Asymptotic Expansions for Two Singularly Perturbed Convection–Diffusion Problems with Discontinuous Data: The Quarter Plane and the Infinite Strip

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We consider a singularly perturbed convection–diffusion equation, $-\epsilon\Delta u + \vec{v}\cdot\vec{\nabla}u = 0$, defined on two domains: a quarter plane, $(x, y) \in (0, \infty) \times (0, \infty)$, and an infinite strip, $(x, y) \in (-\infty, \infty) \times (0, 1)$. We consider for both problems discontinuous Dirichlet boundary conditions: $u(x, 0) = 0$ and $u(0, y) = 1$ for the first one and $u(x, 0) = \chi_{[a,b]}(x)$ and $u(x, 1) = 0$ for the second. For each problem, asymptotic expansions of the solution are obtained from an integral representation in two limits: (a) when the singular parameter $\epsilon \to 0^+$ (with fixed distance r to the discontinuity points of the boundary condition) and (b) when that distance $r \to 0^+$ (with fixed ϵ). It is shown that in both problems, the first term of the expansion at $\epsilon = 0$ is an error function or a combination of error functions. This term characterizes the effect of the discontinuities on the ϵ-behavior of the solution and its derivatives in the boundary or internal layers. On the other hand, near the discontinuities of the boundary condition, the solution $u(x, y)$ of both problems is approximated by a linear function of the polar angle at the discontinuity points.

1. Introduction

Mathematical models that involve a combination of convective and diffusive processes are quite important in all of science, engineering, and other fields

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where mathematical modeling is required. Very often the dimensionless parameter that measures the relative strength of the diffusion is quite small. This implies that thin boundary and interior layers are present in the solution and singular perturbation problems arise. This kind of problem appears, for example, in fluid or gas dynamics [20, 32] heat transfer [2, 3], theory of plates and shells [18], or magnetohydrodynamic flow [10, 25]. An extensive selection of singularly perturbed convection–diffusion problems of the physics or engineering may be found in [22]: pollutant dispersal in a river estuary, vorticity transport in the incompressible Navier–Stokes equations, atmospheric pollution, groundwater transport, turbulence transport, etc.

Mathematically speaking, a singularly perturbed convection–diffusion problem is a boundary value problem of second-order in which the coefficients of the second order derivatives are small. In this paper, we focus our attention on two-dimensional linear elliptic problems of the form: find a function $u \in \mathcal{C}(\bar{\Omega}) \cap \mathcal{D}^2(\Omega)$ such that

$$\begin{cases} -\epsilon\Delta u + \vec{v}\cdot\vec{\nabla} u = f(x), & x \in \Omega \subset \mathbb{R}^2, \\ \quad u(x)|_{\partial\Omega} = g(\tilde{x}), & \tilde{x} \in \partial\Omega, \end{cases} \tag{P}$$

where ϵ is a small parameter and g represents the Dirichlet boundary conditions of the problem.

The location and shape of the boundary layers of u depend, among other things, on the prescribed velocity field $\vec{v}$, on the shape of the boundary $\partial\Omega$ and on the existence of discontinuities in $g(\tilde{x})$. For example, regular boundary layers of size $\mathcal{O}(\epsilon)$ appear on the outflow boundary, whereas parabolic boundary layers of size $\mathcal{O}(\sqrt{\epsilon})$ appear along the characteristic boundaries. For more details on the shape and nature of boundary layers see for example [5–7, 12, 13] and references therein.

The knowledge of an asymptotic expansion for the solution may help in the development of a suitable numerical method for these kind of problems because it gives the qualitative behavior of the solution [33, p. 6]. An ϵ-uniformly convergent method requires the analysis of uniform convergence and then, accurate error bounds for the local error. The accuracy of these error bounds depends on the precision in the approximation given by the first terms of the asymptotic expansion. The design of the numerical technique is based on the exact integration of the first terms of the asymptotic expansion or of functions that have a similar behavior in the singular layer. Along this line, some references which propose exponential fitting techniques or special meshes based on asymptotic expansions are [4, 15]. Classical references are [9, 21].

There is an extensive literature devoted to the construction of asymptotic expansions of the solution of singular perturbation problems based on matching techniques. The book of Il'in [13] contains a quite exhaustive and general analysis for different equations and domains. Other important references are for example [6, 8, 17, 23, 33]. But a perturbative analysis based on an expansion of the solution in powers of the perturbation parameter does not always work for discontinuous Dirichlet boundary conditions $g(\tilde{x})$ [31]. This is so because the coefficients of the expansion contain derivatives of $g(\tilde{x})$, whereas the solution of the elliptic problem (P) is smooth inside the domain. Therefore, it is of interest to consider an approach based on the exact solution [31].

The solution of (P) may be decomposed in the form $u = u_1 + u_2$, where u_1 is the solution of (P) with $f(x) = 0$ and u_2 is the solution of (P) with $g(\tilde{x}) = 0$. Therefore, the effect of the discontinuous data on the solution u is contained in u_1 and must be analyzed for the homogeneous problem (P) with $f(x) = 0$.

There are some useful examples of approaches based on an integral representation for the solution of some singular convection–diffusion problems with constant coefficients. Grasman analyzed the problem $\epsilon\Delta u - \partial_y u = 0$ on a quarter plane with continuous Dirichlet boundary conditions [11]. The solution u is represented as a Bessel transform by using the Green's function for the associated Helmholtz problem. A uniformly valid asymptotic expansion in the whole quarter plane is derived for $\epsilon \to 0$. Cook and Ludford analyze the same problem on a rectangle by considering first the problem defined on a semi-infinite strip [5]. Asymptotic expansions for $\epsilon \to 0$ are obtained from the exact solution written as a Fourier sine transform of the Green's function of the transformed problem.

But, in order to analyze boundary layers originated by discontinuous boundary data, it is interesting to find the exact solution of a problem with a discontinuity in the boundary condition. Hedstrom and Osterheld [12] studied the problem $\epsilon\Delta u - \partial_y u = 0$ on the positive quarter plane with boundary conditions $u(x, 0) = 0$ and $u(0, y) = 1$. They obtained the first two terms of the asymptotic expansion of u for $\epsilon \to 0$ from a Fourier integral representation of u. The first term of this expansion is an error function. A more detailed investigation on this subject has been developed by Temme [29–31]. The problem $\epsilon\Delta u - \partial_y u = 0$ on the positive quarter plane with boundary conditions $u(x, 0) = 0$ and $u(0, y) = \phi(y)$ is analyzed in [29]. An integral representation for u is obtained from the associated Helmholtz equation. A complete asymptotic expansion of u for $\epsilon \to 0$ is derived from this integral representation for some particular cases of boundary conditions $\phi(y)$. The same problem, but in a generic sector, is considered in [30], where an integral representation for u is obtained from an integral representation of the solution of the associated Helmholtz equation. Different asymptotic expansions as $\epsilon \to 0$ are obtained depending on the angle of the sector and again the error function plays an important role in the analysis. A similar problem defined in the interior of a circle is analyzed in [31].

Using similar techniques to those of Temme, Shih has analyzed parabolic equations with discontinuous Dirichlet boundary conditions defined over certain unbounded domains. The problem $u_t - \epsilon u_{xx} + pu_x = 0$, with initial condition $u(x, 0) = f(x)$, boundary condition $u(0, t) = g(t)$, and defined on the quarter plane $x > 0$, $t > 0$, is considered in [27]. It is shown there that the solution $u(x, t)$ has an asymptotic expansion in powers of ϵ whose coefficients are error functions. Similar results have been obtained in [26, 28] for the same problem, but with a more general coefficient for the term u_x and a nonhomogeneous equation: $u_t - \epsilon u_{xx} + p(x, t)u_x = F(x, t)$. It is shown that the solution is approximated by the integral of the error function.

The role of the error function in problems with discontinuous boundary data has also been pointed out in [16, 17]. The exact solution of the problem $u_t - u_{xx} = 0$ in the quarter plane $x > 0$, $t > 0$, with initial condition $u(x, 0) = c$ and boundary condition $u(0, t) = 0$ is an error function [16, Section 1.4.2]. The solution of the problem $\epsilon\Delta u = u_y$ in the unit square with discontinuous boundary data is approximated by error functions [16, Section 8.3.3]. The same conclusion is obtained in [17, Section 3.1] for a general elliptic equation with constant coefficients defined on a more general domain. Apart from discontinuities in the boundary condition, another source of error functions in the dominant part of the solution of convection–diffusion problems is a change of sign in the convection vector [14, Section 6.2.3].

In this paper, we try to shed light on the influence that the discontinuities of the boundary conditions have on the boundary or interior layers. For that purpose, we consider two homogeneous singular perturbation problems with discontinuous boundary data: problems (P_1) and (P_2) defined below, in Sections 2 and 3 respectively. Problem (P_1) displays a boundary or an interior layer, whereas (P_2) displays boundary and interior layers. As in the references mentioned, the starting point is an integral representation for the solution. From this integral, we derive complete asymptotic expansions for the solution, not only in the singular limit $\epsilon \to 0^+$, but also in the limit $r \to 0^+$, where r represents the distance to the discontinuity at the boundary. Then, we approximate the solution on the whole domain, including the neighborhood of the discontinuity point(s). Moreover, error bounds are obtained at any order of approximation for both expansions.

In Sections 2 and 3, we study the convection–diffusion problem in a quarter plane and in an infinite strip, respectively. Some comments and a few conclusions are postponed to Section 4.

2. Convection–diffusion in a quarter plane

We consider a singularly perturbed convection–diffusion problem in the first quadrant with Dirichlet boundary conditions:

$$\begin{cases} -\epsilon\Delta U + \vec{v}\cdot\vec{\nabla}U = 0, & (x, y) \in \Omega_1 \equiv (0, \infty) \times (0, \infty) \\ U(x, 0) = f(x), U(0, y) = g(y), & U \in \mathcal{C}(\bar{\Omega}_1 \backslash \{(0, 0)\}) \cap \mathcal{D}^2(\Omega_1), \end{cases}$$

where $\vec{v} \equiv |\vec{v}|(\sin\beta, \cos\beta)$ is a constant vector and $\epsilon > 0$ is a small parameter. If the boundary conditions are compatible, $f(0) = g(0)$, asymptotic approximations of $U(x, y)$ may be obtained by means of standard techniques. For example, we can try the method of matched asymptotic expansions [23], or asymptotic methods of integrals using an integral representation of $U(x, y)$ [34]. If $f(0) \neq g(0)$, we can decompose the problem into two simpler problems: one problem with compatible boundary conditions plus a second problem with a discontinuous boundary condition at the corner point (0, 0):

$$\begin{cases} -\epsilon\Delta U + \vec{v}\cdot\vec{\nabla}U = 0, & (x, y) \in \Omega_1 \\ U(x, 0) = 0, U(0, y) = 1, & U \in \mathcal{C}(\bar{\Omega}_1\backslash\{(0, 0)\}) \cap \mathcal{D}^2(\Omega_1), \end{cases} \tag{P$_1$}$$

Therefore, in the remainder of this section we consider the problem (P_1).

Based on the method proposed by Temme for $\beta = 0$ in [31], after the change of the dependent variable $U(x, y) = F(x, y)\exp(\vec{v}\cdot\vec{r}/(2\epsilon))$, where $\vec{r} \equiv (x, y)$, (P_1) is transformed to the Helmholtz equation for $F(x, y)$:

$$\begin{cases} \triangle F - w^2 F = 0, & (x, y) \in \Omega_1 \\ F(x, 0) = 0, F(0, y) = e^{-wy\cos\beta}, & F \in \mathcal{C}(\bar{\Omega}_1\backslash\{(0, 0)\}) \cap \mathcal{D}^2(\Omega_1), \end{cases} \tag{1}$$

where $w \equiv |\vec{v}|/(2\epsilon)$.

The problem (P_1) may not have a unique solution unless we impose a convenient condition upon $U(x, y)$ concerning its growth at infinity. For that purpose we use the following lemma, proved in [30],

LEMMA 1. *Assume that F is a regular function in a region Ω satisfying:* (i) $\Delta F - w^2 F = 0$, (ii) $F = 0$ *on the boundary of* Ω, (iii) $\lim_{r\to\infty} F(x, y)/I_0(wr) = 0$, *where* $r \equiv \sqrt{x^2 + y^2}$ *and* $I_0(z)$ *is the modified Bessel function of order* 0. *Then* $F = 0$ *in the whole domain* Ω.

The appearance of the modified Bessel function $I_0(wr)$ in Lemma 1 is not surprising, because $I_0(wr)$ satisfies the Helmholtz equation $\Delta F - w^2 F = 0$. It has an exponential behavior at ∞: $I_0(wr) = \mathcal{O}(\exp(wr)/\sqrt{2\pi wr})$ when $r \to \infty$. Therefore, just the additional condition $F = \mathrm{o}(\exp(wr)/\sqrt{2\pi wr})$ when $r \to \infty$ is enough to guarantee uniqueness of (1) (a more general discussion can be found in [13, p. 150, Theorem 2.1]).

Then, the problem: find $U \in \mathcal{C}(\bar{\Omega}_1\backslash\{(0, 0)\}) \cap \mathcal{D}^2(\Omega_1)$ such that

$$\begin{cases} -\epsilon\Delta U + \vec{v}\cdot\vec{\nabla}U = 0, & (x, y) \in \Omega_1 \\ U(x, 0) = 0, \quad U(0, y) = 1, & \lim\limits_{r\to\infty} U(x, y)/I_0(wr) = 0, \end{cases} \tag{P$'_1$}$$

has a unique solution.

In the following proposition we obtain the solution of (P'_1) by means of an integral representation.

Proposition 1. *Consider polar variables* $x = r\cos\phi$, $y = r\sin\phi$ *with* $0 < r < \infty$, $0 \le \phi \le \pi/2$ *and write* $w \equiv |\vec{v}|/(2\epsilon)$. *Then, for* $0 \le \beta < \pi/2$, *the solution* $U_\beta(x, y)$ of (P_1') *is*

$$U_\beta(x,y) = \chi_{(\pi/2,\pi)}(\phi+\beta) + \tfrac{1}{2}\delta_{\phi+\beta,\pi/2} + e^{wr\sin(\phi+\beta)} I(r,\phi,\beta), \qquad (2)$$

where χ *represents the characteristic function*, δ *the Kronecker delta*, *so*

$$\chi_{(a,b)}(x) \equiv \begin{cases} 1 & \text{if } x \in (a,b) \\ 0 & \text{if } x \notin (a,b), \end{cases} \qquad \delta_{a,b} \equiv \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \ne b \end{cases}$$

and

$$I(r,\phi,\beta) \equiv \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\infty} e^{-wr\cosh t} \frac{\sinh(t+\mathrm{i}\phi)\cosh(t+\mathrm{i}\phi)\,dt}{\cos^2\beta + \sinh^2(t+\mathrm{i}\phi)}. \qquad (3)$$

When $\phi + \beta = \pi/2$, *this integral must be understood as a principal value integral.*

Proof: By applying the sine transform in the variable y to equation (1), we obtain an explicit representation of $F(x, y)$ valid for $|\beta| < \pi/2$:

$$F(x,y) = \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\infty} e^{\mathrm{i}ty\cos\beta - x\sqrt{w^2+t^2\cos^2\beta}} \frac{t\,dt}{t^2+w^2}.$$

(The condition $F(0, y) = e^{-wy\cos\beta}$ is checked by using the Cauchy residue theorem. But it holds only if $\cos\beta > 0$, that is, $|\beta| < \pi/2$.) Then, the function $U_\beta(x, y) \equiv e^{\vec{v}\cdot\vec{r}/(2\epsilon)} F(x, y)$, with $F(x, y)$ defined above, is the solution of (P_1'). After the change of variable $t\cos\beta = w\sinh u$ and using polar variables, the solution $U_\beta(x, y)$ is

$$U_\beta(x,y) = \frac{e^{wr\sin(\phi+\beta)}}{\pi \mathrm{i}} \int_{-\infty}^{\infty} e^{-wr\cosh(u-\mathrm{i}\phi)} \frac{\sinh u\cosh u\,du}{\sinh^2 u + \cos^2\beta}. \qquad (4)$$

In order to make the following discussion simpler, we further restrict the angle β to $0 \le \beta < \pi/2$. The poles of the integrand are situated at the points $u = \mathrm{i}(\pi/2 \pm \beta) + \mathrm{i}n\pi, n \in \mathbb{Z}$. Then, if we shift the contour in the complex plane up to the line $\Im u = \phi$, we just cross the pole $u = \mathrm{i}(\pi/2 - \beta)$ if $\phi > \pi/2 - \beta$ and we do not cross any pole if $\phi < \pi/2 - \beta$. On the other hand, the real part of the exponent of the integrand is $-rw\cosh(\Re u)\cos(\phi - \Im u)$. Therefore, shifting the integration contour up to the line $\Im u = \phi$ and using the Cauchy residue theorem we obtain the desired result (see Figure 1). ■

Observation 1. The explicit representation given in Proposition 1 is only valid when the angle β between the convection vector $\vec{v}$ and the y-axis is restricted to the interval $[0, \pi/2]$. Nevertheless, an explicit integral representation for the solution $U(x, y)$ of the problem (P_1') whatever the direction of $\vec{v}$ is, may be

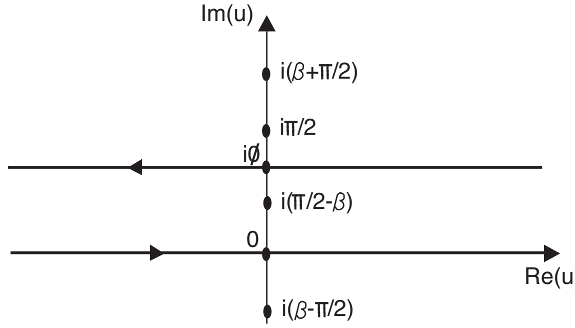

Figure 1. The integrand in (4) has just a single pole situated between the straight lines $\Im u = 0$ and $\Im u = \pi/2$: $u = \mathrm{i}(\pi/2 - \beta)$.

obtained by means of symmetry arguments:

$$U(x, y) = \begin{cases} U_\beta(x, y) & \text{if } 0 \le \beta < \pi/2, \\ e^{2wx \sin\beta} U_{-\beta}(x, y) & \text{if } -\pi/2 < \beta \le 0, \\ 1 - e^{2wy \cos\beta} U_{\beta-\pi/2}(y, x) & \text{if } \pi/2 \le \beta < \pi, \\ e^{2wx \sin\beta}\left[1 - e^{2wy \cos\beta} U_{-\beta-\pi/2}(y, x)\right] & \text{if } -\pi < \beta \le -\pi/2, \\ 1 + e^{-2wy}\left[U_0(x, y) - 1\right] & \text{if } \beta = \pi, \end{cases}$$

where $U_\beta(x, y)$ is given in (2) and (3). Therefore, in the remainder of this section, we restrict ourselves to $\beta \in [0, \pi/2)$.

From the representation of $U(x, y)$ given above is difficult to infer the behavior of the solution of (P_1'). But, for $\epsilon \to 0^+$, $r \ge r_0 > 0$, we can approximate the integral $I(r, \phi, \beta)$ in (3) by an error function plus an asymptotic expansion in powers of ϵ, whereas for $r \to 0^+$, $\epsilon \ge \epsilon_0 > 0$, we can approximate $I(r, \phi, \beta)$ by an asymptotic expansion in powers of r. This is the subject of the two following subsections.

2.1. Asymptotic expansion of U(x, y) in the singular limit

In this section, we denote by Ω_1^* the upper half plane indented at the point (0, 0) (see Figure 2a): $\Omega_1^* \equiv \Omega_1 \backslash D_{r_0}(0, 0)$.

THEOREM 1. *Let* $w \equiv |\vec{v}|/(2\epsilon)$ *and* $x = r\cos\phi$, $y = r\sin\phi$. *Then, for* $(x, y) \in \Omega_1^*$, *the solution* $U_\beta(x, y)$ *of* (P_1') *reads*

$$U_\beta(x, y) = U_\beta^0(x, y) + \frac{e^{wr(\sin(\phi+\beta)-1)}}{\pi\sqrt{2wr}} U_\beta^1(r, \phi), \tag{5}$$

where

$$U_0^0(x, y) \equiv \operatorname{erfc}\left(\sqrt{w(r - y)}\right), \tag{6}$$

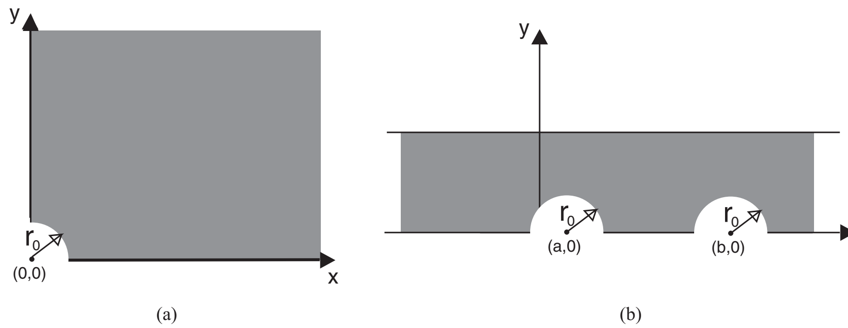

Figure 2. (a) Indented region Ω_1^* in Theorem 1. (b) Indented region Ω_2^* in Theorems 3 and 4.

$$U_\beta^0(x,y) \equiv \frac{1}{2}\begin{cases} \operatorname{erfc}\left(\sqrt{wr(1-\sin(\phi+\beta))}\right) & \text{if } \phi+\beta < \dfrac{\pi}{2} \\ 1 & \text{if } \phi+\beta = \dfrac{\pi}{2}, \qquad \beta > 0. \\ 2-\operatorname{erfc}\left(\sqrt{wr(1-\sin(\phi+\beta))}\right) & \text{if } \phi+\beta > \dfrac{\pi}{2} \end{cases} \tag{7}$$

The function $U_\beta^1(r,\phi)$ *has the asymptotic expansion*

$$U_\beta^1(r,\phi) = \sum_{k=0}^{n-1}(-1)^k T_k(\phi,\beta)\frac{\Gamma(k+1/2)}{(2wr)^k} + R_n(r,\phi,\beta), \tag{8}$$

where empty sums must be understood as zero. For $\phi+\beta \neq \pi/2$ *and* $k = 0, 1, 2, \dots,$

$$\begin{aligned} T_k(\phi,\beta) \equiv{} & \frac{\cos(\phi+\beta)\cos(\phi-\beta)}{\sin(2\beta)}A_k + 4\cot(2\beta)(A_{k-2}-A_{k-1}) \\ & - \frac{1+\delta_{\beta,0}}{2\sin^{2k+1}\left(\frac{1}{2}\left(\frac{\pi}{2}-\phi-\beta\right)\right)}, \end{aligned} \tag{9}$$

where $A_{-1} = A_{-2} = 0$ *and, for* $k = 0, 1, 2, \dots,$

$$A_k \equiv \sum_{j=0}^{k}\frac{2^j(1/2)_{k-j}}{(k-j)!}\sum_{l=0}^{[j/2]}\binom{j+1}{2l+1}\left[\frac{\sin^{2l}(\phi+\beta)}{\cos^{2j+2}(\phi+\beta)} - \frac{\sin^{2l}(\phi-\beta)}{\cos^{2j+2}(\phi-\beta)}\right]. \tag{10}$$

On the other hand, if $\phi+\beta=\pi/2$ *and* $\phi\neq\pi/2$,

$$T_k(\phi,\beta)\equiv\frac{\sin(2\phi)}{4}\sum_{j=0}^{k}\frac{(1/2)_{k-j}}{(k-j)!}\left[\frac{1}{\cos^{2j+2}\phi}-\frac{1}{\sin^{2j+2}\phi}\right],\qquad k=0,1,2,\ldots \tag{11}$$

An error bound for the remainder term is given by

$$|R_n(r,\phi,\beta)|\leq\frac{M_n(\phi,\beta)\Gamma(n+1/2)}{(2wr)^n}, \tag{12}$$

where

$$M_n(\phi,\beta)\equiv\left|\frac{\cos(\phi+\beta)\cos(\phi-\beta)}{\sin(2\beta)}\right||A_n|+4|\cot(2\beta)|\,\mathrm{Max}\{|A_{n-1}|,|A_{n-2}|\}$$
$$+\frac{1+\delta_{\beta,0}}{2\left|\sin^{2n+1}\left(\frac{1}{2}\left(\frac{\pi}{2}-\phi-\beta\right)\right)\right|} \tag{13}$$

if $\phi+\beta\neq\pi/2$ *and* $M_n(\phi,\beta)\equiv|T_n|$ *if* $\phi+\beta=\pi/2$.

Proof: The asymptotic features of the integral (3) are: (i) there is a saddle point at $t=0$, (ii) there is a pole located at $\sin(\phi-\mathrm{i}t)=\cos\beta$, (iii) both coalesce when $\phi+\beta\to\pi/2$. Uniform asymptotic expansions of this kind of integrals are obtained by using the error function as a basic approximant [34, Chapter 7, Section 2]. Therefore, we split off the pole of the integrand at $t=\mathrm{i}(\pi/2-\phi-\beta)$:

$$\frac{\sinh(t+\mathrm{i}\phi)\cosh(t+\mathrm{i}\phi)}{\cos^2\beta+\sinh^2(t+\mathrm{i}\phi)}=\frac{1+\delta_{\beta,0}}{4\sinh\frac{1}{2}(t+\mathrm{i}(\phi+\beta-\pi/2))}+f(t,\phi,\beta),$$

with the obvious definition of $f(t,\phi,\beta)$. Using the complementary error function representation [31],

$$e^{-r\cos\alpha}\mathrm{erfc}\left(\sqrt{2r}\sin\frac{\alpha}{2}\right)=\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}e^{-r\cosh t}\frac{dt}{\sinh\frac{1}{2}(t-\mathrm{i}\alpha)},\qquad 0<\alpha<2\pi, \tag{14}$$

we obtain

$$I(r,\phi,\beta)=\frac{1}{2}\mathrm{sign}\left(\frac{\pi}{2}-\phi-\beta\right)(1+\delta_{\beta,0})e^{-wr\sin(\phi+\beta)}$$
$$\times\,\mathrm{erfc}\left(\sqrt{wr(1-\sin(\phi+\beta))}\right)+\frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty}e^{-wr\cosh t}f(t,\phi,\beta)\,dt,$$

where sign(0) must be understood as zero. Therefore, from Equation (2) we obtain (5), where $U^0_\beta(x, y)$ is given in (6) and (7) and

$$U^1_\beta(r, \phi) \equiv -\mathrm{i}e^{wr}\sqrt{2wr}\int_{-\infty}^{\infty} e^{-wr\cosh t} f(t, \phi, \beta)\, dt. \tag{15}$$

When $\phi + \beta = \pi/2$, this last integral is a principal value integral. To obtain the asymptotic expansion of $U^1_\beta(r, \phi)$ for large w and bounded $r \geq r_0 > 0$, we perform the change of variable $\sinh(t/2) \equiv u$ and remove the odd part of $f(t, \phi, \beta)$ from (15). We obtain

$$U^1_\beta(r, \phi) = 2\sqrt{2wr}\int_0^{\infty} e^{-2wru^2} g(u^2, \phi, \beta)\, du, \tag{16}$$

where

$$\begin{aligned} g(u, \phi, \beta) &= [4\cos(2\beta)u(u+1) + \cos^2(\beta) - \sin^2(\phi)]\frac{g_1(u)}{\sin(2\beta)} \\ &\quad - \frac{1+\delta_{\beta,0}}{2}\left(\sin\frac{1}{2}\left(\frac{\pi}{2} - \phi - \beta\right)\right) g_2(u), \\ g_1(u) &\equiv \frac{1}{\sqrt{1+u}}\frac{\sin(2\phi)\sin(2\beta)}{(2u+1+s_+)(2u+1-s_+)(2u+1+s_-)(2u+1-s_-)}, \end{aligned} \tag{17}$$

where we have denoted $s_\pm \equiv \sin(\phi \pm \beta)$, and

$$g_2(u) \equiv \frac{1}{u + \sin^2\frac{1}{2}\left(\frac{\pi}{2} - \phi - \beta\right)}.$$

Splitting $g_1(u)$ into simple fractions, we derive the Taylor expansion of $g(u, \phi, \beta)$

$$g(u, \phi, \beta) = \sum_{k=0}^{n-1}(-1)^k T_k(\phi, \beta)u^k + r_n(u, \phi, \beta), \tag{18}$$

where $r_n(u, \phi, \beta) = \mathcal{O}(u^n)$ as $u \to 0$ and coefficients $T_k(\phi, \beta)$ are given in (9)–(11). Introducing the expansion (18) in (16) we obtain (8), where the remainder term

$$R_n(r, \phi, \beta) \equiv \int_0^{\infty} e^{-2wru^2} r_n(u^2, \phi, \beta)\, du. \tag{19}$$

Finally, we shall derive the error bound (12) which shows the asymptotic character of the expansion (8). Consider first $\phi + \beta \neq \pi/2$. Applying the binomial formula for the derivative of a product we realize that the nth u-derivative of each of the two rational functions $g_1(u)$ and $g_2(u)$ has the same sign as $(-1)^n$ $\forall u \in [\,0, \infty)$. By using the Lagrange formula for the remainders $r_n^{(1)}(u)$ and $r_n^{(2)}(u)$ in the Taylor expansions of these two functions at $u = 0$,

$$g_1(u) = \sum_{k=0}^{n-1} (-1)^k A_k u^k + r_n^{(1)}(u),$$

$$g_2(u) = \sum_{k=0}^{n-1} \frac{(-u)^k}{\sin^{2k+2} \frac{1}{2}\left(\frac{\pi}{2} - \phi - \beta\right)} + r_n^{(2)}(u),$$

we realize that two consecutive remainder terms in these expansions have opposite sign [19, Lemmas 3 and 4]: $r_n^{(j)}(u) r_{n+1}^{(j)}(u) < 0$, $j = 1, 2$ $\forall u \in [0, \infty)$, $n \in \mathbb{N} \cup \{0\}$. After applying the error test [34, p. 38], we obtain

$$\left|r_n^{(1)}(u)\right| \le |A_n| u^n, \quad \left|r_n^{(2)}(u)\right| \le \frac{u^n}{\sin^{2n+2} \frac{1}{2}\left(\frac{\pi}{2} - \phi - \beta\right)}.$$

Therefore, the remainder term $r_n(u, \phi, \beta)$ for $g(u, \phi, \beta)$ satisfies

$$|r_n(u, \phi, \beta)| \le M_n(\phi, \beta) u^n, \tag{20}$$

where $M_n(\phi, \beta)$ is given by (13). If $\phi + \beta = \pi/2$, then $g(u, \phi, \beta)$ is itself a rational function and, following the same reasoning as before, we obtain (20) with $M_n(\phi, \beta) = |T_n|$. Introducing (20) in (19) we obtain (12). ■

Observation 2. When $\phi = \pi/2$ and $\beta = 0$ or $\phi = \beta = \pi/4$, the integrand in (15) is an odd function of t and the integral $U_\beta^1(r, \phi)$ vanishes. Therefore, for $\beta = 0$, the exact solution verifies $U_0(0, y) = U_0^0(0, y) = 1$ (as it should be) over the characteristic line $\phi = \pi/2$. On the other hand, for $\beta = \pi/4$, $U_{\pi/4}(x, x) = U_{\pi/4}^0(x, x) = 1/2$ over the characteristic line $\phi = \pi/4$. Moreover, for $\beta > 0$, $U_\beta^0(x, y)$ approximately satisfies the boundary conditions: $U_\beta^0(x, 0) = \mathcal{O}(e^{wx(\sin\beta - 1)})$ and $U_\beta^0(0, y) = 1 + \mathcal{O}(e^{wy(\cos\beta - 1)})$, $w \to \infty$.

Remark 1. From (5), (8), and (12) $U_\beta(x, y) = U_\beta^0(x, y) + \mathcal{O}(\sqrt{\epsilon})$ as $\epsilon \to 0^+$, $r \ge r_0 > 0$ and then, the first-order approximation is an error function. This error function exhibits an interior layer of width $\mathcal{O}(\sqrt{\epsilon})$ and parabolic level lines of equation $r - \vec{v} \cdot \vec{r} = C \cdot \epsilon$ near the half-straight line $t\vec{v}, t > 0$ (see Figure 3).

2.2. *Asymptotic expansion of $U(x, y)$ near the discontinuity*

The asymptotic expansion given in (8) breaks down when $r \to 0^+$. The asymptotic approximation of $U_\beta(x, y)$ near the point $(0, 0)$ requires a completely different analysis. An asymptotic approximation of $U_\beta(x, y)$ near the corner point with bounded ϵ ($r \to 0^+$, $\epsilon \ge \epsilon_0 > 0$) is given in the following theorem.

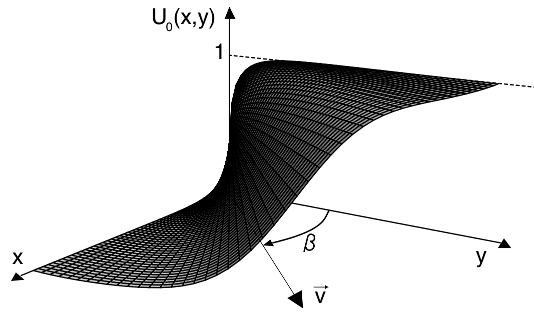

Figure 3. Graph of the first order approximation $U^0_{\pi/4}(x, y)$ to the solution of the problem (P'_1) for $\epsilon = 0.1$ and $\beta = \pi/4$. The convection vector $\vec{v}$ "drags" the discontinuity of the boundary condition at (0, 0) originating a parabolic layer along the direction $\vec{v}$.

THEOREM 2. *Let* $w \equiv |\vec{v}|/(2\epsilon)$ *and* $x = r\cos\phi$, $y = r\sin\phi$. *Then, for* $(x, y) \in \Omega_1$, *the solution* $U_\beta(x, y)$ *of* (P'_1) *satisfies*

$$U_\beta(x, y) = \frac{2\phi}{\pi} + \frac{rw}{\pi} e^{wr(\sin(\phi+\beta)-1)} U^1_\beta(r, \phi), \tag{21}$$

where

$$U^1_\beta(r, \phi) \equiv \frac{T_0}{rw}\left(1 - e^{wr(1-\sin(\phi+\beta))}\right) + \sum_{k=1}^{n-1} \frac{(-1)^k}{k!}\left[T_k - V_k \log(rw)\right](rw)^{k-1} + R_n(r, \phi, \beta), \tag{22}$$

where $V_1 = 0$ *and empty sums must be understood to be zero. For* $k = 2, 3, 4, \ldots$, *the remaining coefficients* V_k *are given by*

$$V_k \equiv \frac{(-2)^{k-1}}{\sin(2\beta)}\left[4\cos(2\beta)(A_{k+1} - A_k) + (\cos^2\beta - \sin^2\phi)A_{k-1}\right], \tag{23}$$

where $A_1 = A_2 = 0$ *and, for* $k = 3, 4, 5, \ldots$,

$$A_k \equiv \sum_{j=3}^{k} \frac{(1/2)_{k-j}}{2^j (k-j)!} \sum_{l=0}^{[(j-1)/2]} \binom{j}{2l+1} (\sin^{2l}(\phi - \beta) - \sin^{2l}(\phi + \beta)). \tag{24}$$

Coefficient T_0 *is given by*

$$T_0 = \begin{cases} 2\phi & \text{if } \phi + \beta < \pi/2, \\ 2\phi - \pi/2 & \text{if } \phi + \beta = \pi/2, \\ 2\phi - \pi & \text{if } \phi + \beta > \pi/2. \end{cases} \tag{25}$$

For $k = 1, 2, 3, \ldots,$

$$T_k = V_k \psi(k+1) + \frac{(-2)^{k-1}\Gamma(k+1/2)}{\sqrt{\pi}(k-1)!}[H'(k+1,\beta) + (\psi(k+1/2) - \psi(k) + \log 2)H(k+1,\beta)], \tag{26}$$

where

$$H(s,\beta) \equiv \cos(\phi+\beta)(M_2(s,\beta) - M_1(s,\beta)) + \cos(\phi-\beta)(M_2(s,-\beta) - M_1(s,-\beta)), \tag{27}$$

$$M_1(s,\beta) \equiv \sin^{2s-3}\left(\frac{\pi}{4} - \frac{\phi+\beta}{2}\right) F\left(\begin{matrix} s-1/2, 1/2 \\ 3/2 \end{matrix} \,\middle|\, \cos^2\left(\frac{\pi}{4} - \frac{\phi+\beta}{2}\right)\right)$$

and

$$M_2(s,\beta) \equiv \cos^{2s-3}\left(\frac{\pi}{4} - \frac{\phi+\beta}{2}\right) F\left(\begin{matrix} s-1/2, 1/2 \\ 3/2 \end{matrix} \,\middle|\, \sin^2\left(\frac{\pi}{4} - \frac{\phi+\beta}{2}\right)\right). \tag{28}$$

In these formulas, $F\left({a,b \atop c} \,|\, z\right)$ *denotes the Gauss hypergeometric function and* $H'(s,\beta)$ *the derivative of* $H(s,\beta)$ *with respect to* s.

For $n = 1, 2, 3, \ldots,$ *an error bound for the remainder term is given by*

$$|R_n(r,\phi,\beta)| \le \frac{nB_{n-1}(\phi,\beta) + B_n(\phi,\beta)(|\log(rw)| + n + e^{-1})}{n!}(rw)^{n-1}, \tag{29}$$

where

$$B_n(\phi,\beta) \equiv \frac{2^{n-1}}{|\sin(2\beta)|}\big[4|\cos(2\beta)|\,\mathrm{Max}\{|\bar{A}_{n+1}|, |\bar{A}_n|\} + |\cos^2\beta - \cos^2\phi||\bar{A}_{n-1}|\big] \tag{30}$$

with

$$\bar{A}_{-1} \equiv \frac{\bar{A}_2}{\cos^2(\phi+\beta)\cos^2(\phi-\beta)}, \qquad \bar{A}_2 \equiv \sin(2\phi)\sin(2\beta),$$

$$\bar{A}_0 \equiv \frac{\bar{A}_2}{\cos^2(\phi-\beta)}, \quad \bar{A}_1 \equiv \frac{\bar{A}_2}{1+\sin(\phi-\beta)}, \quad \text{and } \bar{A}_k = A_k \ \text{ for } k = 3, 4, 5, \ldots \tag{31}$$

Proof: After the change of variable $\cosh\ t = u + 1$ in definition (3) of $I(r,\phi,\beta)$ we have

$$I(r,\phi,\beta) = \frac{e^{-wr}}{\pi}\int_0^\infty e^{-wru} f(u,\phi,\beta)\,du, \tag{32}$$

where

$$f(u,\phi,\beta)=\left[\cos(2\beta)\left(1+\frac{2}{u}\right)+\frac{\cos^2\beta-\sin^2\phi}{u^2}\right]\frac{g(u)}{\sin(2\beta)}, \tag{33}$$

$$g(u)\equiv\frac{\sin(2\phi)\sin(2\beta)}{u^3\sqrt{1+\frac{2}{u}}\left(1+\frac{1+s_+}{u}\right)\left(1+\frac{1-s_+}{u}\right)\left(1+\frac{1+s_-}{u}\right)\left(1+\frac{1-s_-}{u}\right)}, \tag{34}$$

where $s_\pm$ are defined after Equation (17). Splitting $g(u)$ in simple fractions we obtain an expansion of $f(u,\phi,\beta)$ in inverse powers of u:

$$f(u,\phi,\beta)=\sum_{k=0}^{n-1}\frac{V_k}{u^{k+1}}+f_n(u), \tag{35}$$

where $f_n(u)=\mathcal{O}(u^{-n-1})$ as $u\to\infty$, $V_0=V_1=0$ and, for $k=2,3,4,\ldots$, the remaining V_k are given in (23) and (24). Applying [34, Chapter 6, Theorem 13(ii)] to the integral in (32) and using (2), we obtain

$$U_\beta(x,y)=\chi_{(\pi/2,\pi)}(\phi+\beta)+\frac{1}{2}\delta_{\phi+\beta,\pi/2}+\frac{e^{wr(\sin(\phi+\beta)-1)}}{\pi}\times\left[\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}\left[T_k-V_k\log(rw)\right](rw)^k+R_n(r,\phi,\beta)\right], \tag{36}$$

with the following expressions for the coefficients T_k and the remainder $R_n(r,\phi,\beta)$:

$$T_k\equiv V_k\psi(k+1)+\lim_{s\to k+1}\left\{M[f;s]+\frac{V_k}{s-k-1}\right\}, \tag{37}$$

where $M[f;s]$ denotes the Mellin transform of $f(u,\phi,\beta)$ at s, $\int_0^\infty u^{s-1}f(u,\phi,\beta)\,du$, or its analytic continuation. On the other hand, the remainder is

$$R_n(r,\phi,\beta)\equiv(rw)^n\int_0^\infty f_{n,n}(t)e^{-rwt}\,dt, \tag{38}$$

where

$$f_{n,n}(t)\equiv\frac{(-1)^n}{(n-1)!}\int_t^\infty(u-t)^{n-1}f_n(u)\,du. \tag{39}$$

Splitting $f(u,\phi,\beta)$ and using [24, p. 303, Eq. (24)], the Mellin transform $M[f;s]$ may be written as

$$M[f;s]=-\frac{2^{s-2}\Gamma(2-s)\Gamma(s-1/2)}{\sqrt{\pi}}H(s,\beta), \tag{40}$$

where $H(s,\ \beta)$ is given in (27) and (28). Then, using [1, Eq. (15.1.6)] we obtain $M[f;\ 1] = V_0$ with V_0 given in (25), and $M[f;\ 2] = V_1$, with V_1 given in (26) for $k = 1$. For $s \geq 3$, the Mellin transform $M[f;\ s]$ is defined by the analytic continuation of (40). In particular, when $s = k + 1 + \epsilon$, $k = 2, 3, 4, \ldots$, and ϵ a small positive number,

$$M[f; k+1+\epsilon] = \frac{(-2)^{k-1}\Gamma(k+1/2)}{\sqrt{\pi}(k-1)!}\left\{\frac{H(k+1,\beta)}{\epsilon} + H'(k+1,\beta) + [\log 2 - \psi(k) + \psi(k+1/2)] \times H(k+1,\beta) + \mathcal{O}(\epsilon)\right\}, \quad \epsilon \to 0^+. \tag{41}$$

On the other hand, the analytic continuation of $\int_0^\infty u^{s-1} f(u,\phi,\beta)\, du$ at $s = k + 1 + \epsilon$, $k = 2, 3, 4, \ldots$, can also be obtained by performing the change of variable $u = 1/t$ and integrating by parts $k + 2$ times,

$$M[f; k+1+\epsilon] = \frac{\Gamma(\epsilon)}{\Gamma(k+2+\epsilon)} \int_0^\infty t^{-\epsilon} f^{(k+2)}\left(\frac{1}{t}, \phi, \beta\right) dt.$$

Using this formula and (35), we see that $\lim_{\epsilon\to 0}\{\epsilon M[f;\ k+1+\epsilon]\} = -V_k$. Then, introducing this limit in (37) and using (41) we obtain (26) for $k = 2, 3, 4, \ldots$. Rearranging terms in (36) we obtain (21) and (22) with V_k, T_k given in (23)–(26).

Finally, we obtain the error bound (29), which shows the convergent character of the expansion (22). The function $g(u)$ satisfies

$$g(u) = \sum_{k=0}^{n-1} \frac{(-2)^{k+1} A_{k+1}}{u^{k+1}} + g_n(u),$$

with $g_n(u) = \mathcal{O}(u^{-n-1})$ as $u \to \infty$ and A_k given in (24). Following a similar reasoning as above, we conclude that $g_n(u)$ satisfies the error test, and therefore, for $n = 0, 1, 2, \ldots$,

$$|g_n(u)| \leq \frac{2^{n+1}|A_{n+1}|}{u^{n+1}}, \qquad |g_n(u)| \leq \frac{2^n |A_n|}{u^n},$$

where the A_k are given in (31). Then,

$$|f_n(u)| \leq \frac{B_n(\phi,\beta)}{u^{n+1}}, \qquad |f_n(u)| \leq \frac{B_{n-1}(\phi,\beta)}{u^n}, \tag{42}$$

where $B_n(\phi,\ \beta)$ is given in (30). Introducing the second bound of (42) in (39) we obtain [19, Eq. (23)],

$$|f_{n,n}(t)| \leq \frac{1}{(n-1)!}(B_n(\phi,\beta) - B_{n-1}(\phi,\beta)\log t) \qquad \forall t \in [0, 1]$$

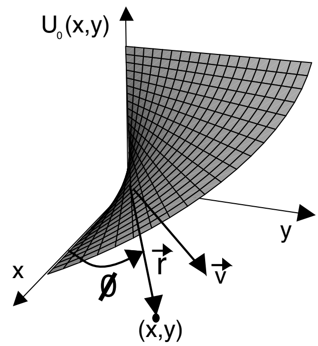

Figure 4. Graph of the first order approximation, $2\phi/\pi$, to the solution of the problem (P_1') near the corner for $\epsilon = 0.1$ and $\beta = \pi/4$.

and introducing the first bound of (42) in (39) we obtain [20, Eq. (24)],

$$|f_{n,n}(t)| \leq \frac{B_n(\phi, \beta)}{n!t} \qquad \forall t \in [0, \infty).$$

We divide the integral on the right-hand side of (38) at the point $t = 1$ and use the first bound of $f_{n,n}(t)$ in the interval $[0, 1]$ and the second one in the interval $[1, \infty)$. Formula (29) follows after simple computations. From (24) and (30) we easily deduce that $|B_n(\phi, \beta)| \leq Cn2^n$ for some $C > 0$ and then $\lim_{n\to\infty} R_n(r, \phi, \beta) = 0$. ■

Remark 2. From (21), (22), and (29) we have

$$U_\beta(x, y) = \frac{2\phi}{\pi} + \mathcal{O}(r), \qquad r \to 0^+, \quad \epsilon \geq \epsilon_0 > 0.$$

The discontinuity at (0, 0) is smoothed inside the corner by a linear function of the angle ϕ between $\vec{r}$ and the x-axis independently of $\vec{v}$ (see Figure 4).

3. Convection–diffusion in an infinite strip

We consider a singularly perturbed convection–diffusion problem in an infinite strip of width 1, parallel to the x-axis in which one of the Dirichlet boundary conditions has two discontinuities:

$$\begin{cases} -\epsilon \Delta U + \vec{v} \cdot \vec{\nabla} U = 0, & (x, y) \in \Omega_2 \equiv (-\infty, \infty) \times (0, 1) \\ U(x, 0) = \chi_{[a,b]}(x),\ U(x, 1) = 0, & U \in \mathcal{C}(\bar{\Omega}_2 \backslash \{(a, 0), (b, 0)\}) \cap \mathcal{D}^2(\Omega_2) \end{cases} \quad (\mathrm{P}_2)$$

where $\vec{v} \equiv |\vec{v}|(\sin\beta, \cos\beta)$ is a constant vector and again, $\epsilon > 0$ is a small parameter. (That is, we consider a finite source of "contamination" in the inflow boundary located at the interval $[a, b]$, $a < b$). As in the problem (P_1), after the change of the dependent variable $U(x, y) = F(x, y)\exp(\vec{v}\cdot\vec{r}/(2\epsilon))$, where $\vec{r} \equiv (x, y)$, problem (P_2) is transformed in the Helmholtz equation for $F(x, y)$:

$$\begin{cases} \triangle F - w^2 F = 0, & (x, y) \in \Omega_2 \\ F(x, 0) = e^{-wx\sin\beta}\chi_{[a,b]}(x), & F \in \mathcal{C}(\bar{\Omega}_2\backslash\{(a, 0), (b, 0)\}) \cap \mathcal{D}^2(\Omega_2) \\ F(x, 1) = 0, & \end{cases} \tag{43}$$

where again, $w \equiv |\vec{v}|/(2\epsilon)$.

As in the problem (P_1), the solution $U(x, y)$ of (P_2) may not be unique unless we impose a convenient condition upon $U(x, y)$ concerning its growth at infinity. In fact, using the maximum principle of elliptic partial differential equations, we can see that the problem: find $U \in \mathcal{C}(\bar{\Omega}_2\backslash\{(a, 0), (b, 0)\}) \cap \mathcal{D}^2(\Omega_2)$ such that

$$\begin{cases} -\epsilon\triangle U + \vec{v}\cdot\vec{\nabla}U = 0, & (x, y) \in \Omega_2 \\ U(x, 0) = \chi_{[a,b]}(x),\ U(x, 1) = 0, & \lim\limits_{|x|\to\infty} U(x, y) = 0. \end{cases} \tag{P_2'}$$

has a unique solution.

In the following proposition we obtain the solution of (P_2') by means of an integral representation.

PROPOSITION 2. *Let* $w \equiv |\vec{v}|/(2\epsilon)$. *Then, for* $(x, y) \in \Omega_2$ *and* $0 \le \beta \le \pi/2$, *the solution* $U_\beta(x, y)$ *of* (P_2') *reads*

$$\begin{aligned} U_\beta(x, y) = {} & \tfrac{1}{2}\left[\chi_A(x, y) + \chi_{A_0}(x, y) - e^{2(y-1)w\cos\beta}\left(\chi_B(x, y) + \chi_{B_0}(x, y)\right)\right] \\ & + e^{w(y\cos\beta + x\sin\beta)}[I(x - a, y) - I(x - a, 2 - y) - I(x - b, y) \\ & + I(x - b, 2 - y) + R(x, y)], \end{aligned} \tag{44}$$

where the functions $I(X, Y)$, $R(x, y)$, $\chi_A(x, y)$, $\chi_{A_0}(x, y)$, $\chi_B(x, y)$, *and* $\chi_{B_0}(x, y)$, *are defined as follows*:

$$I(X, Y) \equiv \frac{e^{w(X-x)\sin\beta}}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} e^{-rw\cosh t}\,\frac{\cosh(t + \mathrm{i}\phi)}{\sinh(t + \mathrm{i}\phi) - \mathrm{i}\sin\beta}\,dt \tag{45}$$

with

$$\begin{cases} X = r\sin\phi, & 0 < r < \infty \\ Y = r\cos\phi, & -\pi/2 \le \phi \le \pi/2 \end{cases} \tag{46}$$

in each integral $I(X, Y)$,

$$\begin{aligned} R(x, y) \equiv {} & \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \frac{e^{-aw\sin\beta + \mathrm{i}(x-a)t} - e^{-bw\sin\beta + \mathrm{i}(x-b)t}}{(t - \mathrm{i}w\sin\beta)e^{2\sqrt{w^2+t^2}}} \\ & \times \frac{\sinh((1 - y)\sqrt{w^2 + t^2})}{\sinh(\sqrt{w^2 + t^2})}\,dt \end{aligned} \tag{47}$$

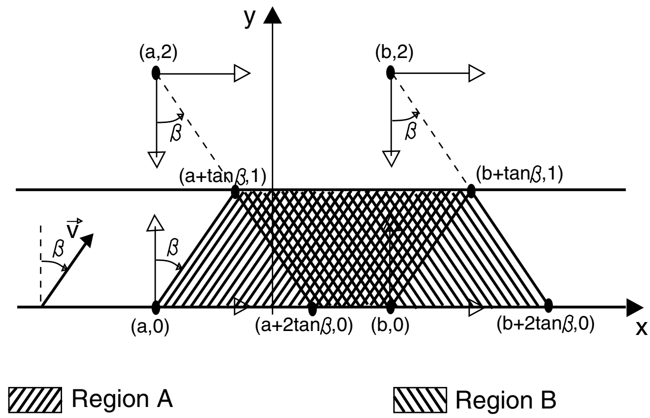

Figure 5. Region A is limited by the straight lines $y = 0$, $y = 1$, $x = a + y \tan\beta$ and $x = b + y \tan\beta$. Region B is limited by the straight lines $y = 0$, $y = 1$, $x = a + (2 - y) \tan\beta$ and $x = b + (2 - y) \tan\beta$. Every one of the four functions $F(X, Y)$ in (51) contributes to $U_\beta(x, y)$ with $I(X, Y)$ and the second term in (53) or (54). The first line in (44) (related to the regions A and B) comes from the addition of those four second terms. The angle ϕ is different in each one of those four terms and is defined by the coordinate system (X, Y) involved in each of them (see Equation (46)): the origin of the four coordinate systems (X, Y) involved in (51) are $(a, 0)$, $(b, 0)$, $(a, 2)$, and $(b, 2)$ and are depicted in the figure.

and $\chi_A(x, y)$, $\chi_{A_0}(x, y)$, $\chi_B(x, y)$, *and* $\chi_{B_0}(x, y)$ *are the characteristic functions of the respective regions* A, A_0, B, *and* B_0 *depicted in Figure* 5:

$$\begin{aligned} \chi_R &\equiv \begin{cases} 1 & \text{if } (x, y) \in R, \\ 0 & \text{if } (x, y) \notin R, \end{cases} \\ A &\equiv \{(x, y) \in \mathbb{R}^2, \quad 0 < y < 1, \quad a \le x - y \tan\beta \le b\}, \\ B &\equiv \{(x, y) \in \mathbb{R}^2, \quad 0 < y < 1, \quad a \le x + (y - 2) \tan\beta \le b\}. \end{aligned} \tag{48}$$

Regions A_0 and B_0 *are the interior sets of* A *and* B *respectively. The integral* (45) *must be understood as a principal value integral if* $\phi = \beta$.

Proof: The exact solution of the problem (43), valid for $0 \le \beta \le \pi/2$, may be obtained by taking the Fourier transform of the differential equation with respect to x:

$$F(x, y) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{e^{-aw\sin\beta + i(x-a)t} - e^{-bw\sin\beta + i(x-b)t}}{t - \mathrm{i}w\sin\beta} \times \frac{\sinh((1-y)\sqrt{w^2 + t^2})}{\sinh(\sqrt{w^2 + t^2})}\, dt.$$

Then, the function $U_\beta(x, y) \equiv e^{w(x\sin\beta+y\cos\beta)}F(x, y)$, with $F(x, y)$ defined above, is the solution of (P_2'). We write

$$U_\beta(x, y) = \frac{e^{w(x\sin\beta+y\cos\beta)}}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \left(e^{-aw\sin\beta+\mathrm{i}(x-a)t} - e^{-bw\sin\beta+i(x-b)t}\right)$$

$$\times\left(e^{-y\sqrt{w^2+t^2}} - e^{(y-2)\sqrt{w^2+t^2}}\right)\frac{H(t)\,dt}{t - iw\sin\beta}, \tag{49}$$

where $H(t) \equiv (1 - e^{-2\sqrt{w^2+t^2}})^{-1}$. Splitting $H(t)$ in a constant term plus an exponentially decaying one:

$$H(t) = 1 + \frac{e^{-2\sqrt{w^2+t^2}}}{1 - e^{-2\sqrt{w^2+t^2}}}, \tag{50}$$

the right-hand side of (49) becomes the sum of two integrals. After the change of variable $t = w \sinh u$ in the first of these integrals we obtain

$$U_\beta(x, y) = e^{w(x\sin\beta+y\cos\beta)}[F(x - a, y) - F(x - a, 2 - y)$$

$$- F(x - b, y) + F(x - b, 2 - y) + R(x, y)], \tag{51}$$

valid for $0 < y < 2$, where

$$F(X, Y) \equiv \frac{e^{w(X-x)\sin\beta}}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \exp\{w[\mathrm{i}X\sinh u - Y\cosh u]\}\frac{\cosh u}{\sinh u - \mathrm{i}\sin\beta}\,du$$

and $R(x, y)$ is given in (47). Using the polar variables (46) adapted to each integral $F(X, Y)$ we have

$$F(X, Y) = \frac{e^{w(X-x)\sin\beta}}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} e^{-rw\cosh(u-\mathrm{i}\phi)}\frac{\cosh u}{\sinh u - \mathrm{i}\sin\beta}\,du. \tag{52}$$

The poles of the integrand of $F(X, Y)$ are located at the points $u = \mathrm{i}\beta + 2k\pi\,\mathrm{i}$ and $u = -\mathrm{i}\beta + (2k + 1)\pi\mathrm{i}$, $k \in Z$ and the real part of the exponent reads $-rw\cosh(\Re u)\cos(\phi - \Im u)$. Therefore, we can use the Cauchy's residue theorem for shifting the integration contour in each of the integrals $F(X, Y)$ to the straight line $\Im u = \phi$. We distinguish two cases.

Case 1. $0 < \beta \le \pi/2$ (see Figure 6a). We apply the Cauchy residue theorem to obtain

$$F(X, Y) = I(X, Y) + e^{-w(x\sin\beta+Y\cos\beta)}\left[\chi_{(0,\pi/2)}(\phi - \beta) + \tfrac{1}{2}\delta_{\phi,\beta}\right] \tag{53}$$

where $I(X, Y)$ is given in (45). Therefore, (44) follows.

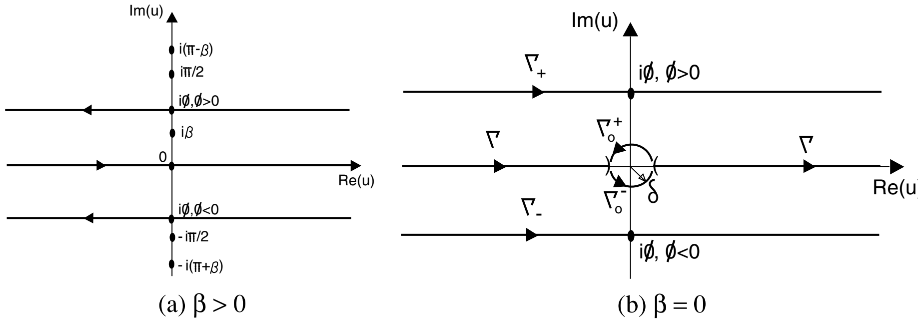

Figure 6. (a) The integrand in (52) has just a single pole situated between the straight lines $u = -i\pi/2$ and $u = i\pi/2$: $u = i\beta$. (b) The path $\Gamma_o \equiv \Gamma_o^+ \cup \Gamma_o^-$ is a small circle of radius δ. The path Γ (with $\delta \to 0$) is the u-integration contour for the integrand $h(u)$ in $F(X, Y)$. With the shift $u \to u + i\phi$, $\Gamma \cup \Gamma_o^+ \to \Gamma_+$ if $\phi > 0$ and $\Gamma \cup \Gamma_o^- \to \Gamma_-$ if $\phi < 0$. Therefore, $\int_\Gamma h(u) - \frac{1}{2}\int_{\Gamma_o} h(u) = \int_{\Gamma_+} h(u)$ and $\int_\Gamma h(u) + \frac{1}{2}\int_{\Gamma_o} h(u) = \int_{\Gamma_-} h(u)$.

Case 2. $\beta = 0$ (see Figure 6b). Now we also apply the Cauchy residue theorem, but taking into account now that the pole $u = 0$ is on the integration contour,

$$F(X, Y) = I(X, Y) + \operatorname{sign}(X)\frac{e^{-wY}}{2}, \tag{54}$$

where $I(X, Y)$ is given in (45) and sign(0) must be understood as zero. Therefore, (44) follows. ■

PROPOSITION 3. *With the same notation as in the preceding proposition, for* $(x, y) \in \Omega_2$ *and* $0 \le \beta \le \pi/2$, *the solution* $U_\beta(x, y)$ *of* (P_2') *is*

$$\begin{aligned} U_\beta(x, y) = \tfrac{1}{2}[\chi_A(x, y) + \chi_{A_0}(x, y)] \cdot \frac{e^{wy\cos\beta}}{2} \cdot \frac{\sinh[(1-y)w\cos\beta]}{\sinh(w\cos\beta)} \\ + e^{w(y\cos\beta + x\sin\beta)}[J(x-a, y) - J(x-b, y)], \end{aligned} \tag{55}$$

where $\chi_A(x, y)$ *and* $\chi_{A_0}(x, y)$ *are defined in Proposition* 2 *and, for* $c = a, b$:

$$\begin{aligned} J(x-c, y) \equiv \frac{e^{-wc\sin\beta}}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} e^{-rw\cosh t + yw\cosh(t+\mathrm{i}\phi)} \frac{\cosh(t+i\phi)}{\sinh(t+i\phi) - \mathrm{i}\sin\beta} \\ \times \frac{\sinh[(1-y)w\cosh(t+i\phi)]}{\sinh[w\cosh(t+i\phi)]}\, dt \end{aligned} \tag{56}$$

with

$$\begin{cases} x - c = r\sin\phi, & 0 < r < \infty \\ y = r\cos\phi, & -\pi/2 \le \phi \le \pi/2. \end{cases} \tag{57}$$

The integral (56) *is a principal value integral if* $\phi = \beta$.

Proof: It is similar to the proof of the preceding proposition, but without splitting the function $H(t)$. ■

Observation 2. The explicit representation given in Proposition 2 is only valid when the angle β between the convection vector $\vec{v}$ and the y-axis is restricted to the interval $[0, \pi/2]$. Nevertheless, an explicit integral representation for the solution $U(x, y)$ of the problem (P_2') whatever the direction of $\vec{v}$ is, may be obtained by means of symmetry arguments:

$$U(x,y) = \begin{cases} U_\beta(x,y) & \text{if } 0 \le \beta \le \pi/2, \\ U_{-\beta}(-x,y) & \text{if } -\pi/2 \le \beta \le 0, \\ e^{2wy\cos\beta} U_{\pi-\beta}(x,y) & \text{if } \pi/2 \le \beta \le \pi, \\ e^{2wy\cos\beta} U_{\pi+\beta}(-x,y) & \text{if } -\pi < \beta \le -\pi/2, \end{cases}$$

where $U_\beta(x, y)$ is given in (44)–(50). Therefore, in the remainder of this section, we restrict ourselves to $\beta \in [0, \pi/2]$.

From the representation of $U(x, y)$ given in Proposition 2 or Proposition 3 it is difficult to infer the behavior of the solution of (P_2'). But, for $\epsilon \to 0^+$, $r \ge r_0 > 0$, we can approximate the integral $I(r, \phi, \beta)$ in (3) by an error function plus an asymptotic expansion in powers of ϵ, whereas for $r \to 0^+$, $\epsilon \ge \epsilon_0 > 0$, we can approximate $I(r, \phi, \beta)$ by an asymptotic expansion in powers of r. This is the subject of the two following subsections. But the explicit computation of the coefficients is now much more involved than for the quarter plane. Therefore, we will just show the form of the expansions and indicate how to calculate the coefficients. Nevertheless, the dominant terms in the expansions are explicitly computed.

3.1. Asymptotic expansion of $U(x, y)$ in the singular limit

In this section, we denote by Ω_2^* the infinite strip indented at the points $(a, 0)$ and $(b, 0)$ (see Figure 3b):

$$\Omega_2^* \equiv \Omega_2 \backslash \{D_{r_0}(a,0) \cup D_{r_0}(b,0)\}.$$

THEOREM 3. *Write* $w \equiv |\vec{v}|/(2\epsilon)$. *Then, for* $(x, y) \in \Omega_2^*$, *the solution* $U(x, y)$ *of* (P_2') *reads*

$$U_\beta(x,y) = U_\beta^0(x,y) + \frac{1}{2\pi\sqrt{2w}} U_\beta^1(x,y), \tag{58}$$

where

$$
\begin{aligned}
&U_\beta^0(x,y)\\
&\equiv \frac{1+\delta_{\beta,\pi/2}}{2}\left\{\operatorname{sign}\left[\beta-\arctan\left(\frac{x-a}{y}\right)\right]\operatorname{erfc}\left(\sqrt{w\zeta(x-a,y)}\right)\right.\\
&\quad -e^{2(y-1)w\cos\beta}\operatorname{sign}\left[\beta-\arctan\left(\frac{x-a}{2-y}\right)\right]\operatorname{erfc}\left(\sqrt{w\zeta(x-a,2-y)}\right)\\
&\quad -\operatorname{sign}\left[\beta-\arctan\left(\frac{x-b}{y}\right)\right]\operatorname{erfc}\left(\sqrt{w\zeta(x-b,y)}\right)\\
&\quad \left.+e^{2(y-1)w\cos\beta}\operatorname{sign}\left[\beta-\arctan\left(\frac{x-b}{2-y}\right)\right]\operatorname{erfc}\left(\sqrt{w\zeta(x-b,2-y)}\right)\right\}\\
&\quad +\frac{1}{2}\left[\chi_A(x,y)+\chi_{A_0}(x,y)-e^{2(y-1)w\cos\beta}\left(\chi_B(x,y)+\chi_{B_0}(x,y)\right)\right].
\end{aligned}
$$

The functions χ_A, χ_{A_0}, χ_B, *and* χ_{B_0} *are defined in Proposition* 2, $\operatorname{sign}(0)=0$ *and*

$$
\zeta(X,Y)\equiv\sqrt{X^2+Y^2}-X\sin\beta-Y\cos\beta. \tag{60}
$$

The function $U_\beta^1(x,y)$ *has an asymptotic expansion*

$$
U_\beta^1(x,y)=\sum_{k=0}^{n-1}\frac{T_k(x,y)}{(2w)^k}+R_n(x,y), \tag{61}
$$

where empty sums must be understood as zero. The coefficients T_k *(defined below) are smooth functions of* x *and* y, $\mathcal{O}(1)$ *when* $w\to\infty$ *uniformly for* $(x,y)\in\Omega_2^*$.

The remainder $R_n(x,y)$ *satisfies a bound*

$$
|R_n(x,y)|\le M\frac{\Gamma(n+1/2)}{(2wd)^n}\left[\frac{e^{-w\zeta(x-a,y)}}{[(x-a)^2+y^2]^{n/2}}+\frac{e^{-w\zeta(x-b,y)}}{[(x-b)^2+y^2]^{n/2}}\right], \tag{62}
$$

for positive constants M *and* d *given below.*

Proof. We consider for $U_\beta(x,y)$ the explicit representation given in Proposition 2. For large w and fixed r, the asymptotic features of the integral $I(X,Y)$ defined in (45) are: (i) there is a saddle point at $t=0$. (ii) The pole situated at $t=\mathrm{i}(\beta-\phi)$ and the saddle point coalesce when $\phi\to\beta$. Once again, uniform asymptotic expansions of this kind of integrals are obtained by using the error function as the basic approximant. Therefore, we split off the pole of the integrand at $t = \mathrm{i}(\beta - \phi)$:

$$\frac{\cosh(t + i\phi)}{\sinh(t + \mathrm{i}\phi) - \mathrm{i}\sin\beta} = \frac{1 + \delta_{\beta,\pi/2}}{2\sinh\frac{1}{2}(t + \mathrm{i}(\phi - \beta))} + f(t, \phi, \beta),$$

with the obvious definition of $f(t, \phi, \beta)$.

Using again the complementary error function representation (14) we obtain that the integral $I(X, Y)$ is

$$I(X, Y) = \operatorname{sign}\left[\beta - \arctan\left(\frac{X}{Y}\right)\right](1 + \delta_{\beta,\pi/2})\frac{e^{-w(Y\cos\beta + x\sin\beta)}}{2} \times \operatorname{erfc}\left(\sqrt{w\zeta(X, Y)}\right) + \bar{I}(X, Y), \tag{63}$$

where using (46),

$$\bar{I}(X, Y) \equiv \frac{e^{w(X-x)\sin\beta}}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} e^{-rw\cosh t} f(t, \phi, \beta)\, dt.$$

Therefore, from (44) we obtain (58) where $U_\beta^0(x, y)$ is given in (59) and (60) and

$$U_\beta^1(x, y) = 2\pi\sqrt{2w}e^{w(y\cos\beta + x\sin\beta)}[\bar{I}(x - a, y) - \bar{I}(x - a, 2 - y) - \bar{I}(x - b, y) + \bar{I}(x - b, 2 - y) + R(x, y)]. \tag{64}$$

In order to obtain the asymptotic expansion of $\bar{I}(X, Y)$ for large w and bounded $r \geq r_0 > 0$, we perform the change of variable $\sinh(t/2) \equiv u$ and remove the odd part of $f(t, \phi, \beta)$ from (64). We obtain

$$\bar{I}(X, Y) = \frac{e^{w(X-x)\sin\beta - wr}}{\pi}\int_0^{\infty} e^{-2wru^2} g(u^2, \phi, \beta)\, du, \tag{65}$$

where

$$g(u, \phi, \beta) \equiv \frac{2(1 + 2u)\sin\beta\cos\phi - \sin(2\phi)}{[4u^2 + 4(1 + \sin\beta\sin\phi)u + (\sin\phi + \sin\beta)^2]\sqrt{1 + u}} + \frac{(1 + \delta_{\beta,\pi/2})\sin\dfrac{\phi - \beta}{2}}{u + \sin^2\dfrac{\phi - \beta}{2}}, \tag{66}$$

This function has a Taylor expansion at $u = 0$ for each $\phi \in [-\pi/2, \pi/2]$:

$$g(u, \phi, \beta) = \sum_{k=0}^{n-1} \frac{g^{(k)}(0, \phi, \beta)}{k!}u^k + g_n(u, \phi, \beta), \tag{67}$$

where

$$g_n(u, \phi, \beta) \equiv \frac{g^{(n)}(\xi, \phi, \beta)}{n!} u^n$$

for some $\xi \in (0, u)$. The singularities of $g(u, \phi, \beta)$ are away from the positive real axis (let d be the distance from the closest of those singularities to the positive real axis). Therefore, using the Cauchy formula for the derivative $g^{(n)}(\xi, \phi, \beta)$, we see that

$$|g_n(u, \phi, \beta)| \leq M \frac{n!}{d^n} u^n, \tag{68}$$

where M is a bound for $g(w, \phi, \beta)$ on the portion of the complex w-plane surrounding the positive real axis: $\{w \in \mathbb{C}, |w - u| < d, u \in \mathbb{R}^+\}$. Introducing the expansion (67) in (65) and this in (64) we obtain (61) with

$$\begin{aligned} T_k(x, y) \equiv \frac{\Gamma\left(k + \frac{1}{2}\right)}{k!} &[A_k(x - a, y) - A_k(x - a, 2 - y) \\ &- A_k(x - b, y) + A_k(x - b, 2 - y)], \end{aligned} \tag{69}$$

where

$$A_k(X, Y) \equiv \frac{e^{-w\zeta(X,y)}}{r^{k+1/2}} g^{(k)}(0, \phi, \beta).$$

On the other hand,

$$\begin{aligned} R_n(x, y) \equiv 2\sqrt{2w} &[B_k(x - a, y) - B_k(x - a, 2 - y) \\ &- B_k(x - b, y) + B_k(x - b, 2 - y)], \end{aligned} \tag{70}$$

where, using (46):

$$B_k(X, Y) \equiv e^{-w\zeta(X,y)} \int_0^\infty e^{-2wru^2} g_n(u^2, \phi, \beta)\, du.$$

Introducing (68) in (70) we obtain (62). On the other hand, we write

$$R(x, y) = S(x - a, y) - S(x - b, y),$$

where, for $c = a, b$:

$$S(x - c, y) \equiv \frac{e^{-cw\sin\beta}}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}(x-c)t}}{(t - \mathrm{i}w\sin\beta)e^{2\sqrt{w^2+t^2}}} \frac{\sinh((1 - y)\sqrt{w^2 + t^2})}{\sinh(\sqrt{w^2 + t^2})}\, dt.$$

We perform similar manipulations to those performed on $F(x, y)$ in the proof of Proposition 2: a change of variable $t \to s$ given by $t = w \sinh s$ and a shift in the integration contour $s \to u$ with $u = s + i\phi_2$. Then we are led to

$$
\begin{aligned}
S(x-c,y) = &-e^{-wx\sin\beta - 2w\cos\beta}\frac{\sinh[(1-y)w\cos\beta]}{2\sinh[w\cos\beta]} \\
&\times\left[\chi_{(0,\pi/2)}(\phi_2-\beta)+\frac{1}{2}\delta_{\phi_2,\beta}\right] \\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-wr_2\cosh u - y\cosh(u+i\phi_2)}\frac{\sinh(u+i\phi_2)\cosh(u+i\phi_2)}{\sin^2\beta+\sinh^2(u+i\phi_2)} \\
&\times\frac{\sinh[(1-y)w\cosh(u+i\phi_2)]}{\sinh[w\cosh(u+i\phi_2)]}\,du,
\end{aligned}
\tag{71}
$$

where r_2 and ϕ_2 are polar coordinates for the cartesian coordinates $(x-c, 2-y)$. After the change of variable $u \to t$ given by $t = \sinh(u/2)$ in the last integral, I_R, we can see that this integral has a bound of the form

$$
|I_R| \le \bar{M}e^{-wr_2 - w\cos\phi_2}, \tag{72}
$$

where $\bar{M}$ is a positive constant independent of x and y. Then, $|R(x,y)| \le \bar{M}e^{-\alpha w}$ for some $\alpha > 0$ uniformly in Ω_2. This bound has been included in formula (62). ■

THEOREM 4. *With the notation of the preceding theorem and for* $(x,y) \in \Omega_2^*$,

$$
U_\beta(x,y) = \tilde{U}_\beta^0(x,y) + \tilde{U}_\beta^1(x-a,y) - \tilde{U}_\beta^1(x-b,y), \tag{73}
$$

where

$$
\begin{aligned}
\tilde{U}_\beta^0(x,y) \equiv \frac{1}{2}\Bigg\{&(1+\delta_{\beta,\pi/2})\operatorname{sign}\left[\beta-\arctan\left(\frac{x-a}{y}\right)\right]\operatorname{erfc}\left(\sqrt{w\zeta(x-a,y)}\right) \\
&-(1+\delta_{\beta,\pi/2})\operatorname{sign}\left[\beta-\arctan\left(\frac{x-b}{y}\right)\right]\operatorname{erfc}\left(\sqrt{w\zeta(x-b,y)}\right) \\
&+\chi_A(x,y)+\chi_{A_0}(x,y)\Bigg\}e^{wy\cos\beta}\frac{\sinh[(1-y)w\cos\beta]}{\sinh[w\cos\beta]}.
\end{aligned}
\tag{74}
$$

For $c = a, b$, $\tilde{U}_\beta^1(x-c,y)$ *has an asymptotic expansion in powers of* w^{-1}:

$$
\tilde{U}_\beta^1(x-c,y) = \frac{e^{-w\zeta(x-c,y)}}{\sqrt{2wr}}\left[\sum_{k=0}^{n-1}\frac{\tilde{T}_k(x-c,y)}{(2wr)^k}+R_n(x-c,y)\right], \tag{75}
$$

where empty sums are zero. The coefficients $\tilde{T}_k(x-c,y)$ *are smooth functions of* x *and* y *and* $\mathcal{O}(1)$ *when* $w \to \infty$ *uniformly for* $(x,y) \in \Omega_2^*$.

The remainder $R_n(x-c, y)$ *satisfies*

$$|R_n(x-c, y)| \leq M\frac{\Gamma(n+1/2)}{(2wdr)^n} \tag{76}$$

for some positive constants M *and* d.

Proof: It is similar to the proof of the preceding theorem, but using the representation for U given in Proposition 3. ■

Remark 3. From (58), (61), and (62) or from (73), (75), and (76) we see that $U_\beta(x, y) = U_\beta^0(x, y) + \mathcal{O}(\sqrt{\epsilon})$ or $U_\beta(x, y) = \tilde{U}_\beta^0(x, y) + \mathcal{O}(\sqrt{\epsilon})$ as $\epsilon \to 0^+$ and away from the points $(a, 0)$ and $(b, 0)$ $(U_\beta^0(x, y) - \tilde{U}_\beta^0(x, y) = \mathcal{O}(\sqrt{\epsilon}))$. Then, the first-order approximation to the solution of (P_2') is a combination of two or four error functions plus step functions multiplied by exponentials of y. The first and third error functions in (59) exhibit two interior layers of width $\mathcal{O}(\sqrt{\epsilon})$ and level lines with equation $\zeta(x-c, y) =$ constant with $c = a, b$. The last line in (59) represents a regular boundary layer of width $\mathcal{O}(\epsilon)$ near the piece of the outflow boundary situated between the points $(a + \tan\beta, 1)$ and $(b + \tan\beta, 1)$. The combination of the four error functions and the last line in (59) exhibit two corner layers of area $\mathcal{O}(\sqrt{\epsilon}) \times \mathcal{O}(\epsilon)$ near the points $(a + \tan\beta, 1)$ and $(b + \tan\beta, 1)$ (see Figure 7).

3.2. Asymptotic expansion of U(x, y) near the discontinuities

The asymptotic expansions given in Theorem 3 or Theorem 4 break down when $(x, y) \to (a, 0)$ or $(x, y) \to (b, 0)$ (i.e, when $r \to 0^+$ with the identification

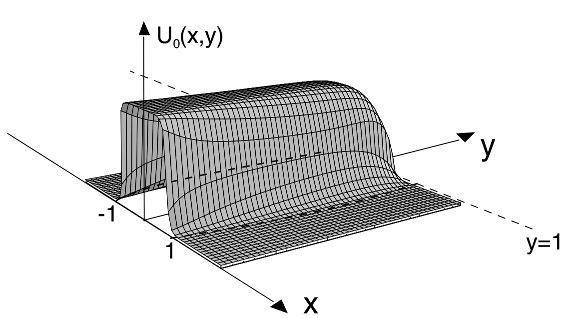

Figure 7. Graph of the first-order approximation, $U_\beta^0(x, y)$, to the solution of the problem (P_2') for $\beta = 0$ and $\epsilon = 0.1$. The convection vector $(0, 1)$ "drags" the two discontinuities of the boundary condition at $(a, 0)$ and $(b, 0)$ originating two parabolic layers along the direction of that vector. Moreover, a regular boundary layer occurs at the outflow boundary between $x = a$ and $x = b$, and two corner layers around the points $(a, 1)$ and $(b, 1)$ in order to satisfy the boundary condition at $y = 1$.

(46)). The asymptotic approximation of $U_\beta(x, y)$ near the points $(a, 0)$ and $(b, 0)$ requires a similar analysis to Section 3.2. An asymptotic approximation of $U_\beta(x, y)$ near the discontinuity points $(a, 0)$ and $(b, 0)$ with $(x, y) \to (a, 0)$ or $(x, y) \to (b, 0)$ faster than $\epsilon \to 0^+$ is given in the following theorem.

THEOREM 5. *Write* $w \equiv |\vec{v}|/(2\epsilon)$ *and* $x - c = r_c \sin\phi_c$, $y = r_c \cos\phi_c$ *with* $c = a, b$. *Then, for* $(x, y) \in \Omega_2$, *the solution* $U_\beta(x, y)$ *of* (P_2') *reads*

$$U_\beta(x, y) = \frac{1}{2} + \frac{\phi_a}{\pi} + \frac{wr_a}{2\pi} e^{-w\zeta(x-a,y)} U^2(x - a, y) + U_a^3(x, y), \tag{77}$$

$$U_\beta(x, y) = \frac{1}{2} - \frac{\phi_b}{\pi} - \frac{wr_b}{2\pi} e^{-w\zeta(x-b,y)} U^2(x - b, y) + U_b^3(x, y), \tag{78}$$

where, for $c = a, b$, $U^2(x - c, y)$ *has a convergent expansion in powers of* wr_c:

$$\begin{aligned} U^2(x - c, y) \equiv {} & \frac{\pi \operatorname{sign}(\phi_c) - 2\phi_c}{wr_c}\left[1 - e^{w\zeta(x-c,y)}\right] \\ & + \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} [T_k(\phi_c, \beta) - V_k(\phi_c, \beta) \log(r_c w)](r_c w)^{k-1} \\ & + R_n(x - c, y), \end{aligned} \tag{79}$$

with $V_1 = 0$, *empty sums are zero and* $\operatorname{sign}(0) = 0$. *Coefficients* $T_k(\phi, \beta)$ *and* $V_k(\phi, \beta)$ *are regular functions of* ϕ *and* β *defined below.*

The remainder term $R_n(x - c, y)$ *satisfies*

$$|R_n(x - c, y)| \le M \frac{n(2 + d) + |\log(wr_c)|}{n!} \left(\frac{r_c w}{d}\right)^{n-1} \tag{80}$$

for some positive constants M *and* d *defined below. On the other hand,*

$$\begin{aligned} U_c^3(x, y) \equiv {} & e^{wy\cos\beta + wx\sin\beta}[\mp I(x - c, y) - I(x - a, 2 - y) + I(x - b, 2 - y) \\ & + R(x, y)] - \frac{1}{2} e^{2wy\cos\beta}(\chi_B(x, y) + \chi_{B_0}(x, y)) \end{aligned} \tag{81}$$

and is of the order $\mathcal{O}(e^{-\alpha/\epsilon})$ *when* $\epsilon \to 0^+$ *uniformly in* Ω_2 *for some* $\alpha > 0$.

Proof: We consider for $U_\beta(x, y)$ the representation given in Proposition 2. After the change of variable $\cosh\, t = u + 1$ in the definition (45) and (46) of $I(x - c, y)$ we have

$$I(x - c, y) = -\frac{e^{-wr_c}}{2\pi} \int_0^\infty e^{-r_c w u} f(u, \phi_c, \beta)\, du, \tag{82}$$

where $f(u, \phi, \beta) = f^{(1)}(u, \phi, \beta) + f^{(2)}(u, \phi, \beta)$,

$$f^{(1)}(u, \phi, \beta) \equiv \left[\frac{\sin^2 \beta - \sin^2 \phi}{u^2} - \cos(2\beta) \left(1 + \frac{2}{u} \right) \right] g(u), \tag{83}$$

$$f^{(2)}(u, \phi, \beta) \equiv \frac{(u+1)\sin \beta}{2 \sin \phi} \left[(1 + \sin^2 \beta \cos^2 \phi) \left(1 + \frac{2}{u} \right) - \frac{\sin^2 \phi}{u^2} \right] g(u), \tag{84}$$

$$g(u) \equiv \frac{\sin(2\phi)}{u^3 \sqrt{1 + \frac{2}{u}} \left(1 + \frac{1+s_+}{u} \right) \left(1 + \frac{1-s_+}{u} \right) \left(1 + \frac{1+s_-}{u} \right) \left(1 + \frac{1-s_-}{u} \right)}, \tag{85}$$

and $s_\pm \equiv \cos(\beta \pm \phi)$.

Splitting $g(u)$ in fractions, we obtain an expansion of $f(u, \phi, \beta)$ in inverse powers of u valid for each $\phi \in [-\pi/2, \pi/2]$:

$$f(u, \phi, \beta) = \sum_{k=0}^{n-1} \frac{V_k(\phi, \beta)}{u^{k+1}} + f_n(u, \phi, \beta), \tag{86}$$

where $f_n(u, \phi, \beta) = \mathcal{O}(u^{-n-1})$ as $u \to \infty$ uniformly in $\phi \in (-\pi/2, \pi/2)$. The coefficients $V_k(\phi, \beta)$ are nothing but the Taylor coefficients of the expansion of the function $s^{-1} f(s^{-1}, \phi, \beta)$ at $s = 0$ ($V_0 = 0$). Applying [34, Chapter 6, Theorem 13(ii)] to the integral in the right-hand side of (82) we obtain

$$I(x - c, y) = \frac{e^{-wr_c}}{2\pi} \left[\sum_{k=0}^{n-1} \frac{(-1)^k}{k!} [T_k - V_k \log(wr_c)](wr_c)^k + R_n(r_c, \phi_c, \beta) \right], \tag{87}$$

with the following expressions for the coefficients T_k and the remainder $R_n(r_c, \phi_c, \beta)$. Here

$$T_k \equiv V_k \psi(k+1) + \lim_{s \to k+1} \left\{ M[f; s] + \frac{V_k}{s - k - 1} \right\}, \tag{88}$$

where $M[f; s]$ denotes the Mellin transform of f at s, $\int_0^\infty u^{s-1} f(u, \phi_c, \beta)\, du$, or its analytic continuation. On the other hand, the remainder is

$$R_n(r_c, \phi_c, \beta) \equiv (r_c w)^n \int_0^\infty f_{n,n}(t) e^{-r_c wt}\, dt, \tag{89}$$

where

$$f_{n,n}(t) \equiv \frac{(-1)^n}{(n-1)!} \int_t^\infty (u - t)^{n-1} f_n(u, \phi_c, \beta)\, du. \tag{90}$$

In particular, the coefficient T_0, which gives the dominant term of the expansion, is

$$T_0 = M[f; 1] = M\big[f^{(1)}; 1\big] + M\big[f^{(2)}; 1\big]. \tag{91}$$

Splitting $f^{(1)}(u, \phi, \beta)$ in simple fractions and using [24, p. 303, Eq. (24)] and [1, Eq. (15.1.6)] we obtain

$$M[f^{(1)}; 1] = \begin{cases} 2\phi & \text{if } \phi < \beta, \\ 2\phi - \pi/2 & \text{if } \phi = \beta, \\ 2\phi - \pi & \text{if } \phi > \beta. \end{cases}$$

On the other hand, undoing the initial change of variable ($\cosh\, t = u + 1$) and using the Cauchy's residue theorem, we have

$$M[f^{(2)}; 1] = \begin{cases} \pi & \text{if } \phi < \beta, \\ \dfrac{\pi}{2} & \text{if } \phi = \beta, \\ 0 & \text{if } \phi > \beta. \end{cases}$$

Therefore,

$$T_0 = \begin{cases} 2\phi + \pi & \text{if } \phi < \beta, \\ 2\phi & \text{if } \phi = \beta, \\ 2\phi - \pi & \text{if } \phi > \beta. \end{cases}$$

Introducing (87) in (44) for $c = a$ and $c = b$ and rearranging terms we obtain (79) and (78) respectively.

Finally, we obtain the error bound (80) which shows that the expansion (79) is convergent. From the Taylor formula for the remainder,

$$f_n(u, \phi, \beta) \equiv \frac{h^{(n)}(\xi)}{n!u^{n+1}}$$

for some $\xi \in (0, u^{-1})$, where $h(s) \equiv s^{-1} f(s^{-1}, \phi, \beta)$. The singularities of $h(s)$ are away from the positive real axis (let us write d for the distance from the closest of those singularities to the positive real axis). Therefore, using the Cauchy formula for the derivative $h^{(n)}(\xi)$, we see that

$$|f_n(u, \phi, \beta)| \leq \frac{M}{d^n u^{n+1}}, \qquad |f_n(u, \phi, \beta)| \leq \frac{M}{d^{n-1} u^n}, \tag{92}$$

where M is a bound for $h(w)$ on the portion of the complex w-plane surrounding the positive real axis: $\{w \in \mathbb{C}, |w - s| < d, s \in \mathbb{R}^+\}$.

Introducing these bounds in (90) we obtain [19, Eq. (2.23)],

$$|f_{n,n}(t)| \leq \frac{M}{d^n (n-1)!}(1 - d \log t) \qquad \forall t \in [0, 1],$$

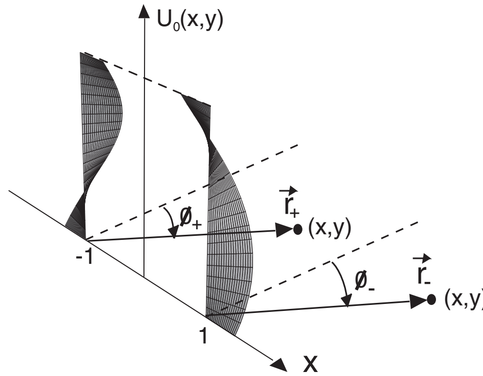

Figure 8. Graph of the first-order approximation, $\frac{1}{2} + \frac{\phi_a}{\pi}$ and $\frac{1}{2} - \frac{\phi_b}{\pi}$, to the solution of the problem (P_2') near the discontinuity points of the boundary condition for $\epsilon = 0.1$.

and introducing the first bound of (92) in (90) we obtain [19, Eq. (2.24)],

$$|f_{n,n}(t)| \le \frac{M}{n! d^n t} \qquad \forall t \in [0, \infty).$$

We divide the integral in the right-hand side of (89) at the point $t = 1$ and use the first bound of $f_{n,n}(t)$ on the interval [0, 1] and the second one on the interval $[1, \infty)$. The bound (80) follows after simple computations. That $U_c^3(x, y)$ is of the order $\mathcal{O}(e^{-\alpha/\epsilon})$ when $\epsilon \to 0^+$ follows from Theorem 3. ■

Remark 4. From Theorems 3 and 5 we see that

$$U_\beta(x, y) = \frac{1}{2} + \frac{\phi_a}{\pi} + \mathcal{O}\left(\frac{r_a}{\epsilon}\right) + \mathcal{O}(e^{-\alpha/\epsilon})$$

as $r_a \to 0^+$ and $\epsilon \to 0^+$ with $r_a/\epsilon \to 0$ and

$$U_\beta(x, y) = \frac{1}{2} - \frac{\phi_b}{\pi} + \mathcal{O}\left(\frac{r_b}{\epsilon}\right) + \mathcal{O}(e^{-\alpha/\epsilon})$$

as $r_b \to 0^+$ and $\epsilon \to 0^+$ with $r_b/\epsilon \to 0$ and some $\alpha > 0$. Once again, the discontinuities of the inflow boundary condition are smoothed inside the domain by a linear function of the angle ϕ_c between $\vec{r}_c \equiv (x - c, y)$ and the line $x = c$, with $c = a$ or $c = b$ (see Figure 8).

4. Conclusions

The singularly perturbed convection–diffusion problems (P_1) and (P_2) are defined including discontinuous Dirichlet boundary conditions, where the discontinuities are located on a corner or on a side of the inflow boundary, respectively. For each problem, we have obtained an integral representation of the solution susceptible to an asymptotic analysis. Then, for each problem, two complementary asymptotic expansions of the solution have been obtained. One expansion is valid in the singular limit $\epsilon \to 0^+$, away from the discontinuities. The other one is valid near the discontinuities with $\epsilon \geq \epsilon_0 > 0$ for (P_1) and with $\epsilon \to 0^+$ and $r_c/\epsilon \to 0^+$ for (P_2), where r_c is the distance to the discontinuity points $(c, 0)$ for $c = a, b$.

These two asymptotic expansions are derived from two quite different asymptotic methods. Whereas the asymptotic expansion in the singular limit is obtained from a classical uniform method, the asymptotic expansion near the discontinuity is derived by a distributional approach. Therefore, two quite different asymptotic principles match into the same problem. On the other hand, despite having these expansions a quite different nature, the error test is present under both expansions, producing accurate error bounds.

The asymptotic expansion in the singular limit shows that the main contribution of the data's discontinuities to the shape of the solution on the boundary layers is contained in a combination of error functions. In the problem (P_1), just one error function is required to reproduce approximately the behavior of the solution in the layer, whereas problem (P_2) requires the combination of two or four error functions and a step function to approach the behavior of the solution on the two interior layers, the regular layer and the corner layers located near the points $(a + \tan\beta, 1)$ and $(b + \tan\beta, 1)$. On the other hand, the asymptotic expansion near the discontinuities shows that the discontinuity on the boundary is smoothed inside the domain by means of a simple linear function of an appropriate angle at the discontinuity.

We want to emphasize the simultaneous dependence of the solution of the two problems (P_1) and (P_2) on the singular parameter ϵ and the distance to the discontinuity point(s) of the boundary data. In problem (P_1), the solution U_β depends on ϵ and the distance r to the discontinuity point $(0, 0)$ through the quotient r/ϵ (see Proposition 1). In problem (P_2), the solution U_β depends on ϵ and the distances r_a and r_b to the discontinuity points $(a, 0)$ and $(b, 0)$ essentially through the quotients r_a/ϵ and r_b/ϵ (see Proposition 2). This is why the expansions for small ϵ (large w) in Theorems 1, 3, and 4 do not hold near the discontinuities. And conversely, the expansions near the discontinuities (small r, r_a, or r_b) in Theorems 2 and 5 only hold when the distances r, r_a, or r_b are smaller than ϵ.

The analysis presented in this paper is applicable to more general problems: Consider any convection–diffusion problem (homogeneous or nonhomogeneous) with constant coefficients defined on the regions considered in Sections 2 or 3 and with Dirichlet conditions discontinuous at the points $(0, 0)$ or $(a, 0)$ and $(b, 0)$, respectively. Then, that problem may be decomposed into several simpler problems. One of those problems is just (P_1) or (P_2) and the remaining ones are defined using continuous Dirichlet data. Moreover, we suspect that, as in the problems analyzed here, the error function plays a fundamental role in the approximation of the solution of many singularly perturbed convection–diffusion problems with discontinuities in the Dirichlet boundary conditions (problems defined over more general domains and through more general coefficients). This will be the subject of further investigations.

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