

ASYMPTOTIC APPROXIMATION OF SINGULARLY PERTURBED  
CONVECTION-DIFFUSION PROBLEMS WITH DISCONTINUOUS  
DERIVATIVES OF THE DIRICHLET DATA

BY

JOSÉ L. LÓPEZ (*Departamento de Matemática e Informática, Universidad Pública de Navarra,  
31006-Pamplona, Spain*)

AND

ESTER PÉREZ SINUSÍA (*Departamento de Matemática e Informática, Universidad Pública de  
Navarra, 31006-Pamplona, Spain*)

**Abstract.** We consider a singularly perturbed convection-diffusion equation,  $-\epsilon \Delta u + \vec{v} \cdot \vec{\nabla} u = 0$ , defined on two domains: a quarter plane,  $(x, y) \in (0, \infty) \times (0, \infty)$ , and a half plane,  $(x, y) \in (-\infty, \infty) \times (0, \infty)$ . We consider for these problems Dirichlet boundary conditions with discontinuous derivatives at some points of the boundary. We obtain for each problem an exact representation of the solution in the form of an integral. From this integral we derive an asymptotic expansion of the solution when the singular parameter  $\epsilon \rightarrow 0^+$  (with fixed distance  $r$  to the points of discontinuity of the boundary condition). It is shown that, in both problems, the first term of the expansion contains the primitive of an error function. This term characterizes the effect of the discontinuities on the  $\epsilon$ -behaviour of the solution and its derivatives in the boundary or internal layers.

**1. Introduction.** Mathematically speaking, a singularly perturbed convection-diffusion problem is a boundary problem of the second order in which the coefficients of the second order derivatives are small. In this paper we focus our attention on two-dimensional linear convection-diffusion (elliptic) problems of the form: find a function  $u \in \mathcal{C}(\bar{\Omega}) \cap \mathcal{D}^2(\Omega)$  such that

$$\begin{cases} -\epsilon \Delta U + \vec{v} \cdot \vec{\nabla} u = 0, & x \in \Omega \subset \mathbb{R}^2, \\ U(x)|_{\partial\Omega} = f(\tilde{x}), & \tilde{x} \in \partial\Omega, \end{cases} \quad (1)$$

---

Received October 28, 2004.

2000 *Mathematics Subject Classification.* Primary 35C20, 41A60.

*Key words and phrases.* Singular perturbation problem, discontinuous boundary data, asymptotic expansion, error function.

*E-mail address:* [jl.lopez@unavarra.es](mailto:jl.lopez@unavarra.es)

*E-mail address:* [ester.perez@unavarra.es](mailto:ester.perez@unavarra.es)

where  $\epsilon$  is a small positive parameter,  $\vec{v}$  is the convection vector,  $\tilde{x}$  is a variable which lives on  $\partial\Omega$ ,  $f(\tilde{x})$  is the Dirichlet data and  $\mathcal{D}^2(\Omega)$  denotes the set of functions with partial derivatives up to order two defined in all points of  $\Omega$ .

The location and shape of the boundary layers of  $u$  depend, among other things, on the velocity field  $\vec{v}$ , on the shape of the boundary  $\partial\Omega$  and on the existence of discontinuities in  $f(\tilde{x})$ . For example, regular boundary layers of size  $\mathcal{O}(\epsilon)$  appear at the outflow boundary, whereas parabolic boundary layers of size  $\mathcal{O}(\sqrt{\epsilon})$  appear along the characteristic boundaries. For more details on the shape and nature of boundary layers, see for example [3], [4], [5], [8], [9] and references therein.

To derive the exact solution of (1) in terms of elementary functions is, in general, an impossible mission. Then, an approximation of the solution of (1) adapted to their singular character (an asymptotic expansion) is of interest because it gives the qualitative behaviour of the solution ([20, p. 6]). Moreover, it helps in the development of suitable numerical methods for these kind of problems. An  $\epsilon$ -uniformly convergent method requires the analysis of uniform convergence, and then accurate error bounds for the local error. The accuracy of these error bounds depends on the precision in the approximation given by the first terms of the asymptotic expansion. The design of the numerical technique is based on the exact integration of the first terms of the asymptotic expansion or of functions which have a similar behaviour in the singular layer. Along this line, some references which propose exponential fitting techniques or special meshes based on asymptotic expansions are [2] or [15]. A classical reference is [13].

There is an extensive amount of literature devoted to the construction of approximated solutions of singular perturbation problems based on a matching of asymptotic expansions. The book of Il'in [9] contains a quite exhaustive and general analysis for different equations and domains. Other important references of the method of matched asymptotic expansions are for example [4], [10] or [14]. With this method, the solution is approximated by an *outer approximation* plus several *inner-layer expansions* (one for every singular layer). We can use this method to derive asymptotic expansions for the solution of (1), but this technique is quite cumbersome: it requires an “a priori” knowledge of the location and nature of the boundary layers in order to choose the correct stretched variable for every inner expansion. In general, this method does not proportionate an expansion uniformly valid in the whole domain, but different expansions for different subdomains of  $\Omega$  (of course, they match smoothly on overlapping subdomains). Moreover, the calculation of the coefficients of those expansions requires solving a boundary problem for every coefficient. Apart from these technical difficulties, the rigour of the method is not clear in all the situations [21].

Several authors have proposed a different method to approximate the solution of linear singularly perturbed convection-diffusion problems based on the knowledge of an exact representation of the solution. This method is extraordinarily useful when the Dirichlet data are not smooth [6], [7], [8], [11], [12], [16], [17], [18], [19]. Its main advantages are: i) it does not require an “a priori” knowledge of the location and nature of the boundary layers, ii) the calculation of the coefficients of the expansion is straightforward and iii) it produces an expansion uniformly valid in the whole domain (except in the vicinity of the discontinuities of the boundary condition). In recent papers [11], [12], we have

analyzed several problems of the form (1) with jump discontinuities at some points of the Dirichlet condition using an integral representation of the solution. We have shown that the solution of those problems is approximated in the singular limit  $\epsilon \rightarrow 0^+$  by error functions. Then, the singular layers originated by the discontinuities in the Dirichlet condition are approximately described by error functions.

In this paper we analyze similar problems to those studied in [11], [12], but with a different kind of discontinuity in the Dirichlet condition: the Dirichlet datum is continuous, but its derivative has jump discontinuities at some points. We analyze the effect of this new kind of discontinuity in the form of the singular layers of the solution. For that purpose, we consider two different homogeneous singular perturbed problems with discontinuous boundary data in sections 2 and 3, respectively. In the first problem the discontinuity is located on a straight side of the boundary, whereas in the second problem it is located on a corner. Both problems display boundary or interior layers. As in the references mentioned above, the starting point is an integral representation for the solution. From this integral, we derive complete asymptotic expansions for the solution in the singular limit  $\epsilon \rightarrow 0^+$ . Some comments and a few conclusions are postponed until section 4.

We use the following notation in the paper:  $\epsilon$  is a small positive number. The position vector is  $\vec{r} \equiv (x, y)$ , where  $(x, y)$  are cartesian coordinates. The convection vector  $\vec{v} = (\sin \beta, \cos \beta)$  has modulus one, where  $\beta$  is the angle between the  $y$  axis and  $\vec{v}$ . We use polar coordinates  $x = r \sin \phi$ ,  $y = r \cos \phi$ , with  $\phi$  being the angle between the  $y$  axis and the position vector  $\vec{r}$ . The parameter  $w \equiv 1/(2\epsilon)$  is the positive asymptotic (large) parameter. The characteristic function of an interval  $(a, b)$  is represented by:

$$\chi_{(a,b)}(t) \equiv \begin{cases} 1 & \text{if } t \in (a, b), \\ 0 & \text{if } t \notin (a, b). \end{cases} \quad (2)$$

**2. The half plane.** In this section we study the following problem (see Fig. 1(a)):

$$\begin{cases} -\epsilon \Delta U + \vec{v} \cdot \vec{\nabla} U = 0, & (x, y) \in \Omega_1 \equiv (-\infty, \infty) \times (0, \infty), \\ U(x, 0) = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases} & U \in \mathcal{C}(\bar{\Omega}_1) \cap \mathcal{D}^2(\Omega_1). \end{cases} \quad (3)$$

Observe that the boundary condition is continuous, but its first derivative has a jump discontinuity at the point  $(0, 0)$ .

After the change of the unknown  $U(x, y) = F(x, y) \exp(w \vec{v} \cdot \vec{r})$ , problem (2) is transformed into the Yukawa equation for  $F(x, y)$ :

$$\begin{cases} \Delta F - w^2 F = 0, & (x, y) \in \Omega_1, \\ F(x, 0) = x e^{-wx \sin \beta} \chi_{[0, \infty)}(x), & F \in \mathcal{C}(\bar{\Omega}_1) \cap \mathcal{D}^2(\Omega_1). \end{cases} \quad (4)$$

Problem (4) may not have a unique solution unless we impose a convenient condition upon  $F(x, y)$  concerning its growth at infinity. Then, we add a radiation condition to

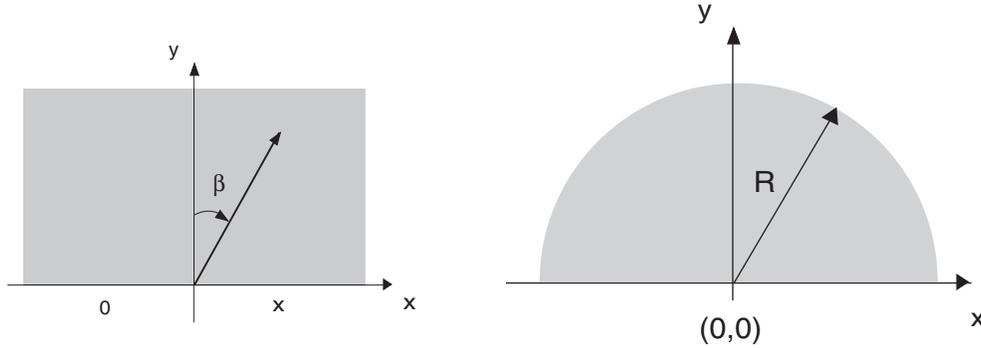


FIG 1. (a) Domain  $\Omega_1$  in problem (3)

(b) Sector  $\Omega_R$  in the proof of Proposition 1

(4) and consider the following problem:

$$\begin{cases} -\epsilon \Delta U + \vec{v} \cdot \vec{\nabla} U = 0, & (x, y) \in \Omega_1, \\ U(x, 0) = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases} \\ U(x, y) = o\left(\frac{e^{w(r+x \sin \beta + y \cos \beta)}}{\sqrt{wr}}\right) & \text{as } r \rightarrow \infty \text{ and } \phi \in (-\pi/2, \pi/2). \end{cases} \quad (5)$$

We have the following uniqueness result:

PROPOSITION 1. Problem (5) has at most one solution.

*Proof.* Suppose that  $U_1(x, y)$  and  $U_2(x, y)$  are two solutions of (5). Then, the function  $G(x, y) \equiv (U_1(x, y) - U_2(x, y)) e^{-w(x \sin \beta + y \cos \beta)}$  satisfies

$$\begin{cases} G \in \mathcal{C}(\bar{\Omega}_1) \cap \mathcal{D}^2(\Omega_1), \\ \Delta G - w^2 G = 0, & (x, y) \in \Omega_1, \\ G(x, 0) = 0, \\ G(x, y) = o\left(\frac{e^{wr}}{\sqrt{wr}}\right) & \text{as } r \rightarrow \infty \text{ and } \phi \in (-\pi/2, \pi/2). \end{cases} \quad (6)$$

*Proof.* Consider the following auxiliary function defined on  $\bar{\Omega}_1$ :

$$V_a(x, y) \equiv \frac{G(x, y)}{H_a(wr)}, \quad H_a(wr) \equiv I_0(wr) + a,$$

where  $I_0$  is a modified Bessel function of order zero and  $a$  is a positive constant. The function  $H_a(wr)$  is positive for  $wr > 0$  and of the order  $\mathcal{O}(e^{wr}/\sqrt{wr})$  as  $wr \rightarrow \infty$  ([1], eqs. 9.7.1). Moreover,  $H_a(wr) \in \mathcal{C}(\bar{\Omega}_1) \cap \mathcal{D}^2(\Omega_1)$  and satisfies the equation  $\Delta H_a - w^2 H_a + aw^2 = 0$  in  $\Omega_1$  ([1], eq. 9.6.1). Therefore, the auxiliary function  $V_a$  is continuous

on  $\bar{\Omega}_1$  and satisfies

$$\begin{cases} \Delta V_a + \frac{2}{H_a} \vec{\nabla} H_a \cdot \vec{\nabla} V_a = \frac{aw^2}{H_a} V_a & \text{in } \Omega_1, \\ V_a(x, 0) = 0 & \forall x \in \mathbb{R}, \\ \lim_{r \rightarrow \infty} V_a(x, y) = 0 & \forall \phi \in [-\pi/2, \pi/2]. \end{cases} \tag{7}$$

Consider the open finite sector of radius  $R$  (see Fig. 1(b)):  $\Omega_R \equiv \{(r \cos \phi, r \sin \phi), 0 < r < R, -\pi/2 \leq \phi \leq \pi/2\}$ . At the points  $(x, y) \in \Omega_R$ , where  $\vec{\nabla} V_a = 0$  and  $V_a \neq 0$ , we have that  $V_a \cdot \Delta V_a > 0$ . Therefore,  $V_a$  has neither positive relative maximums nor negative relative minimums in  $\Omega_R$ . Then  $\text{Sup}_{\Omega_R} |V_a| \leq \text{Sup}_{\partial\Omega_R} |V_a|$ .

Using that  $\lim_{r \rightarrow \infty} V_a(x, y) = 0$  in  $\Omega_1$  we have that,  $\forall \delta > 0$ , there is an  $R > 0$  such that  $|V_a(x, y)| \leq \delta$  for  $x^2 + y^2 = R^2$ . On the other hand,  $V_a(x, 0) = 0 \forall x \in \mathbb{R}$ . Therefore,  $|V_a(x, y)| \leq \delta \forall \delta > 0$  and every  $(x, y) \in \Omega_R$ . Taking the limit  $\delta \rightarrow 0$  ( $R \rightarrow \infty$ ) we have that  $V_a = 0$  on  $\bar{\Omega}_1$ . Therefore,  $G = 0$  and  $U_1 = U_2$  on  $\bar{\Omega}_1$ .  $\square$

In the following proposition we obtain the solution of (5) by means of an integral representation.

PROPOSITION 2. For  $(x, y) \in \Omega_1$ ,  $\beta \in [0, \pi]$  and  $\phi \neq \beta, \pi - \beta$ , the solution  $U_\beta(x, y)$  of (5) is

$$U_\beta(x, y) = I_\beta(x, y) + \begin{cases} (x - y \tan \beta) \chi_{(0, \pi/2)}(\phi - \beta) & \text{if } 0 \leq \beta < \pi/2, \\ 0 & \text{if } \beta = \pi/2, \\ e^{2wy \cos \beta} (x + y \tan \beta) \chi_{(\pi, 3\pi/2)}(\phi + \beta) & \text{if } \pi/2 < \beta \leq \pi, \end{cases} \tag{8}$$

where

$$I_\beta(x, y) \equiv -\frac{e^{wr \cos(\beta-\phi)}}{2w\pi} \int_{-\infty}^{\infty} e^{-wr \cosh u} \frac{\cosh(u + i\phi)}{[\sinh(u + i\phi) - i \sin \beta]^2} du. \tag{9}$$

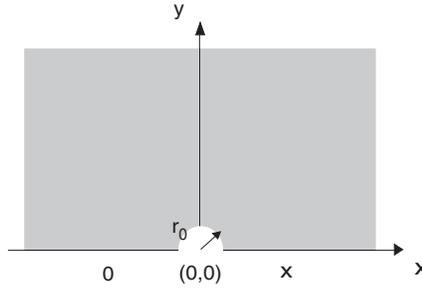
*Proof.* The exact solution of problem (4), valid for  $0 < \beta < \pi$ , may be obtained by taking the Fourier transform of the differential equation with respect to  $x$ :

$$F(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\sqrt{w^2+t^2} y}}{(w \sin \beta + it)^2} e^{itx} dt. \tag{10}$$

(It is easy to check, by direct substitution, that (10) is the solution of problem (4).) Then, the function  $U_\beta(x, y) \equiv e^{w(x \sin \beta + y \cos \beta)} F(x, y)$ , with  $F(x, y)$  defined above, is the solution of (5). After the change of variable  $t = w \sinh u$  in this integral, and using polar variables we obtain

$$U_\beta(x, y) = -\frac{e^{wr \cos(\beta-\phi)}}{2\pi w} \int_{-\infty}^{\infty} e^{-wr \cosh(u-i\phi)} \frac{\cosh u}{(\sinh u - i \sin \beta)^2} du. \tag{11}$$

The poles of the integrand of  $U_\beta(x, y)$  (all of them are double poles except for  $\beta = \pi/2$ ; in this case they are triple poles) are located at the points  $u = i\beta + 2k\pi i$  and  $u = -i\beta + (2k + 1)\pi i$ ,  $k \in \mathbb{Z}$ . The real part of the exponent reads  $-rw \cosh(\Re u) \cos(\phi - \Im u)$ . Therefore, we can use Cauchy's residue theorem for shifting the integration contour in the integral in (11) to the straight line  $\Im u = \phi$ . Applying this technique and distinguishing the cases  $0 < \beta < \pi/2$  and  $\pi/2 < \beta < \pi$ , we obtain (8)–(9). These formulas are also valid for the limit cases  $\beta = 0$  and  $\beta = \pi$ .  $\square$

FIG. 2. Domain  $\Omega_1^*$  used in Theorem 1.

OBSERVATION 1. The explicit representation (8)–(9) is not valid when  $\phi = \beta$  or  $\phi = \pi - \beta$  because of the singularity in the integrand of (9). In these cases the solution  $U_\beta(x, y)$  may be obtained taking the limit  $\phi \rightarrow \beta$  or  $\phi \rightarrow \pi - \beta$  in (8)–(9), respectively ( $U_\beta(x, y)$  is a continuous function of  $r$  and  $\phi$ ).

OBSERVATION 2. The explicit representation given in Proposition 2 is only valid when the angle  $\beta$  between the convection vector  $\vec{v}$  and the  $y$ -axis is restricted to the interval  $[0, \pi]$ . Nevertheless, an explicit integral representation for the solution  $U_\beta(x, y)$  of problem (5), whatever the direction of  $\vec{v}$  is (except for  $\beta = -\pi/2$ ), may be obtained by means of symmetry arguments:

$$U(x, y) = \begin{cases} U_\beta(x, y) & \text{if } \beta \in (0, \pi), \\ x - y \tan \beta + U_{-\beta}(-x, y) & \text{if } \beta \in (-\pi, 0), \quad \beta \neq -\pi/2, \end{cases}$$

where  $U_\beta(x, y)$  is given in (8). Therefore, in the remainder of this section, we restrict ourselves to  $\beta \in [0, \pi]$ .

We denote by  $\Omega_1^*$  the domain  $\Omega_1$  indented at the point  $(0, 0)$  (see Fig. 2):

$$\Omega_1^* \equiv \{(r, \phi), -\pi/2 \leq \phi \leq \pi/2, 0 < r_0 < r < \infty\}.$$

We will need the following lemma to derive the asymptotic behaviour of the solution of problem (5).

LEMMA 1. For  $0 < \alpha < 2\pi$ ,

$$\frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-r \cosh t} \frac{\cosh \frac{1}{2}(t - i\alpha)}{\sinh^2 \frac{1}{2}(t - i\alpha)} dt = -\sqrt{2r} \cos\left(\frac{\alpha}{2}\right) e^{-r \cos \alpha} \operatorname{ierfc}\left(\sqrt{2r} \sin \frac{\alpha}{2}\right) \quad (12)$$

and

$$\begin{aligned} \frac{1}{4\pi i} \int_{-\infty}^{\infty} e^{-r \cosh t} \frac{\frac{1}{2} \sinh^2 \frac{1}{2}(t - i\alpha) - \cosh^2 \frac{1}{2}(t - i\alpha)}{\sinh^3 \frac{1}{2}(t - i\alpha)} dt \\ = e^{-r \cos \alpha} \left[ \sqrt{\frac{r}{2}} \sin \frac{\alpha}{2} \operatorname{ierfc}\left(\sqrt{2r} \sin \frac{\alpha}{2}\right) + 4r \cos^2 \frac{\alpha}{2} i^2 \operatorname{erfc}\left(\sqrt{2r} \sin \frac{\alpha}{2}\right) \right], \end{aligned} \quad (13)$$

where  $\operatorname{ierfc}(x) \equiv \int_x^\infty \operatorname{erfc}(t) dt$  and  $i^2 \operatorname{erfc}(x) \equiv \int_x^\infty \operatorname{ierfc}(t) dt$ .

Consider the following formula [19] valid for  $0 < \alpha < 2\pi$ :

$$e^{-r \cos \alpha} \operatorname{erfc} \left( \sqrt{2r} \sin \frac{\alpha}{2} \right) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-r \cosh t} \frac{dt}{\sinh \frac{1}{2}(t - i\alpha)}. \tag{14}$$

Taking the derivative with respect to  $r$  at both sides of (14) and using [1], eq. 7.2.5,

$$\operatorname{ierfc}(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} - x \operatorname{erfc}(x), \tag{15}$$

we obtain (12). Taking the derivative with respect to  $r$  at both sides of (12) and using [1], eq. 7.2.5,

$$i^2 \operatorname{erfc}(x) = \frac{1}{4} \operatorname{erfc}(x) - \frac{x}{2} \operatorname{ierfc}(x), \tag{16}$$

we obtain (13). □

**THEOREM 1.** For  $(x, y) \in \Omega_1^*$  and  $\beta \in [0, \pi]$ , the solution  $U_\beta(x, y)$  of (5) reads

$$U_\beta(x, y) = U_\beta^0(x, y) + \frac{e^{wr(\cos(\beta-\phi)-1)}}{2\pi w \sqrt{2wr}} U_\beta^1(x, y), \tag{17}$$

where

$$U_\beta^0(x, y) \equiv \begin{cases} \sqrt{\frac{r}{2w}} \frac{\cos\left(\frac{\beta-\phi}{2}\right)}{\cos \beta} \operatorname{ierfc} \left[ \sqrt{2wr} \sin \left( \frac{\beta-\phi}{2} \right) \right] & \text{if } 0 \leq \beta < \pi/2, \\ -e^{2wy \cos \beta} \sqrt{\frac{r}{2w}} \frac{\sin\left(\frac{\beta+\phi}{2}\right)}{\cos \beta} \operatorname{ierfc} \left[ \sqrt{2wr} \cos \left( \frac{\beta+\phi}{2} \right) \right] & \text{if } \pi/2 < \beta \leq \pi, \\ \sqrt{\frac{r}{2w}} \sin \left( \frac{\pi/2 - \phi}{2} \right) \operatorname{ierfc} \left[ \sqrt{2wr} \sin \left( \frac{\pi/2 - \phi}{2} \right) \right] & \text{if } \beta = \pi/2, \\ +4r \cos^2 \left( \frac{\pi/2 - \phi}{2} \right) i^2 \operatorname{erfc} \left[ \sqrt{2wr} \sin \left( \frac{\pi/2 - \phi}{2} \right) \right] & \text{if } \beta = \pi/2, \end{cases} \tag{18}$$

and the function  $U_\beta^1(x, y)$  has an asymptotic expansion in powers of  $(wr)^{-1}$ :

$$U_\beta^1(x, y) = \sum_{k=0}^{n-1} \frac{T_k(x, y)}{(2wr)^k} + R_n(x, y). \tag{19}$$

In this expansion, the coefficients  $T_k$  are smooth functions of  $x$  and  $y$ , and  $\mathcal{O}(1)$  as  $w \rightarrow \infty$  uniformly for  $(x, y) \in \Omega_1^*$ . The remainder  $R_n(x, y)$  satisfies a bound of the form

$$|R_n(x, y)| \leq Mn! \frac{\Gamma(n + \frac{1}{2})}{(2wdr)^n} \tag{20}$$

for some positive constants  $M$  and  $d$ .

*Proof.* For large  $w$  and fixed  $r$ , the asymptotic features of the integral  $I_\beta(x, y)$  defined in (9) are: (i) there is a saddle point at  $u = 0$ . (ii) The poles are situated at  $u_k^1 = i(\beta - \phi + 2k\pi)$  and  $u_k^2 = i(\pi - \beta - \phi + 2k\pi)$ ,  $k \in \mathbb{Z}$ . Then, the saddle point coalesce with  $u_k^1$  when  $\phi \rightarrow \beta + 2k\pi$  or with  $u_k^2$  when  $\phi \rightarrow \pi - \beta + 2k\pi$ . Uniform asymptotic expansions of this kind of integral may be obtained by using the  $i^n \operatorname{erfc}$  function as the basic approximant. Therefore, we need to identify the poles in the integrand of  $I_\beta(x, y)$

which are closest to the point  $u = 0$  (to the real axis). We distinguish three cases:

*Case 1.*  $0 \leq \beta < \pi/2$ . In this case, the double pole  $u_0^1 = i(\beta - \phi)$  is the only one which crosses the real axis when  $\phi$  runs from  $-\pi/2$  to  $\pi/2$ . Therefore, we split off the pole of the integrand in (9) at  $u_0^1$ :

$$\frac{\cosh(u + i\phi)}{(\sinh(u + i\phi) - i \sin \beta)^2} = \frac{\cosh \frac{1}{2}(u - i(\beta - \phi))}{4 \cos \beta \sinh^2 \frac{1}{2}(u - i(\beta - \phi))} + f_1(u, \phi, \beta).$$

(This equation implicitly defines the function  $f_1(u, \phi, \beta)$ .)

Using (12) and the relation [1], eq. 7.2.11,

$$\operatorname{ierfc}(-x) = 2x + \operatorname{ierfc}(x), \quad (21)$$

we obtain

$$\begin{aligned} I_\beta(x, y) = & \sqrt{\frac{r}{2w}} \frac{\cos\left(\frac{\beta-\phi}{2}\right)}{\cos \beta} \operatorname{ierfc} \left[ \sqrt{2wr} \sin\left(\frac{\beta-\phi}{2}\right) \right] \\ & + (y \tan \beta - x) \chi_{(0, \pi/2)}(\phi - \beta) + \bar{I}_\beta^1(x, y) \end{aligned} \quad (22)$$

with

$$\bar{I}_\beta^1(x, y) \equiv -\frac{e^{wr \cos(\beta-\phi)}}{2\pi w} \int_{-\infty}^{\infty} e^{-wr \cosh u} f_1(u, \phi, \beta) du. \quad (23)$$

Using (8), (22) and (23) we obtain (17) with  $U_\beta^0(x, y)$  given in the first line of (18) and

$$U_\beta^1(r, \phi) = 2w\pi\sqrt{2wr} e^{wr(1-\cos(\beta-\phi))} \bar{I}_\beta^1(x, y).$$

*Case 2.*  $\pi/2 < \beta \leq \pi$ . In this case, the double pole  $u_0^2 = i(\pi - \beta - \phi)$  is the only one which crosses the real axis when  $\phi$  runs from  $-\pi/2$  to  $\pi/2$ . Therefore, we split off the pole of the integrand in (9) at  $u_0^2$ :

$$\frac{\cosh(u + i\phi)}{(\sinh(u + i\phi) - i \sin \beta)^2} = -\frac{\cosh \frac{1}{2}(u - i(\pi - \beta - \phi))}{4 \cos \beta \sinh^2 \frac{1}{2}(u - i(\pi - \beta - \phi))} + f_2(u, \phi, \beta).$$

(This equation implicitly defines the function  $f_2(u, \phi, \beta)$ .)

Using (12) and the relation (21), we obtain

$$\begin{aligned} I_\beta(x, y) = & -e^{2wy \cos \beta} \sqrt{\frac{r}{2w}} \frac{\sin\left(\frac{\beta+\phi}{2}\right)}{\cos \beta} \operatorname{ierfc} \left[ \sqrt{2wr} \cos\left(\frac{\beta+\phi}{2}\right) \right] \\ & - e^{2wy \cos \beta} (x + y \tan \beta) \chi_{(\pi, 3\pi/2)}(\phi + \beta) + \bar{I}_\beta^2(x, y) \end{aligned} \quad (24)$$

with

$$\bar{I}_\beta^2(x, y) \equiv -\frac{e^{wr \cos(\beta-\phi)}}{2\pi w} \int_{-\infty}^{\infty} e^{-wr \cosh u} f_2(u, \phi, \beta) du. \quad (25)$$

Using (8), (24) and (25) we obtain (17) with  $U_\beta^0(x, y)$  given in the second line of (18) and

$$U_\beta^1(r, \phi) = 2w\pi\sqrt{2wr} e^{wr(1-\cos(\beta-\phi))} \bar{I}_\beta^2(x, y).$$

*Case 3.*  $\beta = \pi/2$ . In this case, the triple pole  $u_0^1 = u_0^2 = i(\pi/2 - \phi)$  is the closest one to the real axis when  $\phi$  runs from  $-\pi/2$  to  $\pi/2$ . Therefore, we split off the pole of the

integrand in (9) at  $u_0^1$ :

$$\frac{\cosh(u + i\phi)}{(\sinh(u + i\phi) - i)^2} = \frac{i}{2} \frac{\frac{1}{2} \sinh^2 \frac{1}{2}(u - i(\pi/2 - \phi)) - \cosh^2 \frac{1}{2}(u - i(\pi/2 - \phi))}{\sinh^3 \frac{1}{2}(u - i(\pi/2 - \phi))} + f_3(u, \phi).$$

(This equation implicitly defines the function  $f_3(u, \phi)$ .)

Using (13) we obtain

$$I_{\pi/2}(x, y) = \sqrt{\frac{r}{2w}} \sin\left(\frac{\pi/2 - \phi}{2}\right) \operatorname{ierfc}\left[\sqrt{2wr} \sin\left(\frac{\pi/2 - \phi}{2}\right)\right] + 4r \cos^2\left(\frac{\pi/2 - \phi}{2}\right) i^2 \operatorname{erfc}\left[\sqrt{2wr} \sin\left(\frac{\pi/2 - \phi}{2}\right)\right] + \bar{I}_{\pi/2}^3(x, y) \tag{26}$$

with

$$\bar{I}_{\pi/2}^3(x, y) \equiv -\frac{e^{wx}}{2\pi w} \int_{-\infty}^{\infty} e^{-wr \cosh u} f_3(u, \phi) du. \tag{27}$$

Using (8), (26) and (27) we obtain (17) with  $U_{\pi/2}^0(x, y)$  given in the third line of (18) and

$$U_{\pi/2}^1(r, \phi) = 2w\pi\sqrt{2wr} e^{w(r-x)} \bar{I}_{\pi/2}^3(x, y).$$

In order to obtain the asymptotic expansion of  $\bar{I}_{\beta}^1(x, y)$ ,  $\bar{I}_{\beta}^2(x, y)$  and  $\bar{I}_{\pi/2}^3(x, y)$ , given in (23), (25) and (27) respectively, for large  $w$  and bounded  $r \geq r_0 > 0$  we remove the odd part of  $f_1(u, \phi, \beta)$ ,  $f_2(u, \phi, \beta)$  and  $f_3(u, \phi)$  from (23), (25) and (27) respectively, and perform the change of variable  $\sinh(u/2) \equiv t$ . We obtain

$$\bar{I}_{\beta}^i(x, y) = -\frac{e^{wr(\cos(\beta-\phi)-1)}}{\pi w} \int_0^{\infty} e^{-2wrt^2} g_i(t^2, \phi, \beta) dt, \quad i = 1, 2, 3, \tag{28}$$

with

$$g_i(t, \phi, \beta) \equiv 2 \cos \phi \frac{8t^3 + 12t^2 + 2t(\cos^2 \phi + \cos^2 \beta) - (\sin \beta - \sin \phi)^2}{[4t^2 + 4(1 - \sin \beta \sin \phi)t + (\sin \phi - \sin \beta)^2]^2 \sqrt{1 + t}} - \begin{cases} \cos\left(\frac{\beta - \phi}{2}\right) \frac{t - \sin^2 \frac{\beta - \phi}{2}}{2 \cos \beta \left(t + \sin^2 \frac{\beta - \phi}{2}\right)^2} & \text{if } i = 1, \\ -\sin\left(\frac{\beta + \phi}{2}\right) \frac{t - \cos^2 \frac{\beta + \phi}{2}}{2 \cos \beta \left(t + \cos^2 \frac{\beta + \phi}{2}\right)^2} & \text{if } i = 2, \\ \sin\left(\frac{\pi/2 - \phi}{2}\right) \frac{2t^2 + 12 \cos^2 \frac{\pi/2 - \phi}{2} t - (3 + \sin \phi) \sin^2 \frac{\pi/2 - \phi}{2}}{4 \left(t + \sin^2 \frac{\pi/2 - \phi}{2}\right)^3} & \text{if } i = 3. \end{cases} \tag{29}$$

These functions have a Taylor expansion at  $t = 0$  for each  $\phi \in [-\pi/2, \pi/2]$ :

$$g_i(t, \phi, \beta) = \sum_{k=0}^{n-1} \frac{g_i^{(k)}(0, \phi, \beta)}{k!} t^k + g_n^i(t, \phi, \beta) \quad i = 1, 2, 3, \tag{30}$$

where

$$g_n^i(t, \phi, \beta) \equiv \frac{g_i^{(n)}(\xi, \phi, \beta)}{n!} t^n, \quad \xi \in (0, t).$$

The singularities of  $g_i(t, \phi, \beta)$  for  $i = 1, 2, 3$  are away from the positive real axis (let  $d_i$  be the distance from the closest singularity to the positive real axis). Therefore, using the Cauchy formula for the derivative  $g_i^{(n)}(\xi, \phi, \beta)$ , we see that

$$|g_n^i(t, \phi, \beta)| \leq M_i \frac{n!}{d_i^n} t^n, \quad i = 1, 2, 3, \tag{31}$$

where  $M_i$  is a bound for  $g_i(w, \phi, \beta)$  with  $i = 1, 2, 3$ , on the portion of the complex  $w$ -plane surrounding the positive real axis:  $\{w \in \mathbb{C}, |w - u| < d, u \in \mathbb{R}^+\}$ . Introducing the expansion (30) in (28) we obtain that

$$\bar{I}_\beta^i(x, y) = \frac{e^{wr(\cos(\beta-\phi)-1)}}{2\pi w\sqrt{2wr}} \left[ \sum_{k=0}^{n-1} \frac{T_k^i(x, y)}{(2wr)^k} + R_n^i(x, y) \right], \quad i = 1, 2, 3, \tag{32}$$

where

$$T_k^i(x, y) \equiv -\frac{g_i^{(k)}(0, \phi, \beta)}{k!} \Gamma\left(k + \frac{1}{2}\right), \quad i = 1, 2, 3, \tag{33}$$

and

$$R_n^i(x, y) \equiv -2\sqrt{2wr} \int_0^\infty e^{-2wrt^2} g_n^i(t^2, \phi, \beta) dt, \quad i = 1, 2, 3. \tag{34}$$

Using (31) in (34) we obtain the bound (20) which shows the asymptotic character of (19).  $\square$

**3. The quarter plane.** In this section we consider the problem (see Fig. 4(a))

$$\begin{cases} -\epsilon \Delta U + \vec{v} \cdot \vec{\nabla} U = 0, & (x, y) \in \Omega_2 \equiv (0, \infty) \times (0, \infty), \\ U(x, 0) = 0, U(0, y) = y, & U \in \mathcal{C}(\bar{\Omega}_2) \cap \mathcal{D}^2(\Omega_2). \end{cases} \tag{35}$$

As well as in the preceding problem, the boundary conditions at the  $X$  and  $Y$  axes match continuously at the corner point  $(0, 0)$ , but their derivatives do not.

After the change of the unknown  $U(x, y) = F(x, y) \exp(w\vec{v} \cdot \vec{r})$ , problem (35) is transformed into the Yukawa equation for  $F(x, y)$ :

$$\begin{cases} \Delta F - w^2 F = 0, & (x, y) \in \Omega_2, \\ F(x, 0) = 0, F(0, y) = y e^{-wy \cos \beta}, & F \in \mathcal{C}(\bar{\Omega}_2) \cap \mathcal{D}^2(\Omega_2). \end{cases} \tag{36}$$

As well as in the previous problem, uniqueness of the solution of (36) requires a convenient condition upon  $F(x, y)$  concerning its growth at infinity. In fact, the problem: find  $U \in \mathcal{C}(\bar{\Omega}_2) \cap \mathcal{D}^2(\Omega_2)$  such that

$$\begin{cases} -\epsilon \Delta U + \vec{v} \cdot \vec{\nabla} U = 0, & (x, y) \in \Omega_2, \\ U(x, 0) = 0, U(0, y) = y, \\ U(x, y) = o\left(\frac{e^{w(r+x \sin \beta + y \cos \beta)}}{\sqrt{wr}}\right) \text{ as } r \rightarrow \infty \text{ and } \phi \in (0, \pi/2), \end{cases} \tag{37}$$

has a unique solution. The proof of this assertion is almost identical to the proof of Proposition 1.

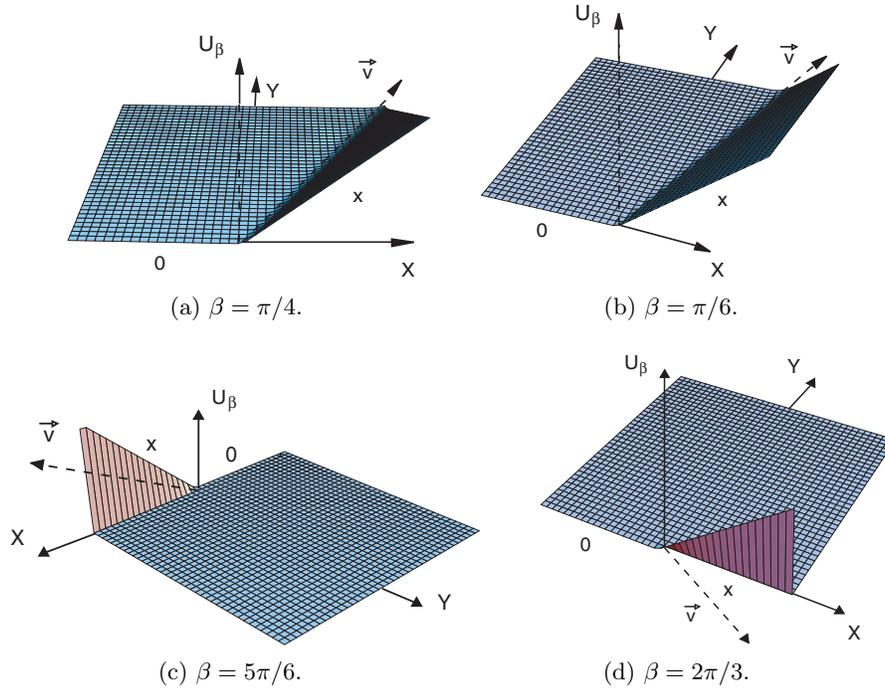


FIG. 3. Graphs of the first order approximation,  $U_\beta^0(r, \phi)$ , to the solution of problem (2) for  $\epsilon = 0.1$  and different values of  $\beta$ .

In the following proposition we obtain the solution of (37) by means of an integral representation.

PROPOSITION 3. For  $(x, y) \in \Omega_2$ ,  $-\pi/2 \leq \beta \leq \pi/2$  and  $\phi \neq \pm\beta, \pi \pm \beta$ , the solution  $U_\beta(x, y)$  of (37) is

$$U_\beta(x, y) = I_\beta(x, y) + \begin{cases} (y - x \cot \beta) \chi_{(0, \pi/2)}(\beta - \phi) & \text{if } 0 < \beta \leq \pi/2, \\ 0 & \text{if } \beta = 0, \\ e^{2wx \sin \beta} (y + x \cot \beta) \chi_{(-\pi/2, 0)}(\beta + \phi) & \text{if } -\pi/2 \leq \beta < 0, \end{cases} \quad (38)$$

with

$$I_\beta(x, y) \equiv -\frac{2 \cos \beta e^{wr \cos(\beta - \phi)}}{w\pi i} \int_{-\infty}^{\infty} e^{-wr \cosh u} \frac{\cosh(u - i\phi) \sinh(u - i\phi)}{[\cosh^2(u - i\phi) - \cos^2 \beta]^2} du. \quad (39)$$

*Proof.* The exact solution of problem (36), valid for  $-\pi/2 < \beta < \pi/2$ , may be obtained by taking the sine transform of the differential equation with respect to  $y$ :

$$F(x, y) = \frac{2w \cos \beta}{\pi i} \int_{-\infty}^{\infty} \frac{t e^{-\sqrt{w^2 + t^2} x}}{(t^2 + w^2 \cos^2 \beta)^2} e^{ity} dt. \quad (40)$$

Then, the function  $U_\beta(x, y) \equiv e^{w(x \sin \beta + y \cos \beta)} F(x, y)$ , with  $F(x, y)$  defined above, is the solution of (37). After the change of variable  $t = w \sinh u$  in this integral, and using

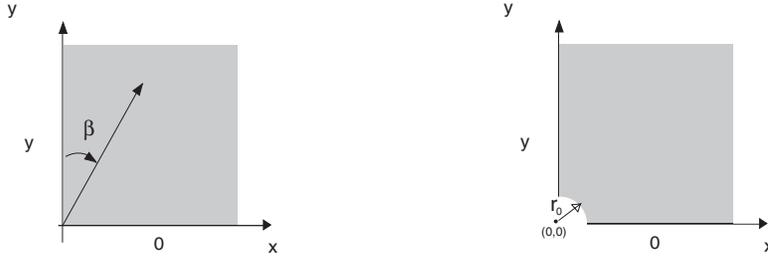


FIG 4. (a) Domain  $\Omega_2$  in problem (2)

(b) Domain  $\Omega_2^*$  in Theorem 2.

polar variables we have

$$U_\beta(x, y) = \frac{2 \cos \beta e^{wr \cos(\beta-\phi)}}{w\pi i} \int_{-\infty}^{\infty} e^{-wr \cosh(u-i(\pi/2-\phi))} \frac{\cosh u \sinh u}{(\sinh^2 u + \cos^2 \beta)^2} du. \quad (41)$$

The poles of the integrand of  $U_\beta(x, y)$  (all of them are double poles except for  $\beta = 0$ ; in this case they are triple poles) are located at the points  $u = i(\frac{\pi}{2} \pm \beta + n\pi)$ ,  $n \in \mathbb{Z}$ , and the real part of the exponent reads  $-rw \cosh(\Re u) \sin(\phi + \Im u)$ . Then, as well as in the previous problem, we can use Cauchy’s residue theorem for shifting the integration contour in the integral in (41) to the straight line  $\Im u = \pi/2 - \phi$ . Applying this technique and distinguishing the cases  $0 \leq \beta < \pi/2$  and  $-\pi/2 < \beta < 0$ , we obtain (38)–(39). These formulas are also valid for the limit cases  $\beta = \pi/2$  and  $\beta = -\pi/2$ .  $\square$

OBSERVATION 3. The explicit representation (38)–(39) is not valid when  $\phi = \pm\beta, \pi \pm \beta$  because of the singularity in the integrand of (39). In these cases the solution  $U_\beta(x, y)$  may be obtained by taking the limit  $\phi \rightarrow \pm\beta$  or  $\phi \rightarrow \pi \pm \beta$  in (38)–(39), respectively ( $U_\beta(x, y)$  is a continuous function of  $r$  and  $\phi$ ).

OBSERVATION 4. The explicit representation given in Proposition 3 is only valid when the angle  $\beta$  between the convection vector  $\vec{v}$  and the y-axis is restricted to the interval  $[-\pi/2, \pi/2]$ . Nevertheless, an explicit integral representation for the solution  $U_\beta(x, y)$  of problem (37) whatever the direction of  $\vec{v}$  is (except for  $\beta = \pi$ ), may be obtained by means of symmetry arguments:

$$U(x, y) = \begin{cases} U_\beta(x, y) & \text{if } \beta \in [-\pi/2, \pi/2], \\ y - x \cot \beta + \cot \beta U_{\pi/2-\beta}(y, x) & \text{if } \beta \in (\pi/2, \pi), \\ e^{2wx \sin \beta} [y + x \cot \beta - \cot \beta U_{\pi/2+\beta}(y, x)] & \text{if } \beta \in (-\pi, -\pi/2), \end{cases}$$

where  $U_\beta(x, y)$  is given in (38). Therefore, in the remainder of this section, we restrict ourselves to  $\beta \in [-\pi/2, \pi/2]$ .

We denote by  $\Omega_2^*$  the domain  $\Omega_2$  indented at the point  $(0, 0)$  (see Fig. 4(b)):

$$\Omega_2^* \equiv \{(r, \phi), 0 \leq \phi \leq \pi/2, 0 < r_0 < r < \infty\}.$$

THEOREM 2. For  $(x, y) \in \Omega_2^*$  and  $\beta \in [-\pi/2, \pi/2]$ , the solution  $U_\beta(x, y)$  of (37) reads

$$U_\beta(x, y) = U_\beta^0(x, y) + \frac{e^{wr(\cos(\beta-\phi)-1)}}{2w\pi \sqrt{2w}} U_\beta^1(x, y), \quad (42)$$

where

$$U_\beta^0(x, y) \equiv \begin{cases} \sqrt{\frac{r}{2w}} \frac{\cos\left(\frac{\phi-\beta}{2}\right)}{\sin\beta} \operatorname{ierfc}\left[\sqrt{2wr} \sin\left(\frac{\phi-\beta}{2}\right)\right] & \text{if } 0 < \beta \leq \pi/2, \\ -e^{2wx \sin\beta} \sqrt{\frac{r}{2w}} \frac{\cos\left(\frac{\phi+\beta}{2}\right)}{\sin\beta} \operatorname{ierfc}\left[\sqrt{2wr} \sin\left(\frac{\phi+\beta}{2}\right)\right] & \text{if } -\pi/2 \leq \beta < 0, \\ \sqrt{\frac{r}{2w}} \sin\left(\frac{\phi}{2}\right) \operatorname{ierfc}\left[\sqrt{2wr} \sin\left(\frac{\phi}{2}\right)\right] \\ + 4r \cos^2\left(\frac{\phi}{2}\right) i^2 \operatorname{erfc}\left[\sqrt{2wr} \sin\left(\frac{\phi}{2}\right)\right] & \text{if } \beta = 0. \end{cases} \tag{43}$$

The function  $U_\beta^1(x, y) = \mathcal{O}(1)$  as  $w \rightarrow \infty$  uniformly for  $(x, y) \in \Omega_2^*$ . Moreover, it admits an asymptotic expansion in inverse powers of  $wr$  similar to (19).

*Proof.* For large  $w$  and fixed  $r$ , the asymptotic features of the integral  $I_\beta(x, y)$  defined in (39) are: (i) there is a saddle point at  $u = 0$ . (ii) The poles are situated at  $u = i(\phi \pm \beta + k\pi)$  with  $k \in \mathbb{Z}$ . Then, the saddle point coalesce with a pole when  $\phi \rightarrow \mp\beta - k\pi$ . Uniform asymptotic expansions of this kind of integral may be obtained by using the  $i^n \operatorname{erfc}$  function as the basic approximant. Therefore, we need to identify the poles in the integrand of  $I_\beta(x, y)$  which are closest to the point  $u = 0$  (to the real axis). We distinguish three cases:

*Case 1.*  $0 < \beta \leq \pi/2$ .

In this case, the double pole  $u = i(\phi - \beta)$  is the only one which crosses the real axis when  $\phi$  runs from 0 to  $\pi/2$ . Therefore, we split off the pole of the integrand in (39) at  $u = i(\phi - \beta)$ :

$$\frac{\cosh(u - i\phi) \sinh(u - i\phi)}{[\cosh^2(u - i\phi) - \cos^2\beta]^2} = -\frac{\cosh\frac{1}{2}[u - i(\phi - \beta)]}{16i \sin\beta \cos\beta \sinh^2\frac{1}{2}[u - i(\phi - \beta)]} + f_1(u, \phi, \beta).$$

(This equation implicitly defines the function  $f_1(u, \phi, \beta)$ .)

Using (12) and the relation (21), we obtain

$$I_\beta(x, y) = \sqrt{\frac{r}{2w}} \frac{\cos\frac{1}{2}(\phi - \beta)}{\sin\beta} \operatorname{ierfc}\left[\sqrt{2wr} \sin\frac{1}{2}(\phi - \beta)\right] + (x \cot\beta - y)\chi_{(0, \pi/2)}(\beta - \phi) + \bar{I}_\beta^1(x, y), \tag{44}$$

with

$$\bar{I}_\beta^1(x, y) \equiv -\frac{2 \cos\beta e^{wr \cos(\beta - \phi)}}{w\pi i} \int_{-\infty}^{\infty} e^{-wr \cosh u} f_1(u, \phi, \beta) du. \tag{45}$$

Using (38), (44) and (45) we obtain (42) with  $U_\beta^0(x, y)$  given in the first line of (43) and

$$U_\beta^1(r, \phi) = 2w\pi\sqrt{2wr} e^{wr(1 - \cos(\beta - \phi))} \bar{I}_\beta^1(x, y).$$

*Case 2.*  $-\pi/2 \leq \beta < 0$ .

In this case, the double pole  $u = i(\phi + \beta)$  is the only one which crosses the real axis when  $\phi$  runs from 0 to  $\pi/2$ . Therefore, we split off the pole of the integrand in (39) at  $u = i(\phi + \beta)$ :

$$\frac{\cosh(u - i\phi) \sinh(u - i\phi)}{[\cosh^2(u - i\phi) - \cos^2 \beta]^2} = \frac{\cosh \frac{1}{2} [u - i(\phi + \beta)]}{16i \sin \beta \cos \beta \sinh^2 \frac{1}{2} [u - i(\phi + \beta)]} + f_2(u, \phi, \beta).$$

(This equation implicitly defines the function  $f_2(u, \phi, \beta)$ .)

Using (12) and the relation (21), we obtain

$$I_\beta(x, y) = e^{2wx \sin \beta} \sqrt{\frac{r}{2w}} \frac{\cos \frac{1}{2} (\phi + \beta)}{\sin \beta} \operatorname{ierfc} \left[ \sqrt{2wr} \sin \frac{1}{2} (\phi + \beta) \right] - e^{2wx \sin \beta} (x \cot \beta + y) \chi_{(-\pi/2, 0)}(\beta + \phi) + \bar{I}_\beta^2(x, y) \tag{46}$$

with

$$\bar{I}_\beta^2(x, y) \equiv -\frac{2w \cos \beta e^{wr \cos(\beta - \phi)}}{w\pi i} \int_{-\infty}^{\infty} e^{-wr \cosh u} f_2(u, \phi, \beta) du. \tag{47}$$

Using (38), (46) and (47) we obtain (42) with  $U_\beta^0(x, y)$  given in the second line of (43) and

$$U_\beta^1(r, \phi) = 2w\pi \sqrt{2wr} e^{wr(1 - \cos(\beta - \phi))} \bar{I}_\beta^2(x, y).$$

*Case 3.*  $\beta = 0$ .

In this case, the triple pole  $u = i\phi$  is the closest one to the real axis when  $\phi$  runs from 0 to  $\pi/2$ . Therefore, we split off the pole of the integrand in (39) at  $u = i\phi$ :

$$\frac{\cosh(u - i\phi) \sinh(u - i\phi)}{[\cosh^2(u - i\phi) - 1]^2} = -\frac{1}{8} \frac{\frac{1}{2} \sinh^2 \frac{1}{2} (u - i\phi) - \cosh^2 \frac{1}{2} (u - i\phi)}{\sinh^3 \frac{1}{2} (u - i\phi)} + f_3(u, \phi).$$

(This equation implicitly defines the function  $f_3(u, \phi)$ .)

Using (13) we obtain

$$I_0(x, y) = \sqrt{\frac{r}{2w}} \sin \left( \frac{\phi}{2} \right) \operatorname{ierfc} \left[ \sqrt{2wr} \sin \left( \frac{\phi}{2} \right) \right] + 4r \cos^2 \left( \frac{\phi}{2} \right) i^2 \operatorname{erfc} \left[ \sqrt{2wr} \sin \left( \frac{\phi}{2} \right) \right] + \bar{I}_0^3(x, y) \tag{48}$$

with

$$\bar{I}_0^3(x, y) \equiv -\frac{2e^{wy}}{w\pi i} \int_{-\infty}^{\infty} e^{-wr \cosh u} f_3(u, \phi) du. \tag{49}$$

Using (38), (48) and (49) we obtain (42) with  $U_0^0(x, y)$  given in the third line of (43) and

$$U_0^1(r, \phi) = 2w\pi \sqrt{2wr} e^{w(r-y)} \bar{I}_0^3(x, y).$$

From here, the derivation of the asymptotic expansion of  $U_\beta^1(x, y)$  in inverse powers of  $wr$  (and bounded  $r \geq r_0 > 0$ ) is similar to the proof of Theorem 1.  $\square$

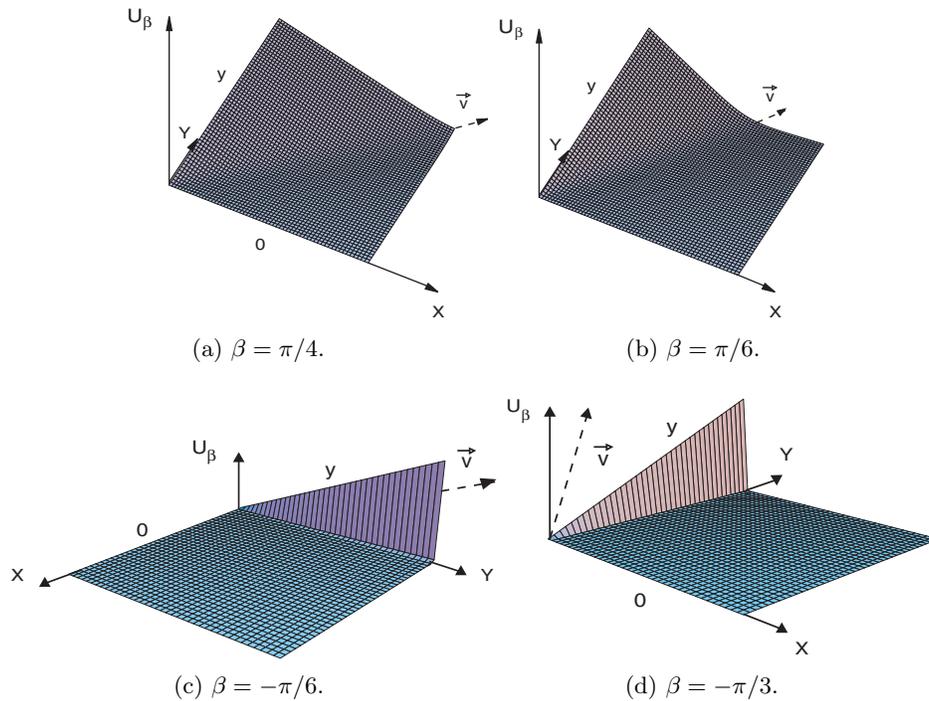


FIG. 5. Graphs of the first order approximation  $U_\beta^0(r, \phi)$ , to the solution of problem (35) for  $\epsilon=0.1$  and different values of  $\beta$ .

**4. Conclusions.** The Dirichlet data of the two singularly perturbed convection-diffusion problems analyzed in sections 3 and 4 have discontinuous derivatives at a corner and at a side of the inflow boundary, respectively. For each problem, we have obtained an integral representation of the solution susceptible of an asymptotic analysis. Then, for each problem, we have derived an asymptotic expansion of the solution in the singular limit  $\epsilon \rightarrow 0^+$  (and away from the discontinuities). This asymptotic expansion is obtained from a modification of the standard uniform method: “saddle point near a pole”. As a difference with respect to the standard situation where the multiplicity of the pole is one, in our problems the pole has multiplicity two. Then, the asymptotic approximant is not an error function, but a primitive of the error function.

The asymptotic expansions derived above show that the main contribution of the discontinuities in the derivatives of the data to the shape of the solution on the boundary or interior layers is contained in the primitives of an error function.

We want to emphasize the simultaneous dependence of the solution of these problems with the singular parameter  $\epsilon$  and the distance to the discontinuity point of the derivative of the boundary data. The solution  $U_\beta$  depends on  $\epsilon$ , and the distance  $r$  to the discontinuity point  $(0,0)$  through the quotient  $r/\epsilon$ . This is why the asymptotic expansions derived above do not hold near these discontinuities.

It was shown in [11], [12] that, when the Dirichlet condition is discontinuous, the solution to these kinds of problems is approximated in the singular limit by error functions.

We have shown here that when the Dirichlet condition is continuous, but its derivative is discontinuous, then the solution is approximated by a primitive of the error function. We conjecture that when the  $n - 1$ -th derivative of the Dirichlet condition is continuous, but its  $n$ -th derivative is discontinuous, then the solution to these kinds of problems is approximated by an  $n$ -th primitive of the error function.

We suspect that, as in the problems analyzed here, the error function plays a fundamental role in the approximation of the solution of singularly perturbed convection-diffusion problems with discontinuities in the Dirichlet boundary conditions defined over more general domains and through more general coefficients. This will be the subject of further investigations.

**5. Acknowledgments.** The *Sec. Est. Educ. y Univ., Programa Salvador de Mada-riaga, res. 28/03/03* is acknowledged for its financial support.

#### REFERENCES

- [1] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, *Dover, New York*, 1970.
- [2] C. Clavero, J. C. Jorge and F. Lisbona, Uniformly convergent schemes for singular perturbation problems combining alternating directions and exponential fitting techniques, *Boole Press, Ireland*, 1993, pp. 33-52. MR1245731 (94j:65106)
- [3] L. P. Cook and G. S. S. Ludford, The behavior as  $\epsilon \rightarrow 0^+$  of solutions to  $\epsilon \nabla^2 w = \partial w / \partial y$  on the rectangle  $0 \leq x \leq l$ ,  $|y| \leq 1$ , *SIAM J. Math. Anal.*, **4**, no. 1 (1973) 161-184. MR0364828 (51:1082)
- [4] W. Eckhaus, Matched Asymptotic Expansions and Singular Perturbations, *North-Holland, Amsterdam*, 1973. MR0670800 (58:32369)
- [5] W. Eckhaus and E. M. de Jager, Asymptotic solutions of singular perturbation problems for linear differential equations of elliptic type, *Arch. Rat. Mech. Anal.*, **23**, (1966) 26-86. MR0206464 (34:6283)
- [6] J. Grasman, On singular perturbations and parabolic boundary layers, *J. Eng. Math.*, **2**, no. 2 (1968) 163-172. MR0233059 (38:1382)
- [7] J. Grasman, An elliptic singular perturbation problem with almost characteristic boundaries, *J. Math. Anal. Appl.*, **46**, (1974) 438-446. MR0352672 (50:5159)
- [8] G. W. Hedstrom and A. Osterheld, The effect of cell Reynolds number on the computation of a boundary layer, *J. Comput. Phys.*, **37** (1980) 399-421. MR0588260 (82a:76023)
- [9] A.M. Il'in, Matching of Asymptotic Expansions of Solutions of Boundary Value Problems, *AMS, Providence*, 1992. MR1182791 (93g:35016)
- [10] J. Kevorkian and J.D. Cole, Multiple Scale and Singular Perturbation Methods, *Springer-Verlag, New York*, 1996. MR1392475 (97k:34001)
- [11] J.L. López and E. Pérez Sinusía, Asymptotic expansions for two singularly perturbed convection-diffusion problems with discontinuous data: the quarter plane and the infinite strip. *Stud. Appl. Math.*, **113**, (2004) 57-89. MR2061650
- [12] J.L. López and E. Pérez Sinusía, Analytic approximations for a singularly perturbed convection-diffusion problems with discontinuous data in a half-infinite strip. *Acta Appl. Math.*, **82**, (2004) 101-117. MR2061479 (2005e:35048)
- [13] J. H. Miller, E. O'Riordan and G. I. Shishkin, Fitted Numerical Methods for Singular Perturbation Problems, *World Scientific*, 1996. MR1439750 (98c:65002)
- [14] R. E. O'Malley, Introduction to Singular Perturbations, *Academic Press, New York*, 1974. MR0402217 (53:6038)
- [15] S.-D. Shih and R. B. Kellogg, Asymptotic analysis of a singular perturbation problem, *SIAM J. Math. Anal.*, **18**, no. 5 (1987) 1467-1511. MR0902346 (88j:35020)
- [16] S.-D. Shih, A Novel uniform expansion for a singularly perturbed parabolic problem with corner singularity, *Meth. Appl. Anal.*, **3**, no. 2 (1996) 203-227. MR1420366 (98c:35011)
- [17] N.M. Temme, Analytical methods for a singular perturbation problem. The quarter plane, *C.W.I. Report*, **125**, (1971).

- [18] N.M. Temme, Analytical methods for a singular perturbation problem in a sector, *SIAM J. Math. Anal.*, **5**, no. 6 (1974) 876-887. MR0369876 (51:6105)
- [19] N.M. Temme, Analytical methods for a selection of elliptic singular perturbation problems, *Recent advances in differential equations (Kunming, 1997)*, 131-148, *Pitman Res. Notes Math. Ser.*, 386, *Longman, Harlow*, 1998. MR1628077 (99c:35018)
- [20] A. B. Vasil'eva, V. F. Butuzov and L. V. Kalachev, *The Boundary Function Method for Singular Perturbation Problems*, *SIAM, Philadelphia*, 1995. MR1316892 (96a:34119)
- [21] C. H. Ou and R. Wong, *On a two-point boundary value problem with spurious solution*. *Stud. Appl. Math.* **111** no. 4 (2003) 377-408. MR2004239 (2004f:34081)