implicitly understood in the above formula that, when $\phi = \pm\beta + 2k\alpha$ with $k \in \mathbb{Z}$, the following Cauchy principal value must be taken on the integral:

$$\begin{aligned} I_{\alpha,\beta}(r, \pm\beta + 2k\alpha) \equiv\ & \pm \frac{e^{wr\cos(\beta\mp\beta-2k\alpha)}}{2\alpha} \\ & \times \lim_{\varepsilon\to 0^+} \left\{ \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right\} \frac{\sin(\mu\beta) e^{-wr\cosh t}}{\cosh[\mu(t - i\beta)] - \cos(\mu\beta)}\, dt. \end{aligned} \tag{6}$$

Lemma 1. *The function $I_{\alpha,\beta}(r, \phi)$ is well-defined for any $(r, \phi) \in \bar{\Omega}$, $\beta \in [0, 2\pi)$ and $\alpha \in (0, 2\pi)$.*

Proof. If $\phi \neq \pm\beta + 2k\alpha$, $k \in \mathbb{Z}$, the integral in (5) is convergent because the integrand is a continuous function of t $\forall t \in \mathbb{R}$ and it is exponentially decaying at $t = \pm\infty$.

If $\phi = \pm\beta + 2k\alpha$, $k \in \mathbb{Z}$, the integrand in (5) has a pole at $t = 0$ which is removed by taking the Cauchy principal value as given in (6). Indeed,

$$\lim_{\varepsilon\to 0^+} \left\{ \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right\} \frac{e^{-wr\cosh t}}{\cosh[\mu(t-i\beta)] - \cos(\mu\beta)}\, dt$$
$$= \int_{-\infty}^{\infty} e^{-wr\cosh t} \frac{\cos(\mu\beta)[\cosh(\mu t) - 1]}{\cos^2(\mu\beta)[\cosh(\mu t)-1]^2 + \sin^2(\mu\beta)\sinh^2(\mu t)}\, dt.$$

This last integral is convergent because the integrand is a continuous function of t $\forall t \in \mathbb{R}\backslash\{0\}$ with a removable singularity at $t = 0$ and it is exponentially decaying at $t = \pm\infty$. $\square$

In the following lemma, we construct an explicit solution of problem (3) as a generalization of the solution derived in [21] for $\beta = \pi/2$. From this solution we will obtain the unique solution of problem (P) in Proposition 2.

Lemma 2. *Let $\mu = \pi/\alpha$. The function*

$$F(r,\phi) = \begin{cases} e^{-wr\cos\beta} & if\ \ \phi = 0 \\ \dfrac{1}{2\alpha}\displaystyle\int_{-\infty+i\beta}^{\infty+i\beta} e^{-wr\cosh t} \frac{\sin(\mu\phi)}{\cosh[\mu(t-i\beta)] - \cos(\mu\phi)}\, dt & if\ \ \phi \in (0,\alpha] \end{cases} \tag{7}$$

is a solution of problem (3).

Proof. For $\phi \neq 0$ the integrand in the second line of (7) is a continuous function of t and exponentially decaying at $t = \pm\infty$ for $r \geqslant 0$ and $\phi \in (0, \alpha]$. Therefore, $F(r, \phi)$ is well defined on $\bar{\Omega}$ and bounded at $r = 0$. On the other hand, using uniform convergence, we see that F is twice differentiable in $(0, \infty) \times (0, \alpha)$. Moreover, the function

$$W(t,\phi) \equiv \frac{1}{2\alpha} \frac{\sin(\mu\phi)}{\cosh[\mu(t-i\beta)] - \cos(\mu\phi)}$$

satisfies the equation

$$\frac{\partial^2 W}{\partial t^2} + \frac{\partial^2 W}{\partial \phi^2} = 0, \quad \forall t \in \mathbb{C} \text{ and } \phi \in (0,\alpha).$$

Using these facts, it can be easily shown that the function $F(r, \phi)$ verifies the equation $\Delta F - w^2 F = 0$ in Ω. The condition $F(r, \alpha) = 0$ is satisfied trivially and

$$\lim_{\phi\to 0} F(r,\phi) = \lim_{\phi\to 0} \frac{1}{2\alpha} \int_{-\infty}^{\infty} e^{-wr\cosh(t+i\beta)} \frac{\sin(\mu\phi)}{\cosh(\mu t) - \cos(\mu\phi)}\, dt$$
$$= \lim_{\phi\to 0} \frac{1}{2\alpha} \int_{-\infty}^{\infty} e^{-wr\cos\beta} \frac{\sin(\mu\phi)}{\cosh(\mu t) - \cos(\mu\phi)}\, dt$$
$$+ \lim_{\phi\to 0} \frac{\sin(\mu\phi)}{2\alpha} \int_{-\infty}^{\infty} \frac{[e^{-wr\cosh(t+i\beta)} - e^{-wr\cos\beta}]}{\cosh(\mu t) - \cos(\mu\phi)}\, dt$$
$$= \lim_{\phi\to 0} e^{-wr\cos\beta} \frac{\alpha - \phi}{\alpha} + 0 = e^{-wr\cos\beta}. \qquad \square$$

In the following proposition we obtain the explicit solution of problem (P) in a more tractable form than in Lemma 2. In what follows, empty sums must be understood as zero.

Proposition 2. *Let $w \equiv 1/(2\varepsilon)$. Then, for $(r,\phi) \in \Omega$ and $\beta \in [0, 2\pi)$, the solution $U_{\alpha,\beta}(r,\phi)$ of problem (P) is*:

1. *If $\beta = 0$,*
$$U_{\alpha,0}(r,\phi) = 1 - I_{\alpha,0}(r,\phi); \tag{8}$$
2. *If $0 < \beta < \alpha$,*
$$U_{\alpha,\beta}(r,\phi) = \chi_{(\beta,\alpha]}(\phi) + \tfrac{1}{2}\delta_{\phi,\beta} - I_{\alpha,\beta}(r,\phi); \tag{9}$$
3. *If $\beta = \alpha$,*
$$U_{\alpha,\alpha}(r,\phi) = -I_{\alpha,\alpha}(r,\phi); \tag{10}$$
4. *If $\alpha < \beta < \alpha + \pi$,*
$$\begin{aligned} U_{\alpha,\beta}(r,\phi) = \mathrm{e}^{wr\ \cos(\beta-\phi)} \Bigg\{ & \sum_{k=1}^{[\frac{\beta+\phi}{2\alpha}]} \mathrm{e}^{-wr\ \cos(\beta+\phi-2k\alpha)} - \sum_{k=1}^{[\frac{\beta-\phi}{2\alpha}]} \mathrm{e}^{-wr\ \cos(\beta-\phi-2k\alpha)} \\ & + \frac{1}{2}\mathrm{e}^{-wr} \left(\delta_{\frac{\beta-\phi}{2\alpha},[\frac{\beta-\phi}{2\alpha}]} - \delta_{\frac{\beta+\phi}{2\alpha},[\frac{\beta+\phi}{2\alpha}]} \right) \Bigg\} - I_{\alpha,\beta}(r,\phi); \end{aligned} \tag{11}$$
5. *If $\alpha + \pi \leqslant \beta$,*
$$\begin{aligned} U_{\alpha,\beta}(r,\phi) = \mathrm{e}^{wr\ \cos(\beta-\phi)} \Bigg\{ & \sum_{k=1}^{[\frac{2\pi+\phi-\beta}{2\alpha}]} \mathrm{e}^{-wr\ \cos(-\phi+\beta+2k\alpha)} - \sum_{k=0}^{[\frac{2\pi-\phi-\beta}{2\alpha}]} \mathrm{e}^{-wr\ \cos(\phi+\beta+2k\alpha)} \\ & + \frac{1}{2}\mathrm{e}^{-wr} \left(\delta_{\frac{2\pi-\phi-\beta}{2\alpha},[\frac{2\pi-\phi-\beta}{2\alpha}]} - \delta_{\frac{2\pi+\phi-\beta}{2\alpha},[\frac{2\pi+\phi-\beta}{2\alpha}]} \right) \Bigg\} + 1 - I_{\alpha,\beta-2\pi}(r,\phi); \end{aligned} \tag{12}$$

where $I_{\alpha,\beta}(r,\phi)$ is given in (5).

Proof. For convenience, we consider in this proof the angle
$$\tilde{\beta} \equiv \begin{cases} \beta & \text{if } 0 \leqslant \beta \leqslant \pi, \\ \beta - 2\pi & \text{if } \pi < \beta < 2\pi \end{cases} \tag{13}$$
and observe that $-\pi < \tilde{\beta} \leqslant \pi$. For reasons that will be clear in a moment, in principle, we restrict ourselves to $-\pi/2 < \tilde{\beta} < \pi/2$.

A solution of (3) has just been obtained in Lemma 2. Then, the function $U_{\alpha,\beta}(r,\phi) \equiv 1 - \mathrm{e}^{wr\ \cos(\tilde{\beta}-\phi)} F(r,\phi)$, with $F(r,\phi)$ given in (7) is a solution of (2).

The poles of the integrand in (7) are located at the points $t_k^1 \equiv \mathrm{i}(\tilde\beta - \phi + 2k\alpha)$ and $t_k^2 \equiv \mathrm{i}(\tilde\beta + \phi + 2k\alpha)$, $k \in \mathbb{Z}$ and the real part of the exponent reads $-rw\cosh(\Re t)\cos(\Im t)$ with $|\Im t| < \pi/2$ if $|\Im t| \leqslant |\tilde\beta|$ and $-\pi/2 < \tilde\beta < \pi/2$. We can use the Cauchy Residue Theorem for shifting the integration contour in the integral $F(r,\phi)$ to the straight line $\Im t = 0$. Therefore, we can write:

$$F(r,\phi) = \frac{1}{2\alpha}\int_{-\infty}^{\infty} \mathrm{e}^{-wr\cosh t}\frac{\sin(\mu\phi)}{\cosh[\mu(t-\mathrm{i}\tilde\beta)] - \cos(\mu\phi)}\,\mathrm{d}t \\ - 2\pi\mathrm{i}\,\mathrm{sign}(\tilde\beta)\sum_{a\in R}\mathrm{Res}\{\mathrm{e}^{-wr\cosh t}W(t,\phi); a\},$$

where R is the set of the poles of $W(t,\phi)$ located between the lines $\Im t = 0$ and $\Im t = \mathrm{i}\tilde\beta$. Therefore, we can write finally

$$U_{\alpha,\beta}(r,\phi) = 1 - I_{\alpha,\tilde\beta}(r,\phi) + 2\pi\mathrm{i}\,\mathrm{sign}(\tilde\beta)\mathrm{e}^{wr\cos(\tilde\beta-\phi)}\sum_{a\in R}\mathrm{Res}\{\mathrm{e}^{-wr\cosh t}W(t,\phi); a\}, \tag{14}$$

where $\mathrm{sign}(0) = 0$ (in what follows we will use this convention). Formulas (8)–(12) for $-\pi/2 < \tilde\beta < \pi/2$ ($0 \leqslant \beta < \pi/2$ or $3\pi/2 < \beta < 2\pi$) follow from counting the poles in R. We distinguish several cases:

Case 1: $\tilde\beta = 0$.

$R = \emptyset$ (in this case it is not necessary to shift the integration contour to $\Im t = 0$ because we are already on it).

Case 2: $0 < \tilde\beta < \frac{\pi}{2}$ and $\alpha > \tilde\beta$.

$R = \{t_0^1 = \mathrm{i}(\tilde\beta - \phi)\}$ if $\phi \leqslant \tilde\beta$ and $R = \emptyset$ if $\phi > \tilde\beta$.

Case 3: $0 < \tilde\beta < \frac{\pi}{2}$ and $\alpha = \tilde\beta$.

$R = \{t_0^1 = \mathrm{i}(\tilde\beta - \phi)\}$.

Case 4: $0 < \tilde\beta < \frac{\pi}{2}$ and $\alpha < \tilde\beta$.

$$R = \left\{t_k^1 = \mathrm{i}(\tilde\beta - \phi + 2k\alpha), k = 0, 1, \ldots, \left[\frac{\tilde\beta - \phi}{2\alpha}\right]\right\} \\ \cup \left\{t_k^2 = \mathrm{i}(\tilde\beta + \phi + 2k\alpha), k = 1, 2, \ldots, \left[\frac{\tilde\beta + \phi}{2\alpha}\right]\right\}.$$

Case 5.1: $-\frac{\pi}{2} < \tilde\beta < 0$ and $\alpha > -\tilde\beta$.

$R = \{t_0^2 = \mathrm{i}(\tilde\beta + \phi)\}$ if $\phi \leqslant -\tilde\beta$ and $R = \emptyset$ if $\phi > -\tilde\beta$.

Case 5.2: $-\frac{\pi}{2} < \tilde\beta < 0$ and $\alpha < -\tilde\beta$.

$$R = \left\{t_k^1 = \mathrm{i}(\tilde\beta - \phi + 2k\alpha), k = \left[\frac{\tilde\beta - \phi}{2\alpha}\right] \ldots, -1\right\} \\ \cup \left\{t_k^2 = \mathrm{i}(\tilde\beta + \phi + 2k\alpha), k = \left[\frac{\tilde\beta + \phi}{2\alpha}\right], \ldots, 0\right\}.$$

Case 5.3: $-\frac{\pi}{2} \leqslant \tilde{\beta} < 0$ and $\alpha = -\tilde{\beta}$.
$R = \{t_0^2 = \mathrm{i}(\tilde{\beta} + \phi)\}$.

At this moment, formulas (8)–(12) are a solution of (2) for $-\pi/2 < \tilde{\beta} < \pi/2$ ($0 \leqslant \beta < \pi/2$ or $3\pi/2 < \beta < 2\pi$). But it is routine to check that they are also a solution of (2) for the range of values of β appearing in the statement of the proposition. From Theorem 1 below we have that $U_{\alpha,\beta}(r,\phi)$ satisfies the radiation condition given in the last line of (P). Therefore, it is the unique solution of problem (P). □

Remark 2. Observe that the general structure of the solution $U_{\alpha,\beta}(r,\phi)$ of problem (P) given in Proposition 2 can be decomposed as

$$U_{\alpha,\beta} \equiv \Xi_{\alpha,\beta} - \begin{cases} I_{\alpha,\beta} & \text{if } 0 \leqslant \beta < \alpha + \pi, \\ I_{\alpha,\beta-2\pi} & \text{if } \alpha + \pi \leqslant \beta < 2\pi, \end{cases} \tag{15}$$

where the function $\Xi_{\alpha,\beta}$ denotes a linear combination of characteristic functions, Kronecker deltas and exponential functions as it is detailed in the enunciate of Proposition 2. The function $I_{\alpha,\beta}$ is defined in (5).

The solution of (P) cannot be written in terms of known functions. But, for $\varepsilon \to 0^+$ and r away from 0, we can approximate $U_{\alpha,\beta}(r,\phi)$ by an error function and elementary functions plus an asymptotic expansion in powers of ε. For $r \to 0^+$ (and $\varepsilon \geqslant \varepsilon_0 > 0$), we can approximate $U_{\alpha,\beta}(r,\phi)$ by an asymptotic expansion in powers of r. This is the subject of the two following sections.

3. Asymptotic expansion of $U_{\alpha,\beta}(r,\phi)$ in the singular limit

In this section we denote by Ω^* the sector shaped domain indented at the point $r = 0$ (see Fig. 1(b)): $\Omega^* \equiv (r_0, \infty) \times (0, \alpha)$, $r_0 > 0$.

The proof of the main theorem of this section uses the following definition and lemma.

Definition 2. We define the functions

$$g(t,\phi,\alpha,\beta) \equiv \frac{\sin(\mu\phi)}{2\alpha[\cosh(\mu(t - \mathrm{i}\beta)) - \cos(\mu\phi)]}; \quad \mu = \frac{\pi}{\alpha} \tag{16}$$

and

$$h(t,\theta) \equiv \frac{1}{4\pi\mathrm{i}\,\sinh\frac{1}{2}(t - \mathrm{i}\theta)}, \tag{17}$$

where θ is not an independent variable but a function of ϕ, α and β:

$$\theta(\phi,\alpha,\beta) \equiv \begin{cases} \beta - \mathrm{sign}(\beta)\left(\phi + 2\alpha\left\lfloor\dfrac{|\beta|}{2\alpha}\right\rfloor\right) & \text{if } \mathrm{Frac}\left(\dfrac{|\beta|}{2\alpha}\right) \leqslant \dfrac{1}{2}, \\ \beta + \mathrm{sign}(\beta)\left(\phi - 2\alpha\left(\left\lfloor\dfrac{|\beta|}{2\alpha}\right\rfloor + 1\right)\right) & \text{if } \mathrm{Frac}\left(\dfrac{|\beta|}{2\alpha}\right) > \dfrac{1}{2}, \end{cases} \tag{18}$$

with $\mathrm{sign}(0) = 1$. We define also

$$\bar{I}(x,\phi,\alpha,\beta) \equiv \int_{-\infty}^{\infty} \mathrm{e}^{-x\,\cosh\,t} f(t,\phi,\alpha,\beta)\,\mathrm{d}t, \tag{19}$$

with

$$f(t,\phi,\alpha,\beta) \equiv g(t,\phi,\alpha,\beta) + \mathrm{sign}(\beta)\mathrm{sign}\left(\frac{1}{2} - \mathrm{Frac}\left(\frac{|\beta|}{2\alpha}\right)\right) \times [h(t,\theta(\phi,\alpha,\beta)) - [\delta_{\beta,0} + \delta_{\beta,\alpha} + \delta_{\beta,2\pi-\alpha}]h(t,-\theta(\phi,\alpha,\beta))]. \tag{20}$$

(Observe that $g(t,\phi,\alpha,\beta)$ is part of the integrand in (5)).

Lemma 3. *Let* $(r,\phi) \in \Omega^*$, $\beta \in [0,2\pi)$, $\alpha \in (0,2\pi)$. *Then, the function* $\bar{I}(x,\phi,\alpha,\beta)$ *given in Definition* 2 *has the following asymptotic expansion for large positive* x:

$$\bar{I}(x,\phi,\alpha,\beta) = \frac{e^{-x}}{2\sqrt{2x}}\left[\sum_{k=0}^{n-1}\frac{\Gamma(k+\frac{1}{2})}{k!(2x)^k}T^{(k)}(0,\phi,\alpha,\beta) + \tilde{R}_n(x,\phi,\alpha,\beta)\right], \tag{21}$$

where

$$T(u,\phi,\alpha,\beta) = \frac{2}{\sqrt{1+u^2}} f(2\,\mathrm{arcsinh}\,u,\phi,\alpha,\beta) \tag{22}$$

and $T^{(k)}(u,\phi,\alpha,\beta)$ *is the kth derivative of the function* $T(u,\phi,\alpha,\beta)$ *with respect to u.*

The remainder $\tilde{R}_n(x,\phi,\alpha,\beta)$ *satisfies a bound of the form*

$$|\tilde{R}_n(x,\phi,\alpha,\beta)| \leqslant \tilde{M}\frac{\Gamma(n+1/2)}{n!(2xd)^n} \tag{23}$$

for some positive constants $\tilde{M}$ *and d.*

Proof. It is easy to check that the imaginary part of the function $f(t,\phi,\alpha,\beta)$ given in (20) is an odd function of t whereas the real part is even. Then, removing the odd part of $f(t,\phi,\alpha,\beta)$ and performing the change of variable $\sinh(t/2) \equiv u$ in (19) we obtain:

$$\bar{I}(x,\phi,\alpha,\beta) = e^{-x}\int_0^\infty e^{-2xu^2}T(u,\phi,\alpha,\beta)\,du, \tag{24}$$

with $T(u,\phi,\alpha,\beta)$ given in (22) or, more explicitly:

$$T(u,\phi,\alpha,\beta) \equiv \frac{2}{\alpha}\frac{\sin(\mu\phi)[c(u)\cos(\mu(\beta-2\pi)) - \cos(\mu\phi)](1+u^2)^{-1/2}}{[c(u)\cos(\mu(\beta-2\pi)) - \cos(\mu\phi)]^2 + s^2(u)\sin^2(\mu(\beta-2\pi))} + \mathrm{sign}(\beta)\mathrm{sign}\left(\frac{1}{2} - \mathrm{Frac}\left(\frac{|\beta|}{2\alpha}\right)\right)\frac{(1+\delta_{\beta,0}+\delta_{\beta,\alpha}+\delta_{\beta,2\pi-\alpha})}{2\pi}\frac{\sin\theta}{u^2+\sin^2\theta},$$

θ given in (18) and

$$c(u) \equiv \frac{[u+\sqrt{u^2+1}]^{2\mu} + [u+\sqrt{u^2+1}]^{-2\mu}}{2},$$

$$s(u) \equiv \frac{[u+\sqrt{u^2+1}]^{2\mu} - [u+\sqrt{u^2+1}]^{-2\mu}}{2}.$$

In these formulas we have used the equality $\mathrm{arcsinh}\,u = \ln[u+\sqrt{u^2+1}]$ valid for $u \geqslant 1$ [1, Eq. (4.6.20)].

The function $T(u, \phi, \alpha, \beta)$ has a Taylor expansion at $u = 0$ for each $\phi \in (0, \alpha)$, $\beta \in [0, 2\pi)$ and $\alpha \in (0, 2\pi)$:

$$T(u, \phi, \alpha, \beta) = \sum_{k=0}^{n-1} \frac{T^{(k)}(0, \phi, \alpha, \beta)}{k!} u^k + T_n(u, \phi, \alpha, \beta), \tag{25}$$

where $T^{(k)}$ means the kth derivative of the function $T(u, \phi, \alpha, \beta)$ with respect to u and $T_n(u, \phi, \alpha, \beta)$ is the Taylor remainder. The points of singularity of $T(u, \phi, \alpha, \beta)$ are away from the positive real axis. Using the Cauchy formula for the remainder $T_n(u, \phi, \alpha, \beta)$, we see that

$$|T_n(u, \phi, \alpha, \beta)| \leqslant \tilde{M} \frac{u^n}{d^n}, \tag{26}$$

where $\tilde{M}$ is a bound for $T(u, \phi, \alpha, \beta)$ on the portion of the complex w-plane surrounding the positive real axis: $\{w \in \mathbb{C}, |w - u| < d, u \in \mathbb{R}^+\}$ and d represents the distance from the closest of the singularities of $T(u, \phi, \alpha, \beta)$ to the positive real axis. Introducing the expansion (25) in (24) we obtain that $\bar{I}(x, \phi, \alpha, \beta)$ has expansion (21) with

$$\tilde{R}_n(x, \phi, \alpha, \beta) \equiv 2\sqrt{2x} \int_0^\infty e^{-2xu^2} T_n(u, \phi, \alpha, \beta)\, du. \tag{27}$$

Introducing (26) in (27) we obtain (23). □

Theorem 1. *Let* $w \equiv 1/(2\varepsilon)$, $(r, \phi) \in \Omega^*$, $\beta \in [0, 2\pi)$ *and* $\alpha \in (0, 2\pi)$. *Then, the solution* $U_{\alpha,\beta}(r, \phi)$ *of problem* (P) *given in Proposition* 2 *reads*

$$U_{\alpha,\beta}(r, \phi) = U^0_{\alpha,\beta}(r, \phi) + \frac{e^{wr(\cos(\beta-\phi)-1)}}{2\sqrt{2wr}} U^1_{\alpha,\beta}(r, \phi), \tag{28}$$

where:

1. *If* $\beta = 0$:

$$U^0_{\alpha,0}(r, \phi) = 1 - \operatorname{erfc}\sqrt{wr(1 - \cos\phi)}. \tag{29}$$

2. *If* $0 < \beta < \alpha$:

$$U^0_{\alpha,\beta}(r, \phi) = \chi_{(\beta,\alpha]}(\phi) + \tfrac{1}{2}\delta_{\phi,\beta} + \tfrac{1}{2}\operatorname{sign}(\beta - \phi)\operatorname{erfc}\sqrt{wr(1 - \cos(\beta - \phi))}. \tag{30}$$

3. *If* $\beta = \alpha$:

$$U^0_{\alpha,\alpha}(r, \phi) = \operatorname{erfc}\sqrt{wr(1 - \cos(\alpha - \phi))}. \tag{31}$$

4. *If* $\alpha < \beta < \alpha + \pi$:

$$U^0_{\alpha,\beta}(r,\phi) = e^{wr\ \cos(\beta-\phi)}\left\{\sum_{k=1}^{[\frac{\beta+\phi}{2\alpha}]} e^{-wr\ \cos(\beta+\phi-2k\alpha)} - \sum_{k=1}^{[\frac{\beta-\phi}{2\alpha}]} e^{-wr\ \cos(\beta-\phi-2k\alpha)}\right.$$
$$\left. + \frac{1}{2}e^{-wr}\left(\delta_{\frac{\beta-\phi}{2\alpha},[\frac{\beta-\phi}{2\alpha}]} - \delta_{\frac{\beta+\phi}{2\alpha},[\frac{\beta+\phi}{2\alpha}]}\right)\right\}. \tag{32}$$

5. *If* $\alpha + \pi \leqslant \beta$:

$$U^0_{\alpha,\beta}(r,\phi) = e^{wr\ \cos(\beta-\phi)}\left\{\sum_{k=1}^{[\frac{2\pi+\phi-\beta}{2\alpha}]} e^{-wr\ \cos(-\phi+\beta+2k\alpha)} - \sum_{k=0}^{[\frac{2\pi-\phi-\beta}{2\alpha}]} e^{-wr\ \cos(\phi+\beta+2k\alpha)}\right.$$
$$\left. + \frac{1}{2}e^{-wr}\left(\delta_{\frac{2\pi-\phi-\beta}{2\alpha},[\frac{2\pi-\phi-\beta}{2\alpha}]} - \delta_{\frac{2\pi+\phi-\beta}{2\alpha},[\frac{2\pi+\phi-\beta}{2\alpha}]}\right)\right\} + 1. \tag{33}$$

The function $U^1_{\alpha,\beta}(r,\phi)$ *has an asymptotic expansion in powers of* w^{-1}:

$$U^1_{\alpha,\beta}(r,\phi) = \sum_{k=0}^{n-1} \frac{\Gamma(k+1/2)}{k!}\frac{T^{(k)}(0,\phi,\alpha,\beta)}{(2wr)^k} + R_n(wr,\phi,\alpha,\beta), \tag{34}$$

where the coefficients $T^{(k)}(0,\phi,\alpha,\beta)$ *are given in Lemma* 3 *and are regular functions of r and* ϕ *for* $(r,\phi) \in \Omega^*$.

The remainder $R_n(wr,\phi,\alpha,\beta)$ *satisfies a bound of the form*

$$|R_n(wr,\phi,\alpha,\beta)| \leqslant M\frac{\Gamma(n+1/2)}{n!(2wdr)^n}, \tag{35}$$

where M and d are positive constants.

Proof. For large w and fixed r, the asymptotic features of the integral $I_{\alpha,\beta}(r,\phi)$ defined in (5) are: (i) there is a saddle point at $t = 0$. (ii) The poles are situated at $t^1_k = i(\beta - \phi + 2k\alpha)$ and $t^2_k = i(\beta + \phi + 2k\alpha)$, $k \in \mathbb{Z}$. Then, the saddle point coalesce with t^1_k when $\phi \to \beta + 2k\alpha$ or with t^2_k when $\phi \to -(\beta + 2k\alpha)$. Uniform asymptotic expansion of this kind of integrals are obtained by using the error function as the basic approximant [25, Chapter 7, Section 2]. Therefore, we need to identify the poles in the integrand of $I_{\alpha,\beta}(r,\phi)$ which are closest to the point $t = 0$ (to the real axis). We distinguish several cases:

Case 1: $\beta = 0$.

In this case two poles, $t^1_0 = -i\phi$ and $t^2_0 = i\phi$, touch the real axis when ϕ runs from 0 to α. Therefore, we split off both poles from the integrand of $I_{\alpha,0}(r,\phi)$ if we use (18) and (20):

$$\theta = -\phi, \quad g(t,\phi,\alpha,0) = h(t,\phi) - h(t,-\phi) + f(t,\phi,\alpha,0),$$

where the functions f, g and h are given in Definition 2.

Using the complementary error function representation [20]

$$\mathrm{e}^{-r\cos\alpha}\,\mathrm{erfc}\left(\sqrt{2r}\,\sin\frac{\alpha}{2}\right)=\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{e}^{-r\cosh t}\frac{\mathrm{d}t}{\sinh\frac{1}{2}(t-\mathrm{i}\alpha)},\qquad 0<\alpha<2\pi, \tag{36}$$

we obtain that the integral $I_{\alpha,0}(r,\phi)$ reads

$$I_{\alpha,0}(r,\phi)=\mathrm{erfc}\sqrt{wr(1-\cos\phi)}+\mathrm{e}^{wr\cos(\beta-\phi)}\bar{I}(wr,\phi,\alpha,0), \tag{37}$$

where $\bar{I}(wr,\phi,\alpha,0)$ is defined in (19).

Therefore, from (8) we obtain (28) with $U^0_{\alpha,0}(r,\phi)$ given in (29) and $U^1_{\alpha,0}(r,\phi)=-2\sqrt{2wr}\mathrm{e}^{wr}\bar{I}(wr,\phi,\alpha,0)$.

Case 2: $0<\beta<\alpha$.

In this case, the pole $t^1_0=\mathrm{i}(\beta-\phi)$ is the only one which crosses the real axis when ϕ runs from 0 to α. Therefore, we split off the pole of the integrand at t^1_0 if we use (18) and (20):

$$\theta=\beta-\phi,\qquad g(t,\phi,\alpha,\beta)=-h(t,\beta-\phi)+f(t,\phi,\alpha,\beta).$$

Using (36) we obtain that the integral $I_{\alpha,\beta}(r,\phi)$ equals

$$I_{\alpha,\beta}(r,\phi)=-\tfrac{1}{2}\,\mathrm{sign}(\beta-\phi)\mathrm{erfc}\sqrt{wr(1-\cos(\beta-\phi))}+\mathrm{e}^{wr\cos(\beta-\phi)}\bar{I}(wr,\phi,\alpha,\beta). \tag{38}$$

Therefore, from (9) we obtain (28) with $U^0_{\alpha,\beta}(r,\phi)$ given in (30) and $U^1_{\alpha,\beta}(r,\phi)=-2\sqrt{2wr}\mathrm{e}^{wr}\bar{I}(wr,\phi,\alpha,\beta)$.

Case 3: $\beta=\alpha$.

In this case both poles, $t^1_0=\mathrm{i}(\alpha-\phi)$ and $t^2_{-1}=\mathrm{i}(\phi-\alpha)$, touch the real axis when ϕ runs from 0 to α. Therefore, we split off these two poles from the integrand if we use (18) and (20):

$$\theta=\alpha-\phi,\qquad g(t,\phi,\alpha,\alpha)=h(t,\phi-\alpha)-h(t,\alpha-\phi)+f(t,\phi,\alpha,\alpha).$$

From (36) we obtain

$$I_{\alpha,\alpha}(r,\phi)=-\mathrm{erfc}\sqrt{wr(1-\cos(\alpha-\phi))}+\mathrm{e}^{wr\cos(\beta-\phi)}\bar{I}(wr,\phi,\alpha,\alpha). \tag{39}$$

Therefore, from (10) we obtain (28) with $U^0_{\alpha,\alpha}(r,\phi)$ given in (31) and $U^1_{\alpha,\alpha}(r,\phi)=-2\sqrt{2wr}\mathrm{e}^{wr}\bar{I}(wr,\phi,\alpha,\alpha)$.

Case 4: $\alpha<\beta<\alpha+\pi$.

As in the preceding cases, we look for the pole t^1_k or t^2_k that crosses the real axis when ϕ runs from 0 to α. For that purpose we choose an integer n satisfying:

$$\frac{\beta}{2\alpha}-1<n<\frac{\beta}{2\alpha}.$$

We distinguish two cases:

Case 4.1: If $\mathrm{Frac}[\frac{\beta}{2\alpha})\leqslant\frac{1}{2}$, just the pole $t^1_n=\mathrm{i}(\beta-\phi-2n\alpha)$ crosses the real axis when ϕ runs from 0 to α. Therefore, we split off the pole of the integrand at t^1_n if we use (18) and (20):

$$\theta=\beta-\phi-2n\alpha,\qquad g(t,\phi,\alpha,\beta)=-h(t,\beta-\phi-2n\alpha)+f(t,\phi,\alpha,\beta).$$

Using (36), we obtain that the integral $I_{\alpha,\beta}(r,\phi)$ can be written as

$$I_{\alpha,\beta}(r,\phi)=\mathrm{e}^{wr\,\cos(\beta-\phi)}[\bar{I}(wr,\phi,\alpha,\beta)+R(wr,\phi,\alpha,\beta)], \tag{40}$$

where

$$\begin{aligned}R(wr,\phi,\alpha,\beta)\equiv\;&-\tfrac{1}{2}\mathrm{e}^{-wr\,\cos(\beta-\phi-2n\alpha)}\,\mathrm{sign}(\beta-\phi-2n\alpha)\\&\times\,\mathrm{erfc}\sqrt{wr(1-\cos(\beta-\phi-2n\alpha))}.\end{aligned} \tag{41}$$

Therefore, from (11) we obtain (28) with $U^0_{\alpha,\beta}(r,\phi)$ given in (32) and

$$U^1_{\alpha,\beta}(r,\phi)=-2\sqrt{2wr}\,\mathrm{e}^{wr}[\bar{I}(wr,\phi,\alpha,\beta)+R(wr,\phi,\alpha,\beta)].$$

Case 4.2: If $\mathrm{Frac}(\frac{\beta}{2\alpha})>\frac{1}{2}$, just the pole $t^2_{n+1}=\mathrm{i}(\beta+\phi-2(n+1)\alpha)$ crosses the real axis when ϕ runs from 0 to α. Therefore, we split off this pole from the integrand if we use (18) and (20):

$$\theta=\beta+\phi-2(n+1)\alpha,\qquad g(t,\phi,\alpha,\beta)=h(t,\beta+\phi-2(n+1)\alpha)+f(t,\phi,\alpha,\beta).$$

The application of (36) yields

$$I_{\alpha,\beta}(r,\phi)=\mathrm{e}^{wr\,\cos(\beta-\phi)}[\bar{I}(wr,\phi,\alpha,\beta)+R(wr,\phi,\alpha,\beta)], \tag{42}$$

where

$$\begin{aligned}R(wr,\phi,\alpha,\beta)\equiv\;&-\tfrac{1}{2}\mathrm{e}^{-wr\,\cos(\beta+\phi-2(n+1)\alpha)}\,\mathrm{sign}(\beta+\phi-2(n+1)\alpha)\\&\times\,\mathrm{erfc}\sqrt{wr(1-\cos(\beta+\phi-2(n+1)\alpha))}.\end{aligned} \tag{43}$$

Therefore, from (11) we obtain (28) with $U^0_{\alpha,\beta}(r,\phi)$ given in (32) and

$$U^1_{\alpha,\beta}(r,\phi)=-2\sqrt{2wr}\,\mathrm{e}^{wr}[\bar{I}(wr,\phi,\alpha,\beta)+R(wr,\phi,\alpha,\beta)].$$

Case 5: $\alpha+\pi\leqslant\beta$. In this case, instead of $I_{\alpha,\beta}(r,\phi)$, we have to analyze $I_{\alpha,\beta-2\pi}(r,\phi)$. Then, the poles of the integrand of this integral are situated at $t^1_k=\mathrm{i}(\beta-2\pi-\phi+2k\alpha)$ and $t^2_k=\mathrm{i}(\beta-2\pi+\phi+2k\alpha)$, $k\in\mathbb{Z}$. The saddle point $t=0$ coalesce with t^1_k when $\phi\to\beta-2\pi+2k\alpha$ or with t^2_k when $\phi\to-(\beta-2\pi+2k\alpha)$. We divide the study of this case in three subcases: $\alpha>2\pi-\beta$, $\alpha=2\pi-\beta$ and $\alpha<2\pi-\beta$.

Case 5.1: $\alpha+\pi\leqslant\beta$ and $\alpha>2\pi-\beta$.

In this case just the pole $t^2_0=\mathrm{i}(\beta-2\pi+\phi)$ crosses the real axis when ϕ runs from 0 to α. Therefore, we split off this pole from the integrand if we use (18) and (20) with β replaced by $\beta-2\pi$:

$$\theta=\beta-2\pi+\phi,\qquad g(t,\phi,\alpha,\beta-2\pi)=h(t,\beta-2\pi+\phi)+f(t,\phi,\alpha,\beta-2\pi).$$

Using (36) we get

$$I_{\alpha,\beta-2\pi}(r,\phi)=\mathrm{e}^{wr\,\cos(\beta-\phi)}[\bar{I}(wr,\phi,\alpha,\beta-2\pi)+R(wr,\phi,\alpha,\beta)], \tag{44}$$

where

$$R(wr,\phi,\alpha,\beta)\equiv\frac{\mathrm{e}^{-wr\,\cos(\beta+\phi)}}{2}\,\mathrm{sign}(\beta-2\pi+\phi)\mathrm{erfc}\sqrt{wr(1-\cos(\beta+\phi))}. \tag{45}$$

Therefore, from (12) we obtain (28) with $U^0_{\alpha,\beta}(r,\phi)$ given in (33) and

$$U^1_{\alpha,\beta}(r,\phi) = -2\sqrt{2wr}\mathrm{e}^{wr}[\bar{I}(wr,\phi,\alpha,\beta-2\pi) + R(wr,\phi,\alpha,\beta)].$$

Case 5.2: $\alpha+\pi \leqslant \beta$ and $\alpha < 2\pi - \beta$.

We look for the pole t^1_k or t^2_k that crosses the real axis when ϕ runs from 0 to α. For that purpose we choose an integer n satisfying:

$$\frac{2\pi-\beta}{2\alpha} - 1 < n < \frac{2\pi-\beta}{2\alpha}.$$

We distinguish two cases:

(a) If $\mathrm{Frac}[(2\pi-\beta)/2\alpha] \leqslant \frac{1}{2}$, just the pole $t^2_n = \mathrm{i}(\beta - 2\pi + \phi + 2n\alpha)$ crosses the real axis when ϕ changes from 0 to α. Therefore, we split off this pole from the integrand if we use (18) and (20) with β replaced by $\beta - 2\pi$:

$$\theta = \beta - 2\pi + \phi + 2n\alpha, \quad g(t,\phi,\alpha,\beta-2\pi) = h(t,\beta+\phi-2\pi+2n\alpha) + f(t,\phi,\alpha,\beta-2\pi).$$

Using (36), we obtain that the integral $I_{\alpha,\beta-2\pi}(r,\phi)$ reads

$$I_{\alpha,\beta-2\pi}(r,\phi) = \mathrm{e}^{wr\,\cos(\beta-\phi)}[\bar{I}(wr,\phi,\alpha,\beta-2\pi) + R(wr,\phi,\alpha,\beta)], \tag{46}$$

where

$$R(wr,\phi,\alpha,\beta) \equiv -\tfrac{1}{2}\mathrm{e}^{-wr\,\cos(\beta+\phi+2n\alpha)}\,\mathrm{sign}(\beta-2\pi+\phi+2n\alpha) \times \mathrm{erfc}\sqrt{wr(1-\cos(\beta+\phi+2n\alpha))}. \tag{47}$$

Therefore, from (12) we obtain (28) with $U^0_{\alpha,\beta}(r,\phi)$ given in (33) and

$$U^1_{\alpha,\beta}(r,\phi) = -2\sqrt{2wr}\mathrm{e}^{wr}[\bar{I}(wr,\phi,\alpha,\beta-2\pi) + R(wr,\phi,\alpha,\beta)].$$

(b) If $\mathrm{Frac}[(2\pi-\beta)/2\alpha] > \frac{1}{2}$, just the pole $t^1_{n+1} = \mathrm{i}(\beta - 2\pi - \phi + 2(n+1)\alpha)$ crosses the real axis when ϕ runs from 0 to α. Therefore, we split off this pole from the integrand if we use (18) and (20) with β replaced by $\beta - 2\pi$:

$$\theta = \beta - 2\pi - \phi + 2(n+1)\alpha,\ g(t,\phi,\alpha,\beta-2\pi) = -h(t,\beta-2\pi-\phi+2(n+1)\alpha) + f(t,\phi,\alpha,\beta-2\pi).$$

Using the complementary error function representation (36), we obtain that the integral $I_{\alpha,\beta-2\pi}(r,\phi)$ equals

$$I_{\alpha,\beta-2\pi}(r,\phi) = \mathrm{e}^{wr\,\cos(\beta-\phi)}[\bar{I}(wr,\phi,\alpha,\beta-2\pi) + R(wr,\phi,\alpha,\beta)], \tag{48}$$

where

$$R(wr,\phi,\alpha,\beta) \equiv -\tfrac{1}{2}\mathrm{e}^{-wr\,\cos(\beta-\phi+2(n+1)\alpha)}\,\mathrm{sign}(\beta-2\pi-\phi+2(n+1)\alpha) \times \mathrm{erfc}\sqrt{wr(1-\cos(\beta-\phi+2(n+1)\alpha))}. \tag{49}$$

Therefore, from (12) we obtain (28) with $U^0_{\alpha,\beta}(r,\phi)$ given in (33) and

$$U^1_{\alpha,\beta}(r,\phi) = -2\sqrt{2wr}\mathrm{e}^{wr}[\bar{I}(wr,\phi,\alpha,\beta-2\pi) + R(wr,\phi,\alpha,\beta)].$$

Case 5.3: $\alpha+\pi\leqslant\beta$ and $\alpha=2\pi-\beta$.

In this case both poles, $t_1^1=\mathrm{i}(\alpha-\phi)$ and $t_0^2=\mathrm{i}(\phi-\alpha)$, touch the real axe when ϕ runs from 0 to α. Therefore, we split off these poles from the integrand if we use (18) and (20) with β replaced by $\beta-2\pi$:

$$\theta=\phi-\alpha, \quad g(t,\phi,\alpha,-\alpha)=h(t,\phi-\alpha)-h(t,\alpha-\phi)+f(t,\phi,\alpha,-\alpha).$$

The application of the complementary error function representation (36) gives us that

$$I_{\alpha,-\alpha}(r,\phi)=\mathrm{e}^{wr\,\cos(\beta-\phi)}[\bar{I}(wr,\phi,\alpha,-\alpha)+R(wr,\phi,\alpha,-\alpha)], \tag{50}$$

where

$$R(wr,\phi,\alpha,-\alpha)\equiv-\mathrm{e}^{-wr\,\cos(\alpha-\phi)}\,\mathrm{sign}(\alpha-\phi)\mathrm{erfc}\sqrt{wr(1-\cos(\alpha-\phi))}. \tag{51}$$

Therefore, from (12) we obtain (28) with $U^0_{\alpha,\beta}(r,\phi)$ given in (33) and

$$U^1_{\alpha,\beta}(r,\phi)=-2\sqrt{2wr}\mathrm{e}^{wr}[\bar{I}(wr,\phi,\alpha,-\alpha)+R(wr,\phi,\alpha,-\alpha)]. \tag{52}$$

From Lemma 3, the function $\bar{I}(wr,\phi,\alpha,\beta)$ defined in (19) has the asymptotic expansion (21) for large w and bounded $r\geqslant r_0>0$. Therefore, formula (34) holds with $R_n(wr,\phi,\alpha,\beta)=\tilde{R}_n(wr,\phi,\alpha,\beta)$ if $\alpha\geqslant\beta$ and $R_n(wr,\phi,\alpha,\beta)=\tilde{R}_n(wr,\phi,\alpha,\beta)+R(wr,\phi,\alpha,\beta)$ if $\alpha<\beta$, with $\tilde{R}_n(wr,\phi,\alpha,\beta)$ given in (27) and $R(wr,\phi,\alpha,\beta)$ given in (41), (43), (45), (47), (49) or (51). Using the asymptotic behaviour of the complementary error function [1, Eq. (7.1.23)] we see that $R(wr,\phi,\alpha,\beta)=\mathcal{O}(\mathrm{e}^{-\eta wr})$ with $\eta>0$ for the given valuesof β and α in (41), (43), (45), (47), (49) or (51). Therefore, using the bound (23) we obtain (35). The exponentially small bound for $R(wr,\phi,\alpha,\beta)$ is included in the constant M in (35). □

Remark 3. It was pointed out in Remark 2 that the solution $U_{\alpha,\beta}$ of problem (P) has the structure given in (15). In cases 1,2 and 3 of the above theorem, the integral $I_{\alpha,\beta}$ in the right-hand side of (15) is asymptotically equivalent to the complementary error function appearing in the right-hand side of (37), (38) and (39), respectively. Then, the basic approximant $U^0_{\alpha,\beta}$ given in (29), (30) and (31) equals the sum of the term $\Xi_{\alpha,\beta}$ of Eq. (15) plus that complementary error function. In the cases 4 and 5 of the preceding theorem, the integral $I_{\alpha,\beta}$ (or the integral $I_{\alpha,\beta-2\pi}$) is asymptotically irrelevant. Then, the basic approximant $U^0_{\alpha,\beta}$ given in (32) and (33) is just the term $\Xi_{\alpha,\beta}$ given in (15).

Remark 4. From (28), (34) and (35) we see that $U_{\alpha,\beta}(r,\phi)=U^0_{\alpha,\beta}(r,\phi)+\mathcal{O}(\sqrt{\varepsilon})$ as $\varepsilon\to0^+$ away from the point $r=0$. Then, the first-order approximation to the solution of (P) is a linear combination of error functions and elementary functions. When $\overrightarrow{v}$ is inside the sector, the error function in (30) exhibits an interior layer of width $\mathcal{O}(\sqrt{\varepsilon})$ and parabolic level lines of equation $r(1-\cos(\beta-\phi))=$ constant. When $\overrightarrow{v}$ is parallel to one side of the sector, the error functions in (29) and (31) exhibit boundary layers of width $\mathcal{O}(\sqrt{\varepsilon})$ and the same level lines. When $\overrightarrow{v}$ is not in the sector, the exponential functions in (32) and (33) exhibit boundary layers of width $\mathcal{O}(\varepsilon)$ (see Fig. 3).

4. Asymptotic expansion of $U_{\alpha,\beta}(r,\phi)$ near the discontinuity

The asymptotic expansion (34) breaks down when $r\to0^+$. Then, Theorem 1 does not offer a good approximation. The asymptotic approximation of $U_{\alpha,\beta}(r,\phi)$ near the point $r=0$ requires a completely different analysis which is given in the following theorem. Let us introduce first the following definition.

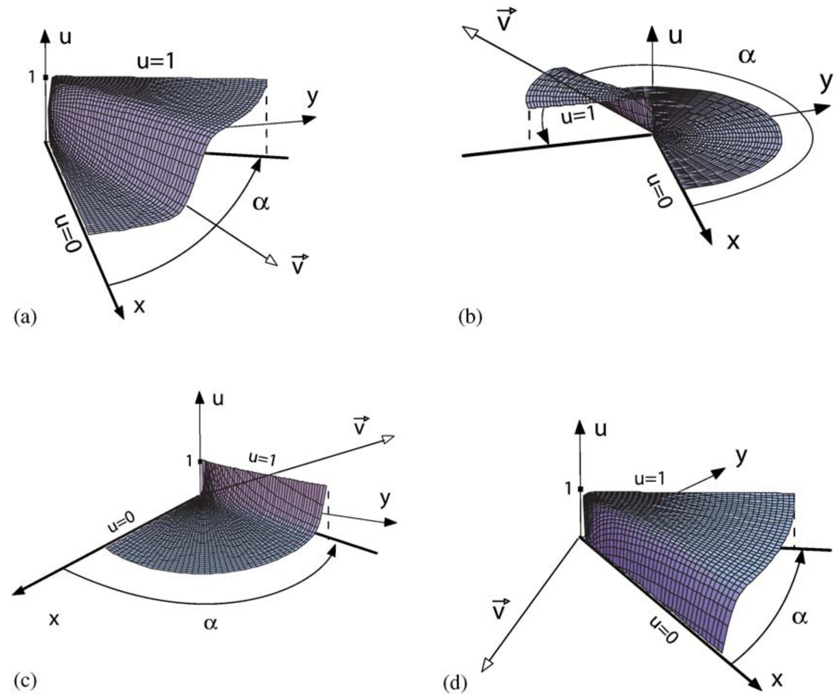

Fig. 3. Graphs of the first-order approximation, $U^0_{\alpha,\beta}(r,\phi)$, to the solution of problem (P) for different values of α and β and $\varepsilon = 0.1$. The convection vector $\vec{v}$ "drags" the discontinuity of the boundary condition at $r = 0$ originating a parabolic layer of size $\mathcal{O}(\sqrt{\varepsilon})$ along $\vec{v}$ if it points into the sector. If $\vec{v}$ points out of the sector, it originates boundary layers of size $\mathcal{O}(\varepsilon)$ along the outflow boundary of the sector. (a) $\beta = \pi/4$, $\alpha = \pi/3$ (case 2); (b) $\beta = 5\pi/4$, $\alpha = 3\pi/2$ (case 2); (c) $\beta = 3\pi/4$, $\alpha = 5\pi/12$ (case 4); (d) $\beta = 7\pi/4$, $\alpha = \pi/4$ (case 5).

Definition 3. We define the function

$$\begin{aligned} U^3_{\alpha,\beta}(r,\phi) \equiv \frac{\alpha}{wr} \Bigg\{ & \Bigg[\sum_{k=1}^{[\frac{\beta+\phi}{2\alpha}]} e^{wr(1-\cos(\beta+\phi-2k\alpha))} - \sum_{k=1}^{[\frac{\beta-\phi}{2\alpha}]} e^{wr(1-\cos(\beta-\phi-2k\alpha))} \\ & + \frac{1}{2}\left(\delta_{\frac{\beta-\phi}{2\alpha},[\frac{\beta-\phi}{2\alpha}]} - \delta_{\frac{\beta+\phi}{2\alpha},[\frac{\beta+\phi}{2\alpha}]}\right) - e^{wr} K(\phi,\alpha,\beta) \Bigg] \chi_{(\alpha,\alpha+\pi)}(\beta) \\ & + \Bigg[\sum_{k=1}^{[\frac{2\pi+\phi-\beta}{2\alpha}]} e^{wr(1-\cos(-\phi+\beta+2k\alpha))} - \sum_{k=0}^{[\frac{2\pi-\phi-\beta}{2\alpha}]} e^{wr(1-\cos(\phi+\beta+2k\alpha))} + e^{wr} \\ & + \frac{1}{2}\left(\delta_{\frac{2\pi-\phi-\beta}{2\alpha},[\frac{2\pi-\phi-\beta}{2\alpha}]} - \delta_{\frac{2\pi+\phi-\beta}{2\alpha},[\frac{2\pi+\phi-\beta}{2\alpha}]}\right) - e^{wr} K(\phi,\alpha,2\pi-\beta) \Bigg] \chi_{[\alpha+\pi,2\pi)}(\beta) \Bigg\}, \end{aligned}$$

where

$$K(\phi,\alpha,\beta)\equiv\begin{cases}1 & \text{if } \left(\dfrac{\beta-\phi}{2\alpha},\dfrac{\beta+\phi}{2\alpha}\right)\cap\mathbb{N}\neq\emptyset,\\ \dfrac{1}{2} & \text{if } \dfrac{\beta-\phi}{2\alpha}\in\mathbb{N} \text{ or } \dfrac{\beta+\phi}{2\alpha}\in\mathbb{N},\\ 0 & \text{if } \left(\dfrac{\beta-\phi}{2\alpha},\dfrac{\beta+\phi}{2\alpha}\right)\cap\mathbb{N}=\emptyset.\end{cases}$$

Observe that $U^3_{\alpha,\beta}(r,\phi)=\mathcal{O}(1)$ as $wr\to 0^+$.

Theorem 2. *Write* $w\equiv 1/(2\varepsilon)$. *Then, for* $(r,\phi)\in\bar{\Omega}$, $\alpha\in(0,2\pi)$ *and* $\beta\in[0,2\pi)$, *the solution* $U_{\alpha,\beta}(r,\phi)$ *of problem* (P) *reads*

$$U_{\alpha,\beta}(r,\phi)=\frac{\phi}{\alpha}+\frac{wr}{\alpha}\mathrm{e}^{wr(\cos(\beta-\phi)-1)}U^1_{\alpha,\beta}(r,\phi), \tag{53}$$

where $U^1_{\alpha,\beta}(r,\phi)=\mathcal{O}(1)$ *as* $wr\to 0^+$. *More precisely*: $U^1_{\alpha,\beta}=U^2_{\alpha,\beta}+U^3_{\alpha,\beta}$ *with* $U^3_{\alpha,\beta}$ *given in Definition* 3 *and, for* $n=1,2,3,\ldots$, $U^2_{\alpha,\beta}(r,\phi)$ *has a convergent expansion in powers of* wr:

$$\begin{aligned}U^2_{\alpha,\beta}(r,\phi)\equiv{}&\frac{T_0(\phi,\beta)}{rw}[\mathrm{e}^{wr(1-\cos(\beta-\phi))}-1]\\&-\sum_{k=1}^{n-1}\frac{(-1)^k}{k!}[T_k(\phi,\beta)-V_k(\phi,\beta)\log(rw)](rw)^{k-1}+R_n(wr,\phi,\alpha,\beta).\end{aligned} \tag{54}$$

The coefficients $T_k(\phi,\beta)$ *and* $V_k(\phi,\beta)$ *are regular functions of* ϕ *and* β *and the remainder term* $R_n(wr,\phi,\alpha,\beta)$ *has a bound of the form*

$$|R_n(wr,\phi,\alpha,\beta)|\leqslant\frac{M}{d^n n!}[n(2+d)+|\log(rw)|](rw)^{n-1} \tag{55}$$

for some positive constants M *and* d.

Proof. Since the imaginary part of the integrand in (5) is an odd function of t and the real part is even, removing the odd part and performing the change of variable $\cosh t=u+1$ in definition (5) of $I_{\alpha,\beta}(r,\phi)$ we have

$$I_{\alpha,\beta}(r,\phi)=\frac{\mathrm{e}^{wr(\cos(\beta-\phi)-1)}}{\alpha}\int_0^\infty \mathrm{e}^{-rwu}f(u,\phi,\alpha,\beta)\,\mathrm{d}u, \tag{56}$$

with

$$f(u,\phi,\alpha,\beta)\equiv\frac{1}{\sqrt{u(u+2)}}\frac{\sin(\mu\phi)[\cosh(\mu t)\cos(\mu\beta)-\cos(\mu\phi)]}{[\cosh(\mu t)\cos(\mu\beta)-\cos(\mu\phi)]^2+\sinh^2(\mu t)\sin^2(\mu\beta)}$$

and $t=\operatorname{arccosh}(u+1)$, $t>0$.

Using the formula $\operatorname{arccosh} u=\ln[u+\sqrt{u^2-1}]$, $u\geqslant 1$ [1, Eq. (4.6.21)], we can write

$$f(u,\phi,\alpha,\beta)=\frac{1}{\sqrt{u(u+2)}}\frac{\sin(\mu\phi)[\tilde{c}(u)\cos(\mu\beta)-\cos(\mu\phi)]}{[\tilde{c}(u)\cos(\mu\beta)-\cos\mu\phi]^2+\tilde{s}^2(u)\sin^2(\mu\beta)},$$

with

$$\tilde{c}(u) \equiv \frac{[1+u+\sqrt{u(u+2)}]^{\mu}+[1+u+\sqrt{u(u+2)}]^{-\mu}}{2},$$
$$\tilde{s}(u) \equiv \frac{[1+u+\sqrt{u(u+2)}]^{\mu}-[1+u+\sqrt{u(u+2)}]^{-\mu}}{2}.$$

From this representation we see that $f(u,\phi,\alpha,\beta)$ has an expansion in inverse powers of u valid for each $\phi \in [0,\alpha]$ and $\beta \in [0,2\pi)$:

$$f(u,\phi,\alpha,\beta) = \sum_{k=0}^{n-1} \frac{V_k(\phi,\beta)}{u^{\mu+k+1}} + f_n(u,\phi,\beta),$$

where $f_n(u,\phi,\alpha,\beta) = \mathcal{O}(u^{-\mu-n-1})$ as $u \to \infty$ uniformly in $\phi \in [0,\alpha]$. The coefficients $V_k(\phi,\beta)$ are the Taylor coefficients of the expansion of the function $u^{-\mu} f(u^{-1},\phi,\alpha,\beta)$ at $u=0$ ($V_0=0$). Applying [25, Chapter 6, Theorem 13(ii)] to the integral in the right-hand side of (56) we obtain

$$I_{\alpha,\beta}(r,\phi) = \frac{e^{wr(\cos(\beta-\phi)-1)}}{\alpha}\left[\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}[T_k - V_k \log(wr)](wr)^k + wrR_n(wr,\phi,\alpha,\beta)\right], \tag{57}$$

with the following expressions for the coefficients T_k and the remainder $R_n(wr,\phi,\alpha,\beta)$. Coefficients T_k read

$$T_k \equiv V_k\psi(k+1) + \lim_{s\to k+1}\left\{M[f;s] + \frac{V_k}{s-k-1}\right\},$$

where $M[f;s]$ denotes the Mellin transform of f at s, $\int_0^\infty u^{s-1} f(u,\phi,\beta)\,du$, or its analytic continuation as a function of s. On the other hand, the remainder $R_n(wr,\phi,\alpha,\beta)$ reads

$$R_n(wr,\phi,\alpha,\beta) \equiv (wr)^{n-1}\int_0^\infty f_{n,n}(t)e^{-wrt}\,dt, \tag{58}$$

with

$$f_{n,n}(t) \equiv \frac{(-1)^n}{(n-1)!}\int_t^\infty (u-t)^{n-1} f_n(u,\phi,\alpha,\beta)\,du. \tag{59}$$

In particular, the coefficient T_0, which gives the dominant term of the expansion, reads

$$\begin{aligned} T_0 = M[f;1] &= \int_0^\infty f(u,\phi,\beta)\,du \\ &= \int_0^\infty \frac{\sin(\mu\phi)[\cosh(\mu t)\cos(\mu\beta)-\cos(\mu\phi)]\,dt}{[\cosh(\mu t)\cos(\mu\beta)-\cos(\mu\phi)]^2+\sinh^2(\mu t)\sin^2(\mu\beta)} \\ &= -\frac{1}{\mu}\arctan\left[\frac{\sin(\mu\phi)\sinh(\mu t)}{\cos(\mu\phi)\cosh(\mu t)-\cos(\mu\beta)}\right]_0^\infty. \end{aligned}$$

From here we see that the value of T_0 depends on the relative value of α and β:

Case 1: If $0 \leqslant \beta \leqslant \alpha$.

$$T_0 = -\phi + \alpha \begin{cases} 1 & \text{if } \phi > \beta, \\ \frac{1}{2} & \text{if } \phi = \beta, \\ 0 & \text{if } \phi < \beta. \end{cases}$$

Case 2: $\alpha < \beta < \alpha + \pi$.

$$T_0 = \alpha K(\phi, \alpha, \beta) - \phi.$$

Case 3: $\alpha + \pi \leqslant \beta$.

$$T_0 = \alpha K(\phi, \alpha, 2\pi - \beta) - \phi.$$

Using these formulas, introducing (57) in (8)–(12) and rearranging terms we obtain (53).

Finally, we obtain the error bound (55) which shows that expansion (54) is not only asymptotic but convergent. From the Taylor formula for the remainder,

$$f_n(u, \phi, \alpha, \beta) \equiv \frac{h^{(n)}(\xi)}{n! u^{n+\mu+1}}$$

for certain $\xi \in (0, u^{-1})$, $u > 0$, where $h(u) \equiv u^{-\mu} f(u^{-1}, \phi, \alpha, \beta)$. The singularities of $h(u)$ are away from the positive real axis. Therefore, using the Cauchy formula for the derivative $h^{(n)}(\xi)$, we see that

$$|f_n(u, \phi, \alpha, \beta)| \leqslant \frac{M}{d^n u^{n+1+\mu}}, \quad |f_n(u, \phi, \alpha, \beta)| \leqslant \frac{M}{d^{n-1} u^{n+\mu}}, \tag{60}$$

where M is a bound for $h(w)$ on the portion of the complex w-plane surrounding the positive real axis: $\{w \in \mathbb{C}, |w - u| < d, u \in \mathbb{R}^+\}$ where d represents the distance from the closest of the singularities of $h(u)$ to the positive real axis.

Introducing these bounds in (59) we obtain [14, Eq. (2.23)],

$$|f_{n,n}(t)| \leqslant \frac{M}{d^n (n-1)!}(1 - d \log t) \quad \forall t \in [0, 1]$$

and introducing the first bound of (60) in (59) we have [14, Eq. (2.24)],

$$|f_{n,n}(t)| \leqslant \frac{M}{n! d^n t} \quad \forall t \in [0, \infty).$$

We divide the integral in the right-hand side of (58) at the point $t = 1$ and use the first bound of $f_{n,n}(t)$ in the interval $[0, 1]$ and the second one in the interval $[1, \infty)$. The bound (55) follows after simple computations. □

Remark 5. From (53) we see that

$$U_{\alpha,\beta}(r, \phi) = \frac{\phi}{\alpha} + \mathcal{O}\left(\frac{r}{\varepsilon}\right) \quad \text{when } \frac{r}{\varepsilon} \to 0^+.$$

The discontinuity of the inflow boundary condition is smoothed inside the domain by a linear function of the polar angle ϕ.

5. Conclusions

The singularly perturbed convection–diffusion problem (P) has been defined on a sector by means of discontinuous Dirichlet boundary conditions with a discontinuity located on the corner of the domain. We have obtained in Proposition 2 an integral representation of the unique solution of problem (P) susceptible of an asymptotic analysis. Then, two complementary asymptotic expansions of the solution have been obtained in Theorems 1 and 2. One expansion is valid in the singular limit $\varepsilon \to 0^+$ and away from the discontinuity $r = 0$. The other one is valid near the discontinuity $r = 0$ for $\varepsilon \geqslant \varepsilon_0 > 0$.

These two asymptotic expansions are derived from two quite different asymptotic procedures. While the asymptotic expansion in the singular limit is obtained from a classical uniform method, the asymptotic expansion near the discontinuity is derived by means of a distributional approach. Two quite different asymptotic principles match into the same problem.

The asymptotic expansion in the singular limit shows that the main contribution from the data's discontinuities to the shape of the solution on the singular layers is contained in a certain combination of error functions, exponential functions and step functions. This combination is necessary to approach the behaviour of the solution on the interior layer of width $\mathcal{O}(\sqrt{\varepsilon})$ or on the boundary layer of width $\mathcal{O}(\varepsilon)$. On the other hand, the asymptotic expansion near the discontinuities shows that the discontinuity on the boundary is smoothed inside the domain by means of a simply linear function of the polar angle.

We want to do emphasis on the simultaneous dependence of the solution of problem (P) with the singular parameter ε and with the distance to the origin (the discontinuity point of the boundary data). The solution $U_{\alpha,\beta}$ depends on ε and the distance r to the origin through the quotient r/ε (see Proposition 2). This is why the expansion for small ε (large w) in Theorem 1 does not hold near the origin. And conversely, the expansion near the origin (small r) in Theorem 2 only holds when the distance r is smaller than ε.

We suspect that, as in the problem analyzed here, the error function plays a fundamental role in the approximation of the solution of many singularly perturbed convection–diffusion problems with discontinuities in the boundary conditions (problems defined over more general domains and by more general coefficients). This will be the subject of further investigations. Then, the asymptotic expansions of the solution of problem (P) presented here may give a qualitative idea about the behaviour of the solutions of more realistic convection–diffusion problems with discontinuous Dirichlet conditions. This should help in the development of suitable numerical methods for those problems [24, p. 6]. For a similar discussion with a parabolic problem see [4,16].

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