Proceedings of the Royal Society of Edinburgh, **137A**, 93–109, *2007*

The role of the error function in a singularly perturbed convection–diffusion problem in a rectangle with corner singularities

José L. López and **Ester Pérez Sinusía**
Departamento de Matemática e Informática,
Universidad Pública de Navarra, 31006-Pamplona, Spain
(jl.lopez@unavarra.es; ester.perez@unavarra.es)

(MS received 1 August 2005; accepted 11 January 2006)

We consider a singularly perturbed convection–diffusion equation, $-\varepsilon\Delta u + \boldsymbol{v}\cdot\boldsymbol{\nabla} u = 0$, defined on a rectangular domain $\Omega \equiv \{(x,y) \mid 0 \leqslant x \leqslant \pi a, 0 \leqslant y \leqslant \pi\}$, $a > 0$, with Dirichlet-type boundary conditions discontinuous at the points $(0,0)$ and $(\pi a, 0)$: $u(x,0) = 1$, $u(x,\pi) = u(0,y) = u(\pi a, y) = 0$. An asymptotic expansion of the solution is obtained from a series representation in two limits, namely when the singular parameter $\varepsilon \to 0^+$ (with fixed distance to the points $(0,0)$ and $(\pi a, 0)$), and when $(x,y) \to (0,0)$ or $(x,y) \to (\pi a, 0)$ (with fixed ε). It is shown that the first term of the expansion at $\varepsilon = 0$ contains a linear combination of error functions. This term characterizes the effect of the discontinuities on the ε-behaviour of the solution $u(x,y)$ in the boundary or the internal layers. On the other hand, near the points of discontinuity $(0,0)$ and $(\pi a, 0)$, the solution $u(x,y)$ is approximated by a linear function of the polar angle.

1. Introduction

A singularly perturbed convection–diffusion problem is a second-order boundary-value problem in which the coefficients of the second-order derivatives are small. In this paper we focus our attention on two-dimensional linear elliptic convection–diffusion problems of the following form. Find a function $u \in \mathcal{C}(\bar{\Omega}) \cap \mathcal{D}^2(\Omega)$ such that

$$\left.\begin{aligned} -\varepsilon\Delta u + \boldsymbol{v}\cdot\boldsymbol{\nabla} u = 0, &\qquad x \in \Omega \subset \mathbb{R}^2, \\ u(x)|_{\partial\Omega} = f(\tilde{x}), &\qquad \tilde{x} \in \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where ε is a small positive parameter, $\boldsymbol{v}$ is the convection vector, $\tilde{x}$ is a variable which lives in $\partial\Omega$, $f(\tilde{x})$ is the Dirichlet date and $\mathcal{D}^2(\Omega)$ is the set of functions with partial derivatives up to order two defined in all points of Ω.

The solution u of this kind of problem typically presents thin layers where u undergoes a fast transition between two different values. The location and shape of these layers depend, among other things, on the velocity field, $\boldsymbol{v}$, on the shape of the boundary, $\partial\Omega$, and on the existence of discontinuities in $f(\tilde{x})$ (for more details on the shape and nature of singular layers, see, for example, [2, 4, 5, 8, 10] and references therein). In this paper we investigate the singular effects that jump-like discontinuities of $f(\tilde{x})$ produce on the solution u.

© 2007 The Royal Society of Edinburgh

It is not usually possible to represent the exact solution of (1.1) in terms of elementary functions, even when the data Ω, $\boldsymbol{v}$ or $f(\tilde{x})$ are simple. Then, an approximation of the solution adapted to the function's singular character (an asymptotic expansion) is of interest. There is an extensive literature devoted to the construction of approximated solutions based on matching of asymptotic expansions. Il'in [10] contains a quite exhaustive and general analysis for different equations and domains. Other important references on the method of matched asymptotic expansions are, for example, [4, 13, 17]. This technique is quite cumbersome: it requires an *a priori* knowledge of the location and nature of the singular layers in order to choose the correct stretched variable for every inner expansion. In general, this method does not give an expansion uniformly valid in the whole domain, but different expansions for different subdomains of Ω. Moreover, the calculation of the coefficients of those expansions require us to solve a boundary problem for every coefficient (see, for example, [11], where this technique is used for a convection–reaction–diffusion problem in a sector with corner singularities).

In many papers on singular perturbations, the establishment of a rigorous mathematical justification is very rare and no proof of the asymptotic properties of the expansion is given. The lack of a rigorous theory has already been pointed out by several authors [3, 5, 10]. More recently, Ou and Wong have shown [18, p. 377–381] that the asymptotic expansions of some well-known boundary-value singular perturbation problems, appearing in classical texts on singular perturbations and derived using the method of matched asymptotic expansions, are incorrect. They have shown that the mistakes in the construction of the asymptotic expansion are mainly due to the matching techniques used.

Several authors [6–8, 14, 15, 19–22] have proposed a different method based on knowledge of an exact representation of the solution. This method does not require an *a priori* knowledge of the location and nature of the boundary layers; the calculation of the coefficients of the expansion is straightforward. The method also produces an expansion uniformly valid in the whole domain (except in the vicinity of the discontinuities of the boundary condition).

The method based on an exact representation of the solution has been applied by several authors to some particular problems of the form (1.1) and with discontinuous Dirichlet data $f(\tilde{x})$ in different domains, such as a quarter-plane [7, 8, 14, 20], a sector [16, 21], an infinite strip [14] or a half-infinite strip [15]. In all of these problems, the solution u is approximated in the singular limit $\varepsilon \to 0^+$ by an error function or combination of error functions. These approximations are not valid near the discontinuities of the boundary condition.

The problems mentioned in the above paragraph are defined on unbounded domains. For these problems, the solution may be written in the form of an integral susceptible to an asymptotic analysis. For problems defined on bounded domains (which are more interesting for practical purposes), the situation is different: exact representations for the solution are, in general, given in terms of Fourier series and an asymptotic analysis is more complicated (see, for example, [22], in which Temme has obtained some information about the asymptotic approximation of the solution of the problem $-\varepsilon\Delta u + u_y = 0$ in the interior of a circle from a series of Bessel functions). Hemker [9] analysed the same equation in the exterior of a circle. He obtained asymptotic information from a similar series of Bessel functions. The asymptotic information given in [9, 22] for these problems is less complete than the asymptotic information given in the problems mentioned in the preceding paragraph. The other way of obtaining an asymptotic approximation for these kinds of problems is of course the method of matched asymptotic expansions. Using this method, several authors have obtained some asymptotic information in specific problems defined on bounded domains (see, for example, [2, 7] and [12, p. 537]). It is shown in these references that the error function plays an important role in the approximation of the solution.

In this paper we consider a problem defined on a bounded domain and a method based on an exact representation: we analyse the problem $-\varepsilon\Delta u + \boldsymbol{v}\cdot\boldsymbol{\nabla} u = 0$ on a rectangle with a discontinuous boundary condition at two of the corners of the rectangle, and we use a method based on a Fourier series representation of the solution. We approximate the solution $u(x, y)$ by deriving asymptotic expansions from this Fourier series, not only in the singular limit $\varepsilon \to 0^+$, but also in the limit $r \to 0^+$, where r denotes the distance from the point (x, y) to the points of discontinuity. Then, we approximate the solution on the whole domain, including the neighbourhood of the points of discontinuity.

In § 2 we obtain a series representation for the solution. In § 3 we derive an asymptotic expansion of the solution for $\varepsilon \to 0^+$. In § 4 we derive asymptotic approximations near the points of discontinuity of the boundary condition. Section 5 contains some comments and concluding remarks.

2. The problem and its exact solution

We consider the problem

$$\left.\begin{aligned} -\varepsilon\Delta U + \boldsymbol{v}\cdot\boldsymbol{\nabla} U = 0 \quad & \text{in } \Omega \equiv \{(x,y) \mid 0 < x < \pi a, 0 < y < \pi\}, \\ U(x,0) = 1, \quad & U \in \mathcal{C}(\tilde{\Omega}) \cap \mathcal{D}^2(\Omega), \\ U(x,\pi) = U(0,y) = U(\pi a, y) = 0, \quad & U \text{ bounded in } \tilde{\Omega}, \end{aligned}\right\} \tag{P}$$

where $\boldsymbol{v} \equiv (\sin\beta, \cos\beta)$ is a constant vector, $0 \leqslant \beta < 2\pi$, $\varepsilon > 0$ is a small parameter and a is a positive constant (observe the discontinuous Dirichlet conditions at the lower corners of the rectangle; see figure 1(a)). $\tilde{\Omega}$ is the closed domain $\bar{\Omega}$ with the discontinuity points of the boundary condition removed: $\tilde{\Omega} \equiv \bar{\Omega} \setminus \{(0,0), (\pi a, 0)\}$.

We have the following uniqueness result.

Proposition 2.1. *Problem (P) has at most one solution.*

Proof. Suppose that U_1 and U_2 are two solutions of (P). Then, the function

$$G(x,y) \equiv (U_1(x,y) - U_2(x,y)) \exp\{-w(x\sin\beta + y\cos\beta)\}$$

verifies

$$\begin{aligned} G \in \mathcal{C}(\tilde{\Omega}) \cap \mathcal{D}^2(\Omega), \quad & G \text{ bounded in } \tilde{\Omega}, \\ \Delta G - w^2 G = 0, \quad & \text{in } \Omega, \\ G(x,y) = 0, \quad & \text{for } (x,y) \in \partial\Omega. \end{aligned}$$

Denote $r = \sqrt{x^2+y^2}$ and $r_a = \sqrt{(x-\pi a)^2+y^2}$ and consider the following auxiliary function defined in $\bar{\Omega}$:

$$V_b(x,y) \equiv \begin{cases} \dfrac{G(x,y)}{H_b(x,y)} & \text{if } r \neq 0 \text{ and } r_a \neq 0, \\ 0 & \text{if } r = 0 \text{ or } r_a = 0, \end{cases}$$

$$H_b(x,y) \equiv K_0(wr) + K_0(wr_a) + b,$$

where K_0 is a modified Bessel function of order zero and b is a positive constant. The function $H_b(x,y)$ is positive for $wr > 0$ and $wr_a > 0$ and of the order $\mathcal{O}(\log(wr))$ as $wr \to 0^+$ and $\mathcal{O}(\log(wr_a))$ as $wr_a \to 0^+$ [1, equation (9.6.13)]. Moreover, $H_b(x,y) \in \mathcal{C}(\tilde{\Omega}) \cap \mathcal{D}^2(\Omega)$ and satisfies the equation: $\Delta H_b - w^2 H_b + bw^2 = 0$ in Ω [1, equation (9.6.1)]. Therefore, the auxiliary function V_b is continuous in $\bar{\Omega}$ and verifies that

$$\Delta V_b + \frac{2}{H_b}\boldsymbol{\nabla} H_b \cdot \boldsymbol{\nabla} V_b = \frac{bw^2}{H_b} V_b, \quad \text{in } \Omega,$$

$$V_b(x,y) = 0, \qquad \text{for } (x,y) \in \partial\Omega.$$

At points $(x,y) \in \Omega$, where $\boldsymbol{\nabla} V_b = 0$ and $V_b \neq 0$, we obtain $V_b \cdot \Delta V_b > 0$. Therefore, V_b has neither positive relative maxima nor negative relative minima in Ω. Then $\operatorname{Sup}_{\Omega} |V_b| \leqslant \operatorname{Sup}_{\partial\Omega} |V_b|$. On the other hand, $V_b(x,y) = 0$ for $(x,y) \in \partial\Omega$. Therefore, we have that $V_b = 0$ in $\bar{\Omega}$. Hence, $G = 0$ and $U_1 = U_2$ in $\tilde{\Omega}$. □

After the change of the unknown $U(x,y) = F(x,y)\exp(\boldsymbol{v} \cdot \boldsymbol{r}/(2\varepsilon))$, where $\boldsymbol{r} \equiv (x,y)$, problem (P) is transformed into the Yukawa equation for $F(x,y)$:

$$\left.\begin{aligned} \Delta F - w^2 F = 0 \qquad & \text{in } \Omega, \\ F(x,0) = \mathrm{e}^{-wx\sin\beta} \qquad & F \in \mathcal{C}(\tilde{\Omega}) \cap \mathcal{D}^2(\Omega), \\ F(x,\pi) = F(0,y) = F(\pi a, y) = 0, \qquad & U \text{ bounded in } \tilde{\Omega}, \end{aligned}\right\} \tag{2.1}$$

where $w \equiv 1/(2\varepsilon)$.

In the following proposition we obtain the explicit solution of problem (P) by means of a series representation. In what follows, empty sums must be treated as zero.

Proposition 2.2. *Let $w \equiv 1/(2\varepsilon)$. Then, for $(x,y) \in \Omega$ and $\beta \in (0, \frac{1}{2}\pi]$, the solution $U_\beta(x,y)$ of (P) is*

$$U_\beta(x,y) = \mathrm{e}^{w(x\sin\beta + y\cos\beta)}[G(x,y) + \mathrm{e}^{-\pi a w \sin\beta} G(\pi a - x, y)], \tag{2.2}$$

where

$$G(x,y) \equiv \sum_{n=-\infty}^{\infty} H_\beta(x + 2n\pi a, y) \tag{2.3}$$

and

$$H_\beta(x,y) \equiv \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\sinh[(\pi - y)\sqrt{w^2+t^2}]}{\sinh[\pi\sqrt{w^2+t^2}]} \frac{t\mathrm{e}^{\mathrm{i}tx}}{w^2\sin^2\beta + t^2}\,\mathrm{d}t. \tag{2.4}$$

On the other hand, for $\beta = 0$,

$$U_0(x,y) = \sum_{n=-\infty}^{\infty} (-1)^n \tilde{H}_0(x + n\pi a, y), \tag{2.5}$$

where

$$\tilde{H}_0(x,y) \equiv \frac{e^{wy}}{2\pi i} \int_{-\infty}^{\infty} \frac{\sinh[(\pi - y)\sqrt{w^2+t^2}]}{\sinh[\pi\sqrt{w^2+t^2}]} \frac{e^{ixt} - e^{i(x-\pi a)t}}{t} \,dt. \tag{2.6}$$

Proof. The exact solution of (2.1) may be obtained by separation of variables:

$$F(x,y) = \sum_{n=1}^{\infty} \frac{2n}{\pi} \frac{\sinh[(\pi - y)\sqrt{w^2+n^2/a^2}]}{\sinh[\pi\sqrt{w^2+n^2/a^2}]} \frac{1-(-1)^n e^{-w\pi a \sin\beta}}{a^2 w^2 \sin^2\beta + n^2} \sin\left(\frac{nx}{a}\right). \tag{2.7}$$

Then, the function

$$U_\beta(x,y) \equiv e^{w(x\sin\beta + y\cos\beta)} F(x,y),$$

with $F(x,y)$ defined above, is the solution of (P). It can be rewritten as follows for $0 < \beta \leqslant \frac{1}{2}\pi$:

$$U_\beta(x,y) = e^{w(x\sin\beta + y\cos\beta)} [J(x,y) + e^{-\pi a w \sin\beta} J(\pi a - x, y)], \tag{2.8}$$

with

$$J(x,y) \equiv \frac{1}{i\pi a} \sum_{n=-\infty}^{\infty} \frac{n/a}{w^2\sin^2\beta + n^2/a^2} \frac{\sinh[(\pi - y)\sqrt{w^2+n^2/a^2}]}{\sinh[\pi\sqrt{w^2+n^2/a^2}]} e^{inx/a}. \tag{2.9}$$

Now we use the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) e^{2i\pi m x} \,dx,$$

which is valid for absolutely integrable functions f on $\mathbb{R}$ and of bounded variation.

By applying this formula to the series in (2.9) and inserting the result in (2.8), we obtain (2.2)–(2.4). Formulae (2.5), (2.6) follow after applying the Poisson summation formula directly to the series on the right-hand side of (2.7) with $\beta = 0$. The convergence of the series in (2.3) and (2.5) will be demonstrated later, in the proofs of theorems 3.1 and 3.2, respectively. □

Remark 2.3. The solution U of problem P for $\beta = 0$ may be obtained by the method of images from the solution V of the problem

$$\left.\begin{aligned} -\varepsilon\Delta V + V_y = 0, \quad & (x,y) \in \Omega_0 \equiv (-\infty,\infty)\times(0,\pi), \\ V(x,0) = \chi_{[0,\pi a]}(x), V(x,\pi) = 0, \quad & V \in \mathcal{C}(\bar{\Omega}_0 \setminus \{(0,0),(\pi a,0)\}) \cap \mathcal{D}^2(\Omega_0). \end{aligned}\right\} \tag{2.10}$$

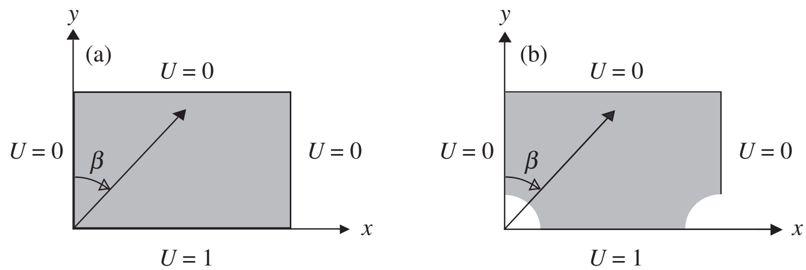

Figure 1. (a) Domain Ω of problem (P). (b) Indented region Ω^* in theorem 3.1.

This is a singularly perturbed convection–diffusion problem defined in an infinite strip Ω_0 of width π and parallel to the x-axis. Its solution is merely $V(x,y) = \tilde{H}_0(x,y)$ [14].

Remark 2.4. The explicit representation given in proposition 2.2 is only valid when the angle β between the convection vector $\boldsymbol{v}$ and the y-axis is restricted to the interval $[0, \frac{1}{2}\pi]$. Nevertheless, an explicit integral representation for the solution $U(x,y)$ of problem (P), whatever the direction of $\boldsymbol{v}$, may be obtained by means of symmetry arguments:

$$U(x,y) = \begin{cases} U_\beta(x,y) & \text{if } 0 \leqslant \beta \leqslant \frac{1}{2}\pi, \\ e^{2wy\cos\beta} U_{\pi-\beta}(x,y) & \text{if } \frac{1}{2}\pi < \beta \leqslant \pi, \\ U_{-\beta}(\pi a - x, y) & \text{if } -\frac{1}{2}\pi \leqslant \beta < 0, \\ e^{2wy\cos\beta} U_{\pi+\beta}(\pi a - x, y) & \text{if } -\pi < \beta < -\frac{1}{2}\pi, \end{cases}$$

where $U_\beta(x,y)$ is given in (2.2) for $\beta \in (0, \frac{1}{2}\pi]$ and $U_0(x,y)$ is given in (2.5). Therefore, in the remainder of the paper, we will restrict ourselves to $\beta \in [0, \frac{1}{2}\pi]$.

The solution of (P) cannot be written in terms of known functions. But, for $\varepsilon \to 0^+$ and (x,y) away from $(0,0)$ and $(\pi a, 0)$, we can approximate $U_\beta(x,y)$ by a combination of error functions plus an asymptotic expansion in powers of ε. For $(x,y) \to (0,0)$ or $(x,y) \to (\pi a, 0)$ (and $\varepsilon \geqslant \varepsilon_0 > 0$), we can approximate $U_\beta(x,y)$ by an asymptotic expansion in powers of r or of $\sqrt{(x-\pi a)^2 + y^2}$, respectively. This is the subject of the following two sections.

3. Asymptotic expansion of $U(x,y)$ in the singular limit

In this section we denote by Ω^* the rectangular domain indented at the points $(0,0)$ and $(\pi a, 0)$ (see figure 1(b)):

$$\Omega^* \equiv \{(x,y) \in \Omega,\ 0 < r_0 < \sqrt{x^2 + y^2},\ 0 < r_0 < \sqrt{(x-\pi a)^2 + y^2}\}.$$

Theorem 3.1. *Let $w \equiv 1/(2\varepsilon)$. Then, for $(x,y) \in \Omega^*$ and $\beta \in (0, \frac{1}{2}\pi]$, the solution $U_\beta(x,y)$ of (P) given in proposition 2.2 is*

$$U_\beta(x,y) = U_\beta^0(x,y) + \frac{1}{\sqrt{w}} U_\beta^1(x,y), \tag{3.1}$$

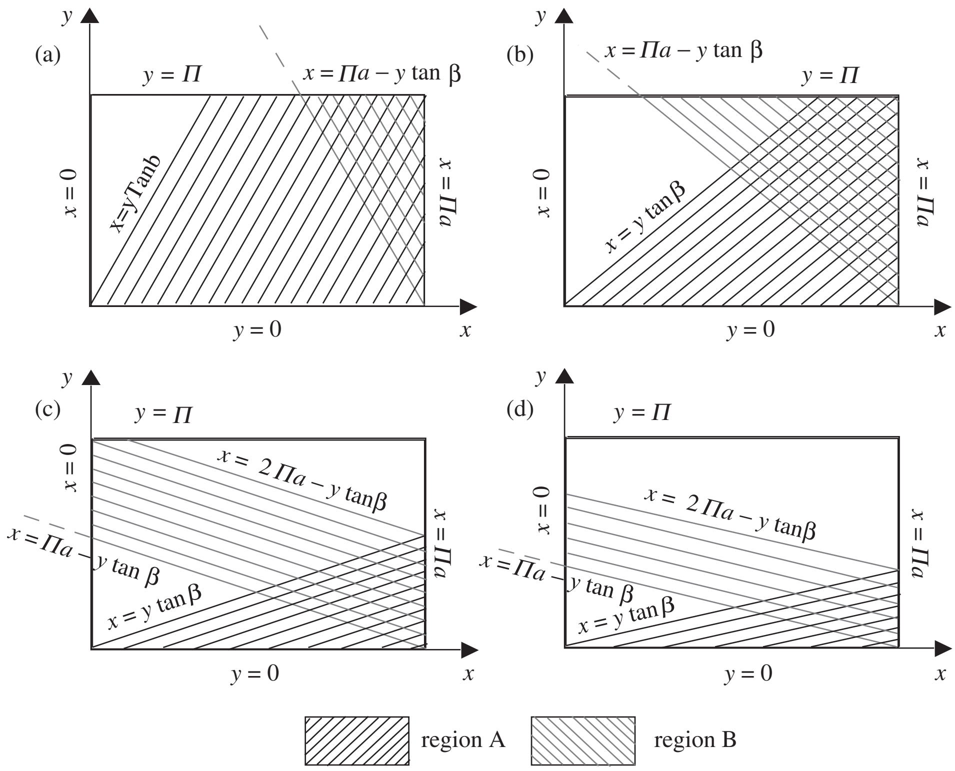

Figure 2. Different aspects of the regions A and B defined in theorem 3.1 depending on the relative value of $\tan\beta$ and a. (a) $0 < \tan\beta \leqslant \frac{1}{2}a$, (b) $\frac{1}{2}a < \tan\beta \leqslant a$, (c) $a < \tan\beta \leqslant 2a$, (d) $2a < \tan\beta$.

where

$$
\begin{aligned}
U^0_\beta(x,y) \equiv \mathrm{e}^{wy\cos\beta}\frac{\sinh[(\pi-y)w\cos\beta]}{\sinh[\pi w\cos\beta]}& \\
\times\Bigg\{\frac{1+\delta_{\beta,\pi/2}}{2}& \\
\times\Bigg[\operatorname{sgn}\left(\beta-\arctan\left(\frac{x}{y}\right)\right)\operatorname{erfc}\sqrt{w\zeta(x,y)}& \\
-\,\mathrm{e}^{2(x-\pi a)w\sin\beta}\operatorname{sgn}\left(\beta-\arctan\left(\frac{2\pi a-x}{y}\right)\right)& \\
\times\operatorname{erfc}\sqrt{w\zeta(2\pi a-x,y)}& \\
+\,\mathrm{e}^{2(x-\pi a)w\sin\beta}\operatorname{sgn}\left(\beta-\arctan\left(\frac{\pi a-x}{y}\right)\right)& \\
\times\operatorname{erfc}\sqrt{w\zeta(\pi a-x,y)}\Bigg]& \\
+\,\chi_A(x,y)-\mathrm{e}^{2w(x-\pi a)\sin\beta}\chi_B(x,y)\Bigg\}.&
\end{aligned}
\tag{3.2}
$$

In these formulae, sgn(0) *must be treated as zero,*

$$\zeta(x,y) \equiv \sqrt{x^2+y^2} - x\sin\beta - y\cos\beta, \tag{3.3}$$

the regions A and B are (see figure 2) defined by

$$\left.\begin{aligned} A &\equiv \{(x,y)\in\Omega, y\tan\beta < x\},\\ B &\equiv \{(x,y)\in\Omega, \pi a - x < y\tan\beta < 2\pi a - x\},\end{aligned}\right\} \tag{3.4}$$

$\chi_A(x,y)$ is the characteristic function of the set A,

$$\chi_A(x,y) \equiv \begin{cases} 1 & \textit{if } (x,y)\in A,\\ 0 & \textit{if } (x,y)\notin A,\end{cases} \tag{3.5}$$

and $\chi_B(x,y)$ is the characteristic function of the set B.

$U^1_\beta(x,y) = \mathcal{O}(1)$ when $w\to\infty$ uniformly in $(x,y)\in\Omega^*$. Moreover, it has an asymptotic expansion in powers of w^{-1}; for $n = 0,1,2,\ldots,$

$$U^1_\beta(x,y) = \sum_{k=0}^{n-1} \frac{T_k(x,y)}{w^k} + R_n(x,y), \tag{3.6}$$

where the coefficients $T_k(x,y)$ (defined below) are smooth functions of x and y. The remainder $R_n(x,y)$ satisfies the bound

$$|R_n(x,y)| \leqslant M\frac{\Gamma(n+\frac{1}{2})}{(2wd\tilde{r})^n}\mathrm{e}^{-w\tilde{\zeta}(x,y)}, \tag{3.7}$$

for some positive constants M and d given below, $\tilde{\zeta}(x,y) = \min\{\zeta(x,y),\zeta(\pi a - x,y)\}$ and $\tilde{r} = \min\{r, r_a\}$.

Proof. We will find first some approximations and useful bounds of the functions $H_\beta(x+k\pi a,y)$ defined in (2.4). After the change of variable $t = w\sinh u$ in the integral (2.4) we obtain

$$H_\beta(x+k\pi a,y) = \frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}w(x+k\pi a)\sinh u}\frac{\sinh u\cosh u}{\sinh^2 u+\sin^2\beta}\frac{\sinh[(\pi-y)w\cosh u]}{\sinh[\pi w\cosh u]}\,\mathrm{d}u. \tag{3.8}$$

Using the polar variables r_k and ϕ_k defined by the relations $x+k\pi a = r_k\sin\phi_k$ and $y = r_k\cos\phi_k$ for $k\in\mathbb{Z}$ we have

$$\begin{aligned} &H_\beta(x+k\pi a,y)\\ &= \frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty} \mathrm{e}^{wy\cosh u - wr_k\cosh(u-\mathrm{i}\phi_k)}\frac{\sinh u\cosh u}{\sinh^2 u+\sin^2\beta}\frac{\sinh[(\pi-y)w\cosh u]}{\sinh[\pi w\cosh u]}\,\mathrm{d}u. \end{aligned} \tag{3.9}$$

The poles of the integrand are located at the points $u = \pm\mathrm{i}\beta + m\pi\mathrm{i}$, $m\in\mathbb{Z}$. The integrand behaves like $\exp\{-r_k w\cosh(\Re(u))\cos(\mathrm{Im}(u)-\phi_k)\}$ when $u\to\pm\infty$. Therefore, we can use the Cauchy's residue theorem for shifting the integration contour in each integral $H_\beta(x+k\pi a,y)$ to the straight line $\operatorname{Im} u=\phi_k$:

$$H_\beta(x+k\pi a,y)=I_\beta(x+k\pi a,y)+\operatorname{sgn}(\phi_k)\mathrm{e}^{-w|x+k\pi a|\sin\beta}\\ \times\frac{\sinh[(\pi-y)w\cos\beta]}{\sinh[\pi w\cos\beta]}[\chi_{(0,\pi/2)}(|\phi_k|-\beta)+\tfrac{1}{2}\delta_{|\phi_k|,\beta}], \quad (3.10)$$

where

$$I_\beta(x+k\pi a,y)\equiv\frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{e}^{wy\cosh(u+\mathrm{i}\phi_k)-r_k w\cosh u}\\ \times\frac{\sinh(u+\mathrm{i}\phi_k)\cosh(u+\mathrm{i}\phi_k)}{\sinh^2(u+\mathrm{i}\phi_k)+\sin^2\beta}\frac{\sinh[(\pi-y)w\cosh(u+\mathrm{i}\phi_k)]}{\sinh[\pi w\cosh(u+\mathrm{i}\phi_k)]}\,\mathrm{d}u. \quad (3.11)$$

When $|\phi_k|=\beta$, this integral must be understood as a principal-value integral.

For large w and fixed r_k, the asymptotic features of this integral are that

(i) there is a saddle point at $u=0$,

(ii) the pole situated at $u=\mathrm{i}(\operatorname{sgn}(\phi_k)\beta-\phi_k)$ and the saddle point coalesce when $|\phi_k|\to\beta$.

Uniform asymptotic expansion of this kind of integral is obtained by using the error function as the basic approximant [23, ch. 7, § 2]. Therefore, we split off the pole of the integrand at $u=\mathrm{i}(\operatorname{sgn}(\phi_k)\beta-\phi_k)$:

$$\mathrm{e}^{wy\cosh(u+\mathrm{i}\phi_k)}\frac{\sinh[(\pi-y)w\cosh(u+\mathrm{i}\phi_k)]}{\sinh[\pi w\cosh(u+\mathrm{i}\phi_k)]}\frac{\sinh(u+\mathrm{i}\phi_k)\cosh(u+\mathrm{i}\phi_k)}{\sinh^2(u+\mathrm{i}\phi_k)+\sin^2\beta}\\ =\frac{\mathrm{e}^{wy\cos\beta}(1+\delta_{\beta,\pi/2})}{4\sinh\frac{1}{2}[u+\mathrm{i}(\phi_k-\operatorname{sgn}(\phi_k)\beta)]}\frac{\sinh[(\pi-y)w\cos\beta]}{\sinh[\pi w\cos\beta]}+f(u,\phi_k,\beta),$$

with the obvious definition of $f(u,\phi_k,\beta)$.

Using the complementary error function representation [20]

$$\mathrm{e}^{-r\cos\alpha}\operatorname{erfc}(\sqrt{2r}\sin\tfrac{1}{2}\alpha)=\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{e}^{-r\cosh u}\frac{\mathrm{d}u}{\sinh\frac{1}{2}(u-\mathrm{i}\alpha)}, \quad 0<\alpha<2\pi, \quad (3.12)$$

we find that the integral $I_\beta(x+k\pi a,y)$ reads

$$I_\beta(x+k\pi a,y)=(1+\delta_{\beta,\pi/2})\operatorname{sgn}\left[\beta\operatorname{sgn}(x+k\pi a)-\arctan\left(\frac{x+k\pi a}{y}\right)\right]\\ \times\frac{\mathrm{e}^{-w|x+k\pi a|\sin\beta}}{2}\frac{\sinh[(\pi-y)w\cos\beta]}{\sinh[\pi w\cos\beta]}\operatorname{erfc}\sqrt{w\zeta(x+k\pi a,y)}\\ +\bar{I}_\beta(x+k\pi a,y), \quad (3.13)$$

where

$$\bar{I}_\beta(x+k\pi a,y)\equiv\frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{e}^{-r_k w\cosh u}f(u,\phi_k,\beta)\,\mathrm{d}u. \quad (3.14)$$

We can write

$$\left.\begin{aligned}
&\mathrm{e}^{wy\cosh(u+\mathrm{i}\phi_k)}\frac{\sinh[(\pi-y)w\cosh(u+\mathrm{i}\phi_k)]}{\sinh[\pi w\cosh(u+\mathrm{i}\phi_k)]}\\
&\qquad\qquad\qquad = 1+\mathrm{e}^{2w(y-\pi)\cosh(u+\mathrm{i}\phi_k)}\psi_1(u,r_k,\phi_k,w),\\
&\mathrm{e}^{wy\cos\beta}\frac{\sinh[(\pi-y)w\cos\beta]}{\sinh[\pi w\cos\beta]} = 1+\mathrm{e}^{2w\cos\beta(y-\pi)}\psi_2(r_k,\phi_k,w),
\end{aligned}\right\} \tag{3.15}$$

where $\psi_1(u,r_k,\phi_k,w)=\mathcal{O}(1)$ and $\psi_2(r_k,\phi_k,w)=\mathcal{O}(1)$ when $w\to\infty$ uniformly in $(x,y)\in\Omega^*$ and $u\in(-\infty,\infty)$. Therefore, we can write

$$f(u,\phi_k,\beta)=f_1(u,\phi_k,\beta)+f_2(u,\phi_k,\beta) \tag{3.16}$$

with

$$\left.\begin{aligned}
f_1(u,\phi_k,\beta) &= \frac{\sinh(u+\mathrm{i}\phi_k)\cosh(u+\mathrm{i}\phi_k)}{\sinh^2(u+\mathrm{i}\phi_k)+\sin^2\beta}-\frac{1+\delta_{\beta,\pi/2}}{4\sinh\{\frac12[u+\mathrm{i}(\phi_k-\mathrm{sgn}(\phi_k)\beta)]\}},\\
f_2(u,\phi_k,\beta) &= -\mathrm{e}^{2w\cos\beta(y-\pi)}\frac{\psi_2(r_k,\phi_k,w)(1+\delta_{\beta,\pi/2})}{4\sinh\{\frac12[u+\mathrm{i}(\phi_k-\mathrm{sgn}(\phi_k)\beta)]\}}+\psi_1(u,r_k,\phi_k,w)\\
&\quad\times \mathrm{e}^{2w(y-\pi)\cosh u\cos\phi_k}\mathrm{e}^{\mathrm{i}2w(y-\pi)\sinh u\sin\phi_k}\\
&\quad\times\frac{\sinh(u+\mathrm{i}\phi_k)\cosh(u+\mathrm{i}\phi_k)}{\sinh^2(u+\mathrm{i}\phi_k)+\sin^2\beta}.
\end{aligned}\right\} \tag{3.17}$$

Inserting (3.16) in (3.14), we obtain

$$\bar I_\beta(x+k\pi a,y) = \underbrace{\frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{e}^{-wr_k\cosh u}f_1(u,\phi_k,\beta)\,\mathrm{d}u}_{\bar I^1_\beta(x+k\pi a,y)}+\underbrace{\frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{e}^{-wr_k\cosh u}f_2(u,\phi_k,\beta)\,\mathrm{d}u}_{\bar I^2_\beta(x+k\pi a,y)}. \tag{3.18}$$

We will analyse the integrals $\bar I^1_\beta$ and $\bar I^2_\beta$ in two different ways: we will derive an asymptotic expansion for $\bar I^1_\beta$ and we will find a useful bound for $\bar I^2_\beta$.

We perform the change of variable $t=\sinh(\frac12 u)$ in $\bar I^1_\beta$:

$$\bar I^1_\beta(x+k\pi a,y)=\frac{\mathrm{e}^{-wr_k}}{\pi}\int_0^\infty \mathrm{e}^{-2wr_kt^2}g(t^2,\phi_k,\beta)\,\mathrm{d}t, \tag{3.19}$$

where

$$\begin{aligned}
g(t,\phi_k,\beta)\equiv{}&\frac{1}{\sqrt{1+t}}\frac{[\sin^2(\beta)-\sin^2(\phi_k)-4\cos(2\beta)t(t+1)]\sin(2\phi_k)}{(2t+1+s_+)(2t+1-s_+)(2t+1+s_-)(2t+1-s_-)}\\
&-\frac{(1+\delta_{\beta,\pi_k/2})\sin(\frac12(\beta-\phi_k))}{2(t+\sin^2\frac12(\beta-\phi_k))}
\end{aligned}$$

and we have denoted $s_\pm\equiv\cos(\beta\pm\phi_k)$.

The function $g(t,\phi_k,\beta)$ has a Taylor expansion at $t=0$ for each $\phi_k\in[0,\frac{1}{2}\pi]$:

$$g(t,\phi_k,\beta)=\sum_{s=0}^{n-1}\frac{g^{(s)}(0,\phi_k,\beta)}{s!}t^s+g_n(t,\phi_k,\beta), \tag{3.20}$$

where

$$g_n(t,\phi_k,\beta)\equiv\frac{g^{(n)}(\xi,\phi_k,\beta)}{n!}t^n,\quad \xi\in(0,t).$$

The singularities of $g(t,\phi_k,\beta)$ are away from the positive real axis (we write d_k for the distance from the closest of those singularities to the positive real axis). Therefore, using the Cauchy formula for the derivative $g^{(n)}(\xi,\phi_k,\beta)$, we see that

$$|g_n(t,\phi,\beta)|\leqslant M\frac{t^n}{d_k^n}, \tag{3.21}$$

where M is a bound for $g(w,\phi_k,\beta)$ on the portion of the complex w-plane surrounding the positive real axis: $\{w\in\mathbb{C},|w-t|<d_k,t\in\mathbb{R}^+\}$.

Substituting (3.20) into (3.19) and interchanging the sum and integral, we obtain

$$\bar{I}^1_\beta(x+k\pi a,y)=\mathrm{e}^{-wr_k}\left[\sum_{s=0}^{n-1}\frac{g^{(s)}(0,\phi_k,\beta)}{2\pi s!}\frac{\Gamma(s+\frac{1}{2})}{(2wr_k)^{s+1/2}}+\bar{R}_n(x+k\pi a,y)\right], \tag{3.22}$$

where

$$\bar{R}_n(x+k\pi a,y)\equiv\frac{1}{\pi}\int_0^\infty \mathrm{e}^{-2wr_kt^2}g_n(t^2,\phi_k,\beta)\,\mathrm{d}u.$$

Therefore, using (3.21) we have

$$|\bar{R}_n(x+k\pi a,y)|\leqslant\bar{M}\frac{\Gamma(n+\frac{1}{2})}{(2wd_kr_k)^{n+1/2}},\quad r_k^2\equiv(x+k\pi a)^2+y^2. \tag{3.23}$$

On the other hand, the integral $\bar{I}^2_\beta$ can be written as

$$\bar{I}^2_\beta(x+k\pi a,y)=\frac{\mathrm{e}^{-wr_k}}{\pi\mathrm{i}}\int_{-\infty}^\infty \mathrm{e}^{-wr_k(\cosh u-1)}f_2(u,\phi_k,\beta)\,\mathrm{d}u. \tag{3.24}$$

Using the fact that $|f_2(u,\phi_k,\beta)|\leqslant M\mathrm{e}^{2w(y-\pi)\alpha}$ with $\alpha\equiv\min\{\cos\phi_k,\cos\beta\}$, we can deduce that

$$|\bar{I}^2_\beta(x+k\pi a,y)|\leqslant M_2\frac{\mathrm{e}^{-w(r_k+2\alpha(\pi-y))}}{\sqrt{wr_k}} \tag{3.25}$$

with $M_2>0$ independent of w and k.

From (2.2), (2.3) and using the polar variables $x=r\sin\phi$, $y=r\cos\phi$ with $0\leqslant r\leqslant\pi\sqrt{1+a^2}$ and $0\leqslant\phi\leqslant\frac{1}{2}\pi$, $U_\beta(x,y)$ can be written as

$$U_\beta(x,y)=\mathrm{e}^{wr\cos(\phi-\beta)}\Bigg[\sum_{n=-\infty}^{\infty}H_\beta(x+2n\pi a,y) - \mathrm{e}^{-\pi aw\sin\beta}\sum_{n=-\infty}^{\infty}H_\beta(x-(2n+1)\pi a,y)\Bigg]$$

$$= \mathrm{e}^{wr\cos(\phi-\beta)}\Bigg[H_\beta(x,y) - H_\beta(2\pi a - x, y) + \mathrm{e}^{-\pi a w\sin\beta} H_\beta(\pi a - x, y) + \sum_{n=-\infty, n\neq 0,-1}^{\infty} H_\beta(x+2n\pi a, y) - \mathrm{e}^{-\pi a w\sin\beta} \sum_{n=-\infty, n\neq 0}^{\infty} H_\beta(x-(2n+1)\pi a, y)\Bigg]. \quad (3.26)$$

From (3.10), (3.13), (3.18), (3.22), (3.23) and (3.25) we can see that

$$|H_\beta(x+k\pi a, y)| \leqslant C\frac{\mathrm{e}^{-w(|x+k\pi a|\sin\beta + y\cos\beta)}}{\sqrt{w}} \quad \forall k\in\mathbb{Z} \quad (3.27)$$

with $C>0$ independent of w and k. Therefore, for large w and $n\in\mathbb{Z}\setminus\{0,-1\}$, the terms $H_\beta(x+2n\pi a, y)$ are exponentially small with respect to $H_\beta(x,y)$ and $H_\beta(x-2\pi a, y)$ for all $(x,y)\in\Omega^*$. Also, for large w and $n\in\mathbb{Z}\setminus\{0\}$, the terms $H_\beta(x-(2n+1)\pi a, y)$ are exponentially small with respect to $H_\beta(x-\pi a, y)$ for all $(x,y)\in\Omega^*$. Moreover, from (3.27) we see that the expansion (2.3) is convergent.

Finally, from (3.26) and (3.27),

$$U_\beta(x,y) = \mathrm{e}^{wr\cos(\phi-\beta)}[H_\beta(x,y) + \mathrm{e}^{-w\pi a\sin\beta}H_\beta(\pi a - x, y) - H_\beta(2\pi a - x, y)] + \mathcal{O}\left(\frac{\mathrm{e}^{-2w\pi a\sin\beta}}{\sqrt{w}}\right). \quad (3.28)$$

From (3.10) and (3.28) and depending on the relation between $\tan\beta$ and a, we can write

$$\begin{aligned} U_\beta(x,y) = \mathrm{e}^{wr\cos(\phi-\beta)}\Bigg[&I_\beta(x,y) + \mathrm{e}^{-w\pi a\sin\beta}I_\beta(\pi a - x, y) - I_\beta(2\pi a - x, y) \\ &+ \mathrm{e}^{-wr\cos(\phi-\beta)}\mathrm{e}^{wy\cos\beta}\frac{\sinh[(\pi-y)w\cos\beta]}{\sinh[\pi w\cos\beta]} \\ &\times [\chi_A(x,y) - \mathrm{e}^{2w(x-\pi a)\sin\beta}\chi_B(x,y)]\Bigg] \\ &+ \mathcal{O}\left(\frac{\mathrm{e}^{-2w\pi a\sin\beta}}{\sqrt{w}}\right), \end{aligned} \quad (3.29)$$

where $I_\beta(x+k\pi a, y)$ is given in (3.11) and χ_A and χ_B are given in (3.4), (3.5).

Therefore, from (3.13) and (3.29) we obtain (3.1) and (3.2) with

$$U_\beta^1(x,y) = \sqrt{w}\mathrm{e}^{wr\cos(\phi-\beta)}[\bar{I}_\beta(x,y) + \mathrm{e}^{-w\pi a\sin\beta}\bar{I}_\beta(\pi a - x, y) - \bar{I}_\beta(2\pi a - x, y)] + \mathcal{O}(\mathrm{e}^{-2w\pi a\sin\beta}). \quad (3.30)$$

Using (3.18), (3.22) and (3.25), we obtain (3.6) with

$$T_s(x,y) \equiv \frac{\Gamma(s+\frac{1}{2})}{2\pi s!}\left[\frac{g^{(s)}(0,w,r,\phi,\beta)}{(2r)^{s+1/2}}\mathrm{e}^{-w\zeta(x,y)} + \frac{g^{(s)}(0,w,r_3,\phi_3,\beta)}{(2r_3)^{s+1/2}}\mathrm{e}^{-w\zeta(\pi a - x, y)} - \frac{g^{(s)}(0,w,r_2,\phi_2,\beta)}{(2r_2)^{s+1/2}}\mathrm{e}^{-w(\zeta(2\pi a - x, y) + 2(\pi a - x)\sin\beta)}\right] \quad (3.31)$$

and

$$R_n(x,y) \equiv \mathrm{e}^{-w\zeta(x,y)}\bar{R}_n(x,y) - \mathrm{e}^{-w(\zeta(2\pi a-x,y)+2(\pi a-x)\sin\beta)}\bar{R}_n(2\pi a-x,y) \\ + \mathrm{e}^{-w\zeta(\pi a-x,y)}\bar{R}_n(\pi a-x,y) \\ + \mathcal{O}(\mathrm{e}^{-w(\zeta(x,y)+2\alpha(\pi-y))}) + \mathcal{O}(\mathrm{e}^{-2w\pi a\sin\beta}). \tag{3.32}$$

In these formulae, $r^2 \equiv x^2+y^2$, $r_2^2 \equiv (2\pi a-x)^2+y^2$, $r_3^2 \equiv (\pi a-x)^2+y^2$, $\phi \equiv \arctan(x/y)$, $\phi_2 \equiv \arctan((2\pi a-x)/y)$ and $\phi_3 \equiv \arctan((\pi a-x)/y)$. The exponentially small terms on the right-hand side of (3.32) include all the terms $H_\beta(x+k\pi a,y)$ for $k\in\mathbb{Z}\setminus\{0,-1,-2\}$ (which have been bounded in (3.27)) and $\bar{I}_\beta^2(x,y)$, $\bar{I}_\beta^2(\pi a-x,y)$ and $\bar{I}_\beta^2(2\pi a-x,y)$ (which are bounded in (3.25)).

From (3.32) and (3.23) we finally obtain (3.7). □

THEOREM 3.2. *Write $w\equiv 1/(2\varepsilon)$. Then, for $(x,y)\in\Omega^*$ and $\beta=0$, the solution $U_0(x,y)$ of (P) given in proposition 2.2 is*

$$U_0(x,y) = U_0^0(x,y) + \frac{1}{\sqrt{w}}U_0^1(x,y), \tag{3.33}$$

where

$$U_0^0(x,y) \equiv \{1-\operatorname{erfc}\sqrt{w\zeta(x,y)} - \operatorname{erfc}\sqrt{w\zeta(x-\pi a,y)}\}\mathrm{e}^{wy}\frac{\sinh[(\pi-y)w]}{\sinh[\pi w]} \tag{3.34}$$

and $\zeta(x,y)\equiv\sqrt{x^2+y^2}-y$. The function $U_0^1(x,y)=\mathcal{O}(1)$ when $w\to\infty$ uniformly in $(x,y)\in\Omega^*$. Moreover, it has an asymptotic expansion in powers of w^{-1}:

$$U_0^1(x,y) = \sum_{k=0}^{n-1}\frac{\tilde{T}_k(x,y)}{w^k} + R_n(x,y). \tag{3.35}$$

The coefficients $\tilde{T}_k(x,y)$ are smooth functions of x and y and $\mathcal{O}(1)$ when $w\to\infty$ uniformly for $(x,y)\in\Omega^*$.

The remainder $R_n(x,y)$ satisfies

$$|R_n(x,y)| \leqslant M\frac{\Gamma(n+\frac{1}{2})}{(2wd\tilde{r})^n}, \tag{3.36}$$

for some positive constants M and d and $\tilde{r}\equiv\min\{r,r_a\}$.

Proof. The solution $U_0(x,y)$ given in (2.5) can be rewritten as

$$U_0(x,y) = \tilde{H}_0(x,y) - \tilde{H}_0(x-\pi a,y) - \tilde{H}_0(x+\pi a,y) + \sum_{\substack{k=-\infty,\\ k\neq 0,\pm 1}}^{\infty}(-1)^k\tilde{H}_0(x+k\pi a,y), \tag{3.37}$$

where $\tilde{H}_0(x,y)$ were defined in (2.6). We will find a bound for the series on the right-hand side of (3.37) and we will derive an asymptotic expansion for the first three terms on the right-hand side of (3.37).

The functions $\tilde{H}_0(x+k\pi a, y)$ in (3.37) are the solution of the respective problems (remark 2.3):

$$
\begin{gathered}
-\varepsilon\Delta V + V_y = 0, \quad (x,y) \in \Omega_0 \equiv (-\infty,\infty)\times(0,\pi), \\
V(x,0) = \chi_{[k\pi a,(k+1)\pi a]}(x), \quad V(x,\pi) = 0,
\end{gathered}
$$

From [14, theorem 4] we see that the terms of the series in (3.37) verify $\tilde{H}_0(x+k\pi a, y) = \mathcal{O}(\mathrm{e}^{-w\zeta(x-k\pi a,y)} + \mathrm{e}^{-w\zeta(x-(k+1)\pi a,y)})$ when $w\to\infty$ uniformly for $(x,y)\in\Omega^*$. Therefore, the series on the right-hand side of (3.37) is convergent and of order $\mathcal{O}(\mathrm{e}^{-w\zeta(\pi a,y)})$. On the other hand, from [14, theorem 4],

$$
\begin{aligned}
\tilde{H}_0(x,y) = \mathrm{e}^{wy}\frac{\sinh[(\pi-y)w]}{\sinh[\pi w]}&[\chi_{(0,\pi a)}(x) - \tfrac{1}{2}\operatorname{sgn}(x)\operatorname{erfc}\sqrt{w\zeta(x,y)} \\
&\quad + \tfrac{1}{2}\operatorname{sgn}(x-\pi a)\operatorname{erfc}\sqrt{w\zeta(x-\pi a,y)}] \\
&+ \tilde{H}_1(x,y) + \tilde{H}_1(x-\pi a,y), \qquad (3.38)
\end{aligned}
$$

where $\tilde{H}_1(x,y)$ has an asymptotic expansion in inverse powers of w,

$$
\tilde{H}_1(x,y) = \frac{\mathrm{e}^{-w\zeta(x,y)}}{\sqrt{wr}}\left[\sum_{k=0}^{n-1}\frac{S_k(x,y)}{(wr)^k} + \tilde{R}_n(x,y)\right].
$$

In this formula, $S_k(x,y)$ are regular functions of x and y for $(x,y)\in\Omega^*$, $r^2\equiv x^2+y^2$ and, for $n=1,2,3,\dots$,

$$
|\tilde{R}_n(x,y)| \leqslant M\frac{\Gamma(n+\frac{1}{2})}{(2wdr)^n}, \qquad (3.39)
$$

where M and d are positive constants. Formulae (3.38) and (3.39) apply also to the functions $H_0(x-\pi a,y)$ and $H_0(x+\pi a,y)$ replacing x by $x-\pi a$ and $x+\pi a$, respectively. Therefore, formulae (3.34)–(3.36) follow with the obvious definitions of $\tilde{T}_k(x,y)$ and $R_n(x,y)$. □

Remark 3.3. From (3.1), (3.6) and (3.7) and from (3.33), (3.35) and (3.36) we see that, for $\beta\in[0,\frac{1}{2}\pi]$, $U_\beta(x,y) = U_\beta^0(x,y)[1+\mathcal{O}(\sqrt{\varepsilon})]$ when $\varepsilon\to0^+$ away from the points $(0,0)$ and $(\pi a,0)$. Then, the first-order approximation to the solution of (P) is a linear combination of error functions and elementary functions. The error functions in (3.2) and (3.34) exhibit interior/boundary layers of width $\mathcal{O}(\sqrt{\varepsilon})$. The exponential factors in (3.2) and (3.34) exhibit boundary layers of width $\mathcal{O}(\varepsilon)$ (see figure 3).

4. Asymptotic expansion of $U(x,y)$ near the corner singularities

The asymptotic expansions given in theorems 3.1 and 3.2 break down when $(x,y)\to(0,0)$ or $(x,y)\to(\pi a,0)$ (that is, $\tilde{r}\to0$ in (3.7) and (3.36)). The asymptotic approximation of $U_\beta(x,y)$ near these points requires a different analysis. An asymptotic approximation of $U_\beta(x,y)$ when $(x,y)\to(0,0)$ or $(x,y)\to(\pi a,0)$ faster than $\varepsilon\to0^+$ is given in the following theorem.

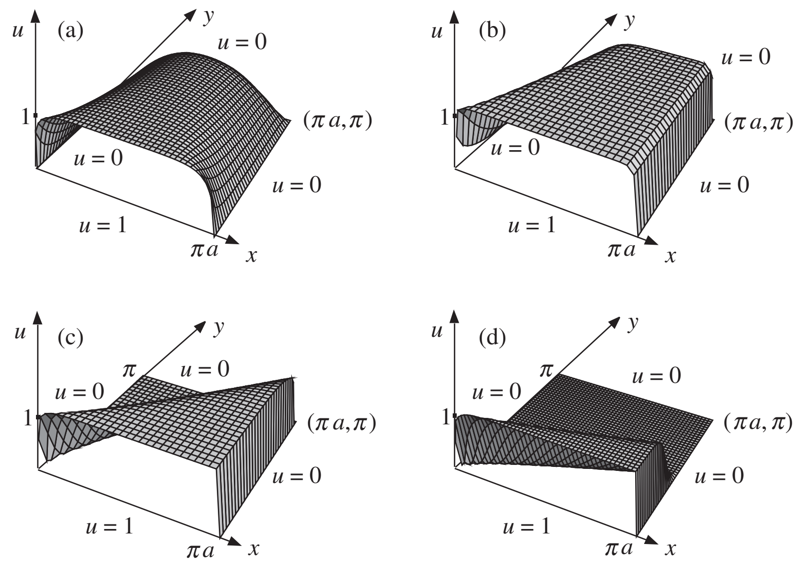

Figure 3. Graphs of the first-order approximation, $U^0_\beta(x,y)$, to the solution of the problem (P) for different values of β and $\varepsilon = 0.1$. The convection vector $\boldsymbol{v}$ 'drags' the discontinuity of the boundary condition at the point $(0,0)$ (and also the discontinuity at the point $(\pi a, 0)$ for $\beta = 0$) originating a parabolic layer of size $\mathcal{O}(\sqrt{\varepsilon})$ along $\boldsymbol{v}$. (a) $\beta = 0$, (b) $\beta = \frac{1}{6}\pi$, (c) $\beta = \frac{1}{3}\pi$, (d) $\beta = \frac{4}{9}\pi$.

THEOREM 4.1. *Let $x = r\sin\phi$, $y = r\cos\phi$ and $\pi a - x = r_a \sin\phi_a$, $y = r_a\cos\phi_a$. Then, for $\beta \in [0, \frac{1}{2}\pi]$ and $(x,y) \in \Omega$, the solution $U_\beta(x,y)$ of (P) verifies*

$$U_\beta(x,y) = \frac{2\phi}{\pi} + \mathcal{O}\left(\frac{r}{\varepsilon}\right) + \mathcal{O}(\mathrm{e}^{-\alpha/\varepsilon}) \tag{4.1}$$

when $r, \varepsilon \to 0^+$ with $r/\varepsilon \to 0^+$, and

$$U_\beta(x,y) = \frac{2\phi_a}{\pi} + \mathcal{O}\left(\frac{r_a}{\varepsilon}\right) + \mathcal{O}(\mathrm{e}^{-\alpha/\varepsilon}) \tag{4.2}$$

when $r_a, \varepsilon \to 0^+$ with $r_a/\varepsilon \to 0^+$. The $\mathcal{O}(\mathrm{e}^{-\alpha/\varepsilon})$ symbols hold uniformly for $(x,y) \in \Omega$ with $\alpha > 0$.

Proof. Consider the representation for $U_\beta(x,y)$ given in (3.28) with the functions $H_\beta(\pm x + k\pi a, y)$, $k = 0, 1, 2$, given in (3.9). The proof follows by applying [14, theorem 5] to the integrals $H_\beta(x,y)$ and $H_\beta(\pi a - x, y)$. □

5. Concluding remarks

The singularly perturbed convection–diffusion problem (P) has been defined on a rectangle by means of discontinuous Dirichlet boundary conditions with two points of discontinuity located at the two lower corners of the domain. We have obtained a series representation of the solution susceptible to an asymptotic analysis. Then, an asymptotic expansion of the solution was obtained in the singular limit $\varepsilon \to 0^+$ and away from the points of discontinuity $(0,0)$ and $(\pi a, 0)$ (theorems 3.1 and 3.2). On the other hand, two asymptotic approximations of the solution near the points of discontinuity $(0,0)$ or $(\pi a, 0)$ (valid for $\varepsilon \geqslant \varepsilon_0 > 0$) have been given in theorem 4.1.

The asymptotic expansion in the singular limit shows that the main contribution from the data's discontinuities to the shape of the solution on the singular layers is contained in a certain combination of error functions, exponential functions and characteristic functions (equations (3.2) and (3.34)). This combination is necessary to approach the behaviour of the solution in the interior/boundary layers of width $\mathcal{O}(\sqrt{\varepsilon})$ or in the boundary layer of width $\mathcal{O}(\varepsilon)$. On the other hand, the asymptotic approximations near the discontinuities (equations (4.1) and (4.2)) show that the points of discontinuity at the boundary are smoothed inside the domain by means of simple linear functions of the polar angle.

In addition to the examples of unbounded domains analysed in the literature, we have found that the error function plays a fundamental role in the approximation of the solution of the problem analysed in this paper defined in a bounded domain. The error function seems to be a universal approximant not only in unbounded domains, but also in bounded domains. In this case the approximation is a little more complicated because it involves two or three error functions. The analysis is also more complicated because it involves a previous asymptotic analysis of a Fourier series.

We suspect that, as in the problem analysed here, the error function plays a fundamental role in the approximation of the solution of many singularly perturbed convection–diffusion problems with discontinuities in the boundary conditions (problems defined over more general domains and by more general coefficients). This will be the subject of further investigation.

Acknowledgments

We thank Nico Temme for his useful comments and improving suggestions. J.L.L. acknowledges financial support from the Secretaría de Estado de Universidades e Investigación, Servicio de Acciones de Promoción y Movilidad, res. 09/03/04.

References

1 M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions* (New York: Dover, 1970).
2 L. P. Cook and G. S. S. Ludford. The behavior as $\varepsilon \to 0^+$ of solutions to $\varepsilon\nabla^2 w = \partial w/\partial y$ on the rectangle $0 \leqslant x \leqslant l$, $|y| \leqslant 1$. *SIAM J. Math. Analysis* **4** (1973), 161–184.
3 E. M. de Jager and J. Furu. *The theory of singular perturbations* (Amsterdam: North-Holland, 1996).
4 W. Eckhaus. *Matched asymptotic expansions and singular perturbations* (Amsterdam: North-Holland, 1973).
5 W. Eckhaus and E. M. de Jager. Asymptotic solutions of singular perturbation problems for linear differential equations of elliptic type. *Arch. Ration. Mech. Analysis* **23** (1966), 26–86.
6 J. Grasman. On singular perturbations and parabolic boundary layers. *J. Engng Math.* **2** (1968), 163–172.
7 J. Grasman. An elliptic singular perturbation problem with almost characteristic boundaries. *J. Math. Analysis Applic.* **46** (1974), 438–446.
8 G. W. Hedstrom and A. Osterheld. The effect of cell Reynolds number on the computation of a boundary layer. *J. Computat. Phys.* **37** (1980), 399–421.

9 P. W. Hemker. A singularly perturbed model problem for numerical computation. *J. Computat. Appl. Math.* **76** (1996), 277–285.

10 A. M. Il'in. *Matching of asymptotic expansions of solutions of boundary-value problems* (Providence, RI: American Mathematical Society, 1992).

11 R. B. Kellogg. Corner singularities and singular perturbations. *Ann. Univ. Ferrara Sez. VII* **47** (2001), 177–206.

12 J. Kevorkian. *Partial differential equations: analytical solution techniques* (Springer, 1990).

13 J. Kevorkian and J. D. Cole. *Multiple scale and singular perturbation methods* (Springer, 1996).

14 J. L. López and E. Pérez Sinusía. Asymptotic expansions for two singularly perturbed convection–diffusion problems with discontinuous data: the quarter plane and the infinite strip. *Stud. Appl. Math.* **112** (2004), 57–89.

15 J. L. López and E. Pérez Sinusía. Analytic approximation for a singularly perturbed convection–diffusion problems with discontinuous data in a half-infinite strip. *Acta Appl. Math.* **82** (2004), 101–117.

16 J. L. López and E. Pérez Sinusía. Asymptotic approximations for a singularly perturbed convection–diffusion problem with discontinuous data in a sector. *J. Computat. Appl. Math.* **181** (2005), 1–23.

17 R. E. O'Malley. *Introduction to singular perturbation* (Academic, 1974).

18 C. H. Ou and R. Wong. On a two-point boundary-value problem with spurious solution. *Stud. Appl. Math.* **111** (2003), 377–408.

19 S.-D. Shih. A novel uniform expansion for a singularly perturbed parabolic problem with corner singularity. *Meth. Applic. Analysis* **3** (1996), 203–227.

20 N. M. Temme. Analytical methods for a singular perturbation problem: the quarter plane. CWI Report no. 125 (1971).

21 N. M. Temme. Analytical methods for a singular perturbation problem in a sector. *SIAM J. Math. Analysis* **5** (1974), 876–887.

22 N. M. Temme. Analytical methods for a selection of elliptic singular perturbation problems. *Recent advances in differential equations* (ed. H.-H. Dai and P. L. Sachdev). Pitman Research Notes in Mathematics, vol. 386, pp. 131–148 (New York: Longman, 1998).

23 R. Wong. *Asymptotic approximations of integrals* (Academic, 1989).

(*Issued 23 February 2007*)