First Order Approximation of an Elliptic 3D Singular Perturbation Problem

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A three-dimensional elliptic singular perturbation problem with discontinuous boundary values is considered. The solution of the problem is written in terms of a double integral. A saddle point analysis is used to obtain a first approximation, which is expressed in terms of a function that can be viewed as a generalization of the complementary error function.

1. Introduction

We consider a singularly perturbed convection-diffusion problem defined on the positive half-space: $\Omega = (-\infty, \infty) \times (-\infty, \infty) \times (0, \infty)$, with a "square-shaped source of contamination" located at the plane $z = 0$ (see Figure 1):

$$\begin{cases} -\varepsilon \Delta U + U_z = 0, & \text{if } (x, y, z) \in \Omega, \\ U(x, y, 0) = \chi_{(-1,1)}(x)\chi_{(-1,1)}(y), & \text{for } -\infty \leq x, y \leq \infty, \end{cases} \tag{1}$$

where ε is a small positive parameter and $\chi_{(a,b)}(x)$ is the characteristic function of the interval (a, b):

$$\chi_{(a,b)}(x) \equiv \begin{cases} 1 & \text{if } x \in (a, b), \\ 0 & \text{if } x \notin (a, b). \end{cases} \tag{2}$$

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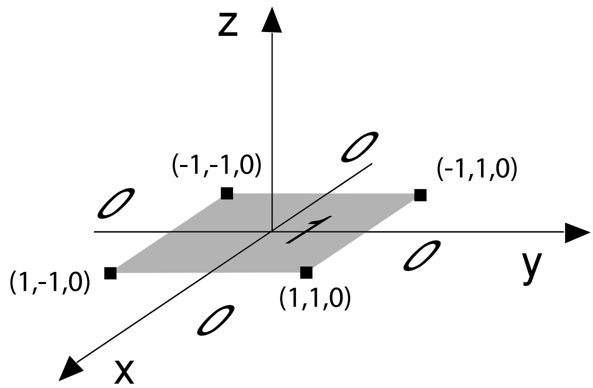

Figure 1. Domain Ω and Dirichlet conditions of problem (1).

Observe that the Dirichlet data at $z = 0$ are discontinuous at the boundary of the unit square in the plane $z = 0$.

The solution of this problem may be derived by using Fourier transforms with respect to x and y, and solving the resulting equation by separating the variables. We obtain

$$U(x, y, z) = \frac{e^{\omega z}}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sin(\omega t)}{t} \frac{\sin(\omega s)}{s} e^{i\omega x t + i\omega y s - z\omega\sqrt{1+t^2+s^2}} \, dt \, ds, \qquad (3)$$

where

$$\omega = \frac{1}{2\varepsilon}. \qquad (4)$$

It is easy to check by direct substitution that this function is a solution of problem (1). But this solution may not be unique unless we impose additional hypotheses on problem (1). In Section 6, we give a more precise formulation of the problem in (1) and prove that (3) is the unique solution of problem (1).

We investigate the behavior of U when ε is small, in particular for x and y values near ± 1. These values correspond with the discontinuous boundary values at $z = 0$, and for $z > 0$ and near $x = \pm 1$, $y = \pm 1$ boundary layers occur. We use saddle point analysis for the double integral in (3) to obtain a first approximation for $U(x, y, z)$. The approximation holds uniformly for $z \geq z_0 > 0$, where z_0 is a fixed number, and for all $-\infty \leq x, y \leq \infty$; in particular it is uniformly valid near the values $x = \pm 1$ and $y = \pm 1$.

This paper is a further step in studying singular perturbation problems with rather simple differential operators, discontinuous boundary conditions, and domains. For these problems, we are able to solve the boundary-value problems in terms of an integral, from which detailed information can be obtained of the asymptotic behavior of the solutions of the problem. For our earlier recent research on this topic for two-dimensional problems, we refer to [4] and [5]; see also [7] and [8]. In these papers, the (complementary) error function plays an important role for describing the asymptotic behavior of the solutions as well as inside as outside the boundary layer, because of the uniform nature of the approximations.

In the present paper, in which we consider a model problem of an elliptic singular pertubation problem in three space dimensions, the role of the complementary error function is taken over by a generalization of this function. We give several properties of this function, and describe how the function $U(x, y, z)$ given by the double integral in (3) can be approximated by this generalization.

The knowledge of the asymptotic behavior of the solutions of model singular perturbation problems is of interest in the development of suitable numerical methods for this kind of problems because it gives the possibility of comparing the values obtained from numerical schemes with those obtained from analytical approximations. Of special interest are boundary- or initial-value problems with discontinuous boundary or initial values; see, for example, [2].

2. Asymptotic analysis

We replace the sine functions in (3) by exponentials, but first we shift the paths of integration slightly upward in the complex s and t planes. In this way, the poles at the origins are avoided. This gives four integrals, and we can write

$$U(x,y,z) = \frac{e^{\omega z}}{4\pi^2}(-U_{1,1} + U_{-1,1} + U_{1,-1} - U_{-1,-1}), \tag{5}$$

where

$$U_{j,k} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega(j+x)t + i\omega(k+y)s - z\omega\sqrt{1+t^2+s^2}}\,\frac{dt\,ds}{t\,s}, \qquad j,k = \pm 1. \tag{6}$$

All four integrals in (6) are of the type

$$V(\xi,\eta,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\omega\phi(s,t)}\,\frac{dt\,ds}{t\,s}, \tag{7}$$

where

$$\phi(s,t) = -i\xi t - i\eta s + z\sqrt{1+t^2+s^2}, \tag{8}$$

where $\xi = \pm 1 + x$, $\eta = \pm 1 + y$, and the paths in the t-plane and s-plane in the integral in (7) run slightly above the real axes.

To start, we assume that $\xi > 0$, $\eta > 0$, and we always assume that $z > 0$. In the two-dimensional saddle point analysis, we try to find saddle points by solving the equations $\partial\phi/\partial s = 0$ and $\partial\phi/\partial t = 0$. That is, we have to solve the equations

$$\frac{\partial \phi}{\partial t} = -i\xi + \frac{zt}{\sqrt{1+t^2+s^2}} = 0, \quad \frac{\partial \phi}{\partial s} = -i\eta + \frac{zs}{\sqrt{1+t^2+s^2}} = 0. \tag{9}$$

Solutions s_0 and t_0 of these equations satisfy

$$s_0^2 = -\frac{\eta^2}{\rho^2}, \quad t_0^2 = -\frac{\xi^2}{\rho^2}, \tag{10}$$

where ρ is the positive number defined by

$$\rho = \sqrt{\xi^2 + \eta^2 + z^2}. \tag{11}$$

Taking square roots in (10) gives several possibilities for s_0 and t_0, but only the following solutions satisfy the equations in (9):

$$s_0 = i\frac{\eta}{\rho}, \quad t_0 = i\frac{\xi}{\rho}. \tag{12}$$

We expand $\phi(s, t)$ at the saddle points up to and including second-order terms. We obtain, because $\phi(s_0, t_0) = \rho$ and the first-order terms vanish at the saddle points,

$$\phi(s,t) = \rho + \frac{1}{2}\frac{\partial^2\phi}{\partial s^2}(s-s_0)^2 + \frac{\partial^2\phi}{\partial s\partial t}(s-s_0)(t-t_0) + \frac{1}{2}\frac{\partial^2\phi}{\partial t^2}(t-t_0)^2 + \cdots, \tag{13}$$

where the partial derivatives are evaluated at (s_0, t_0). That is,

$$\frac{\partial^2\phi}{\partial s^2} = \frac{\rho(\eta^2+z^2)}{z^2}, \quad \frac{\partial^2\phi}{\partial s\partial t} = \frac{\rho\xi\eta}{z^2}, \quad \frac{\partial^2\phi}{\partial t^2} = \frac{\rho(\xi^2+z^2)}{z^2}. \tag{14}$$

For a first approximation, we replace $\phi(s, t)$ in (7) by the first terms in the Taylor expansion given in (13). We also shift the two paths of integration in (7) through the saddle points s_0 and t_0 on the positive imaginary axes, and we introduce the new variables of integration

$$\sigma = s - s_0, \quad \tau = t - t_0. \tag{15}$$

This gives the approximation

$$V_1(\xi,\eta,z) = e^{-\omega\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\lambda(A\sigma^2+2B\sigma\tau+C\tau^2)}\frac{d\tau\, d\sigma}{(\tau+t_0)(\sigma+s_0)}, \tag{16}$$

where the integration is along the real τ and σ axes, and

$$\lambda = \frac{1}{2}\frac{\rho\omega}{z^2}, \quad A = \eta^2 + z^2, \quad B = \xi\eta, \quad C = \xi^2 + z^2. \tag{17}$$

3. Reducing to a standard form

When in the saddle point method the saddle point is coinciding with a pole, the standard methods of asymptotics cannot be used for obtaining a correct approximation. To obtain a uniform expansion that holds when pole and saddle point coalesce the complementary error function can be used. See [6] and [10]. In fact, we need in that case

$$w(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t - z}\, dt, \quad \Im z > 0. \tag{18}$$

Putting $t = -s$ in the integral, we obtain

$$w(z) = -\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{e^{-s^2}}{s + z}\, ds, \quad \Im z > 0. \tag{19}$$

The function $w(z)$ is an entire function, and we have (see [9, p. 275])

$$w(z) = e^{-z^2} \operatorname{erfc}(-iz), \tag{20}$$

where the complementary error function is defined by

$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-t^2}\, dt. \tag{21}$$

Another integral representation of the complementary error function is (see [1, Equation (7.4.11)])

$$\operatorname{erfc} z = \frac{2e^{-z^2}}{\pi} \int_0^{\infty} \frac{e^{-z^2 t^2}}{t^2 + 1}\, dt. \tag{22}$$

Because of the relation

$$\operatorname{erfc}(-z) = 2 - \operatorname{erfc} z, \tag{23}$$

it follows that

$$w(-z) = 2e^{-z^2} - w(z), \quad z \in \mathbb{C}. \tag{24}$$

This relation can also be obtained from (19) by allowing $\Im z < 0$ and at the same time shifting the contour of integration downward in the complex plane. By shifting back the path to the real line, picking up the residue, and using (18) and (19), we obtain again the symmetry relation (24).

The two-dimensional integral in (16) shows also the phenomenon that poles are near saddle points. When ξ and η are small, both poles at $-t_0$ and $-s_0$ are also small. In the singular perturbation problem (1) small values of ξ or η correspond with small values of $\pm 1 + x$ or $\pm 1 + y$. These values correspond with the boundaries of the unit square in the x, y-plane, where we have discontinuous boundary conditions. The poles in (16) lie on imaginary axes in the complex σ and τ planes, and when $\xi = 0$ and $\eta = 0$, the poles coincide with saddle points at the origins of these planes.

In this paper, we consider the following double integral as the two-dimensional analogue of $w(z)$ introduced in (18):

$$W(z,\zeta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-t^2-s^2}}{(t+\alpha s - z)(t+\beta s-\zeta)}\,dt\,ds, \tag{25}$$

where α and β are real and z and ζ are complex. The integral in (16) cannot simply be written as a product of two integrals, because of the term $2B\sigma\tau$ in the exponential function. Also for (25) a simple splitting is not possible.

In the next section, we transform the integral in (16) into an integral of the form (25), in which the poles are located on certain lines in the complex plane, that again will pass through the origins when ξ and η become 0. We evaluate (25) into one-dimensional integrals that can be viewed as standard forms, and as generalizations of the complementary error function defined by (21).

In [3], the two-dimensional integral

$$I(\alpha,\beta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-ia\cosh x - ib\cosh y}}{\sin\frac{1}{2}(ix+\alpha)\sin\frac{1}{2}(ix-iy+\beta)}\,dx\,dy \tag{26}$$

is considered with similar phenomena when α and β tend to zero. Jones considered his integral as a prototype and he introduced the function

$$G(z,\zeta) = \zeta e^{iz^2}\int_z^{\infty} \frac{e^{-it^2}}{t^2+\zeta^2}\,dt \tag{27}$$

that can be used for describing the uniform asymptotic phenomena. This function cannot be expressed in terms of a known special function, and it reduces in certain circumstances to a Fresnel integral.

Jones' function can be viewed as a generalization of the Fresnel integral. In the present case, we have a real phase function (see (16) and (25)), and we write the integral (25) as a sum of two functions of the form

$$F(\lambda,u,v) = \int_0^{\infty} \frac{re^{-\lambda r^2}\,dr}{\sqrt{r^2+u^2}(r^2+v^2)}, \tag{28}$$

where we assume that $\lambda > 0$, $u \ge 0$, and $v > 0$. F can be viewed as a generalization of the error function. For $u = 0$, it becomes, see (22),

$$F(\lambda,0,v) = \frac{\pi}{2v}e^{\lambda v^2}\operatorname{erfc}\bigl(\sqrt{\lambda}v\bigr), \tag{29}$$

where the complementary error function $\operatorname{erfc} z$ is defined in (21). When we change the variable of integration by writing $r^2 + u^2 = s^2$, we obtain

$$F(\lambda,u,v) = e^{\lambda u^2}\int_u^{\infty} \frac{e^{-\lambda s^2}\,ds}{s^2+v^2-u^2}. \tag{30}$$

Observe that F and G are related by

$$\zeta F(i, z, \zeta) = G(z, \zeta) \tag{31}$$

and that our function $F(\lambda, u, v)$ can be viewed as a function of two variables, because

$$F(\lambda, u, v) = \sqrt{\lambda} F\left(1, u\sqrt{\lambda}, v\sqrt{\lambda}\right). \tag{32}$$

When λ is large and $v \to 0$, the saddle point at the origin of the integral in (28) coalesces with poles at $r = \pm iv$. If, in addition, $u \to 0$, the saddle point coalesces also with two algebraic singularities.

In (30), the saddle point is outside the domain of integration, and when $u \to 0$, the saddle point coalesces with an endpoint. If, in addition, $v \to 0$, the saddle point coalesces also with two poles.

In Section 7, some other properties of F are derived.

4. Evaluating $V_1(\xi, \eta, z)$

First we evaluate $V_1(\xi, \eta, z)$ of (16) for the cases $\xi \downarrow 0$ and $\eta \downarrow 0$. In these limits, the quantity B defined in (17) becomes zero, and the double integral can be written as two single integrals. When $\xi \downarrow 0$, we have

$$V_1(0, \eta, z) = e^{-\omega\rho} \int_{-\infty}^{\infty} e^{-\lambda A\sigma^2} \frac{d\sigma}{\sigma + s_0} \int_{-\infty}^{\infty} e^{-\lambda C\tau^2} \frac{d\tau}{\tau}, \tag{33}$$

where in the τ-integral the path runs above the origin. The τ-integral equals (see (19)) $-\, i\pi w(0) = -i\pi$. For the σ-integral we use (19) again, and we obtain

$$V_1(0, \eta, z) = -\pi^2 e^{-\omega\rho + \lambda\eta^2} \operatorname{erfc}\left(\eta\sqrt{\lambda}\right). \tag{34}$$

In a similar way,

$$V_1(\xi, 0, z) = -\pi^2 e^{-\omega\rho + \lambda\xi^2} \operatorname{erfc}\left(\xi\sqrt{\lambda}\right). \tag{35}$$

Hence,

$$V_1(0, 0, z) = -\pi^2 e^{-\omega z}. \tag{36}$$

4.1. Positive values of ξ and η

We can use several transformations for obtaining a pure quadratic form in the exponential function in (16). For example, we can write

$$\begin{aligned} A\sigma^2 + 2B\sigma\tau + C\tau^2 &= A\left(\sigma^2 + \frac{2B\sigma\tau}{A}\right) + C\tau^2 \\ &= A\left(\sigma + \frac{B}{A}\tau\right)^2 + \left(C - \frac{B^2}{A}\right)\tau^2, \end{aligned} \tag{37}$$

and introduce the new variables of integration

$$p = \sqrt{A}\left(\sigma + \frac{B}{A}\tau\right), \quad q = \sqrt{\frac{AC - B^2}{A}}\tau. \tag{38}$$

The inverted relations read, because $AC - B^2 = \rho^2 z^2$,

$$\sigma = \frac{1}{\sqrt{\eta^2 + z^2}}\left(p - \frac{\xi\eta}{\rho z}q\right), \quad \tau = \frac{\sqrt{\eta^2 + z^2}}{\rho z}q. \tag{39}$$

Performing these relations on (16), we obtain

$$V_1(\xi, \eta, z) = e^{-\omega\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\lambda(p^2+q^2)}\frac{dq\,dp}{(p - \alpha q + i\beta)(q + i\gamma)}, \tag{40}$$

where

$$\alpha = \frac{\xi\eta}{\rho z}, \quad \beta = \frac{\eta}{\rho}\sqrt{\eta^2 + z^2}, \quad \gamma = \frac{\xi z}{\sqrt{\eta^2 + z^2}}. \tag{41}$$

Hence, when $\xi > 0$, $\eta > 0$ the quantities α, β, and γ are all positive, and they become small when ξ and η become small.

The location of the poles in the complex p and q planes resembles that of the integral in (26). Also, we can perform the p-integration in terms of the error function by using the function $w(z)$ defined in (18). However, then the q-integral is not easy to handle. In addition, the symmetry with respect to ξ and η, which is obvious in (16), is no longer obvious in (40).

We obtain a symmetric representation by using the transformation of variables

$$\sigma = \frac{1}{\sqrt{\xi^2 + \eta^2}}(\eta p + \xi q), \quad \tau = \frac{1}{\sqrt{\xi^2 + \eta^2}}(\xi p - \eta q), \tag{42}$$

or in inverted form

$$p = \frac{1}{\sqrt{\xi^2 + \eta^2}}(\eta\sigma + \xi\tau), \quad q = \frac{1}{\sqrt{\xi^2 + \eta^2}}(\xi\sigma - \eta\tau). \tag{43}$$

This gives

$$A\sigma^2 + 2B\sigma\tau + C\sigma^2 = \rho^2 p^2 + z^2 q^2, \tag{44}$$

and after scaling $\rho p \to p$, $zq \to q$, we obtain

$$V_1(\xi, \eta, z) = \rho z(\xi^2 + \eta^2)e^{-\omega\rho} \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-\lambda(p^2+q^2)}\,dq\,dp}{(\xi z p - \eta\rho q + \widetilde{t_0})(\eta z p + \xi\rho q + \widetilde{s_0})}, \tag{45}$$

where

$$\widetilde{t_0} = \rho z t_0\sqrt{\xi^2 + \eta^2}, \quad \widetilde{s_0} = \rho z s_0\sqrt{\xi^2 + \eta^2}. \tag{46}$$

4.1.1. *Writing $V_1(\xi, \eta, z)$ in terms of $F(\lambda, u, v)$.* Next we verify how to write $V_1(\xi, \eta, z)$ defined in (45) in terms of the integral in (28). We introduce polar coordinates

$$p = r\cos\theta, \quad q = r\sin\theta, \quad 0 \le \theta \le 2\pi. \tag{47}$$

This gives

$$V_1(\xi, \eta, z) = \rho z(\xi^2 + \eta^2)e^{-\omega\rho}\int_0^\infty e^{-\lambda r^2} f(r)\, r\, dr, \tag{48}$$

where

$$f(r) = \int_0^{2\pi} \frac{d\theta}{(\xi z r\cos\theta - \eta\rho r\sin\theta + \widetilde{t_0})(\eta z r\cos\theta + \xi\rho r\sin\theta + \widetilde{s_0})}. \tag{49}$$

We evaluate this integral by putting $s = e^{i\theta}$ and integrating around the unit circle in the complex s-plane. This gives, because $ds = is\, d\theta$,

$$f(r) = \frac{-4i}{PR}\int_{|s|=1} \frac{s\, ds}{\left(s^2 + 2\widetilde{t_0}s/P + Q/P\right)\left(s^2 + 2\widetilde{s_0}s/R + S/R\right)}, \tag{50}$$

where

$$\begin{aligned} P &= (\xi z + i\eta\rho)r, \quad Q = (\xi z - i\eta\rho)r, \\ R &= (\eta z - i\xi\rho)r, \quad S = (\eta z + i\xi\rho)r. \end{aligned} \tag{51}$$

The zeros of the quadratic factors in (50) are

$$\begin{aligned} s_1 &= \frac{i\sqrt{\xi^2+\eta^2}}{P}\left(-\xi z + \sqrt{\xi^2 z^2 + (\eta^2+z^2)r^2}\right), \\ s_2 &= \frac{i\sqrt{\xi^2+\eta^2}}{P}\left(-\xi z - \sqrt{\xi^2 z^2 + (\eta^2+z^2)r^2}\right), \\ s_3 &= \frac{i\sqrt{\xi^2+\eta^2}}{R}\left(-\eta z + \sqrt{\eta^2 z^2 + (\xi^2+z^2)r^2}\right), \\ s_4 &= \frac{i\sqrt{\xi^2+\eta^2}}{R}\left(-\eta z - \sqrt{\eta^2 z^2 + (\xi^2+z^2)r^2}\right). \end{aligned} \tag{52}$$

Observe that $|s_1 s_2| = 1$ and $|s_3 s_4| = 1$. The zeros s_1 and s_3 are inside the unit circle, and can be used for evaluating the integral by using residues.

First we write

$$\begin{aligned} &\frac{s}{\left(s^2 + 2\widetilde{t_0}s/P + Q/P\right)\left(s^2 + 2\widetilde{s_0}s/R + S/R\right)} \\ &\quad = \frac{a_1 s + a_2}{s^2 + 2\widetilde{t_0}s/P + Q/P} + \frac{a_3 s + a_4}{s^2 + 2\widetilde{s_0}s/R + S/R}. \end{aligned} \tag{53}$$

It is straightforward to verify that

$$a_1 = \frac{-iPRr}{T}, \quad a_2 = \frac{QR\sqrt{\xi^2+\eta^2}}{T}, \quad a_3 = \frac{iPRr}{T}, \quad a_4 = \frac{-PS\sqrt{\xi^2+\eta^2}}{T}, \tag{54}$$

where

$$T = 2r\rho z(\xi^2+\eta^2)(\xi^2+\eta^2+r^2). \tag{55}$$

Calculating the two residues in the integral in (50) gives

$$f(r) = \frac{8\pi}{PR}\left(\frac{a_1 s_1 + a_2}{s_1 - s_2} + \frac{a_3 s_3 + a_4}{s_3 - s_4}\right), \tag{56}$$

which can be evaluated in the form

$$f(r) = -\frac{2\pi}{z(\xi^2+\eta^2)(r^2+v^2)}\left(\frac{\xi\big/\sqrt{\xi^2+z^2}}{\sqrt{r^2+u_1^2}} + \frac{\eta/\sqrt{\eta^2+z^2}}{\sqrt{r^2+u_2^2}}\right), \tag{57}$$

where v, u_1, and u_2 are defined by

$$v = \sqrt{\xi^2+\eta^2}, \quad u_1 = \frac{\eta z}{\sqrt{\xi^2+z^2}}, \quad u_2 = \frac{\xi z}{\sqrt{\eta^2+z^2}}. \tag{58}$$

It follows that in terms of F of (28):

$$V_1(\xi,\eta,z) = -2\pi\rho e^{-\omega\rho}\left[\frac{\xi}{\sqrt{\xi^2+z^2}}F(\lambda,u_1,v) + \frac{\eta}{\sqrt{\eta^2+z^2}}F(\lambda,u_2,v)\right]. \tag{59}$$

When we write this in terms of the integral in (30), we introduce the notation

$$\zeta_1^2 = v^2 - u_1^2 = \frac{\rho^2\xi^2}{\xi^2+z^2}, \quad \zeta_2^2 = v^2 - u_2^2 = \frac{\rho^2\eta^2}{\eta^2+z^2}. \tag{60}$$

This gives

$$V_1(\xi,\eta,z) = -2\pi e^{-\omega\rho}\left[\frac{\xi e^{\lambda u_1^2}}{\sqrt{\xi^2+z^2}}\int_{u_1}^{\infty}\frac{e^{-\lambda s^2}\,ds}{s^2+\zeta_1^2} + \frac{\eta e^{\lambda u_2^2}}{\sqrt{\eta^2+z^2}}\int_{u_2}^{\infty}\frac{e^{-\lambda s^2}\,ds}{s^2+\zeta_2^2}\right]. \tag{61}$$

When we let $\xi \downarrow 0$, we have $v = \eta$, $u_1 = \eta$, and $\rho = \sqrt{\eta^2+z^2}$. It follows that (59) becomes

$$V_1(0,\eta,z) = -\pi^2 e^{-\omega\rho+\lambda\eta^2}\operatorname{erfc}\big(\eta\sqrt{\lambda}\big), \tag{62}$$

where we have used (29). This confirms (34). In a similar way, we find (35) and (36).

5. Negative values of ξ and η

When $\xi < 0$ or $\eta < 0$, the saddle points t_0 or s_0 defined in (12) become negative. With some modifications, we can repeat the evaluations of Section 4. Recall that in (7) the paths run above the real t-axis and s-axis. To obtain a representation through the saddle points t_0 and s_0, of which t_0 or s_0 are on the negative imaginary t-axis or s-axis, we now have to pass the poles at $t = 0$ or $s = 0$. This gives one or two residues in the form of a single integral.

5.1. $\xi < 0,\ \eta > 0$

In this case, we obtain from (7), by shifting the path in the t-plane downward, across the origin,

$$V(\xi, \eta, z) = -2\pi i \int_{-\infty}^{\infty} e^{-\omega\left[-i\eta s + z\sqrt{1+s^2}\right]} \frac{ds}{s} + \tilde{V}(\xi, \eta, z), \tag{63}$$

where $\tilde{V}(\xi, \eta, z)$ is as in (7), now with the path of integration for the t-integral below the origin in the t-plane. By changing $t \to -t$ we see that $\tilde{V}(\xi, \eta, z) = -V(-\xi, \eta, z)$, and we can write

$$V(-\xi, \eta, z) = -2\pi i \int_{-\infty}^{\infty} e^{-\omega\left[-i\eta s + z\sqrt{1+s^2}\right]} \frac{ds}{s} - V(\xi, \eta, z), \quad \xi > 0, \quad \eta > 0, \tag{64}$$

where $V(\xi, \eta, z)$ is as in (7) with both paths running above the origins. The s-integral in (64) runs above the origin and has a saddle point at $i\eta/\sqrt{A}$, where A is defined in (7). By an asymptotic analysis as performed for the double integral in Section 2 it follows that the integral can be approximated by (see (19))

$$-2\pi i e^{-\omega\sqrt{A}} \int_{-\infty}^{\infty} e^{-v^2} \frac{dv}{v + i\sqrt{\mu}\eta} = -2\pi^2 e^{-\omega\sqrt{A}} w\left(i\sqrt{\mu}\eta\right), \tag{65}$$

where

$$\mu = \frac{\omega\sqrt{A}}{2z^2}. \tag{66}$$

By using (20) we obtain

$$V(-\xi, \eta, z) \sim -2\pi^2 e^{-\omega\sqrt{A}} e^{\mu\eta^2} \operatorname{erfc}\left(\eta\sqrt{\mu}\right) - V_1(\xi, \eta, z), \quad \xi > 0, \quad \eta > 0, \tag{67}$$

where $V_1(\xi, \eta, z)$ can be written in terms of the F-function; see (59) and (61).

5.2. $\xi > 0$, $\eta < 0$

In a similar way,

$$V(\xi, -\eta, z) \sim -2\pi^2 e^{-\omega\sqrt{C}} e^{\nu\xi^2} \operatorname{erfc}\big(\xi\sqrt{\nu}\big) - V_1(\xi, \eta, z), \quad \xi > 0, \quad \eta > 0, \tag{68}$$

where C is defined in (17) and

$$\nu = \frac{\omega\sqrt{C}}{2z^2}. \tag{69}$$

5.3. $\xi < 0$, $\eta < 0$

Consider (64) with η replaced by $-\eta$, with $\eta > 0$. Then the function $V(\xi, -\eta, z)$ follows from (68). After the change $\eta \to -\eta$ in (64), the saddle point of the integral is now at $-i\eta/\sqrt{A}$, and for the asymptotic analysis of this integral we shift it downward, across the pole at $s = 0$, giving a residue $-4\pi^2 e^{-\omega z}$. The final result reads for $\xi > 0$ and $\eta > 0$

$$\begin{aligned} V(-\xi, -\eta, z) \sim\ & V_1(\xi, \eta, z) - 4\pi^2 e^{-\omega z} + 2\pi^2 e^{-\omega\sqrt{A}} e^{\mu\eta^2} \operatorname{erfc}\big(\eta\sqrt{\mu}\big) \\ & + 2\pi^2 e^{-\omega\sqrt{C}} e^{\nu\xi^2} \operatorname{erfc}\big(\xi\sqrt{\nu}\big). \end{aligned} \tag{70}$$

In Figure 2 we show graphs of the first order approximation obtained in this section of the solution $U(x, y, z)$ of problem (1) for $\varepsilon = 0.1$ and several values of z. We see smooth surfaces, also near $x = \pm 1$ and $y = \pm 1$, where the boundary layers occur. All combinations of positive and negative values of ξ and η considered in this section are needed to produce these smooth surfaces.

6. Proof of uniqueness of problem (1)

We give a more precise formulation of the problem in (1). To prove uniqueness of the problem in (1) we need extra conditions on the problem. A more precise formulation of problem (1) is then

$$\begin{cases} U \in \mathcal{C}(\tilde{\Omega}) \cap \mathcal{D}^2(\Omega) & U \text{ bounded in bounded subsets of } \tilde{\Omega}, \\ \quad -\varepsilon\, \Delta U + U_z = 0 & \quad \text{in } \Omega, \\ U(x, y, 0) = \chi_{(-1,1)}(x)\chi_{(-1,1)}(y), & \text{for } -\infty \le x, y \le \infty, \\ U(x, y, z) = o\left(\dfrac{e^{\omega(r_k+z)}}{\sqrt{\omega r_k}}\right) & \text{as } r_k \to \infty \text{ in } \Omega \text{ with } k = 1, 2, 3, \end{cases} \tag{71}$$

where $\omega = 1/(2\varepsilon)$, $r_1 \equiv \sqrt{x^2 + z^2}$, $r_2 \equiv \sqrt{y^2 + z^2}$, and $r_3 \equiv \sqrt{x^2 + y^2}$.

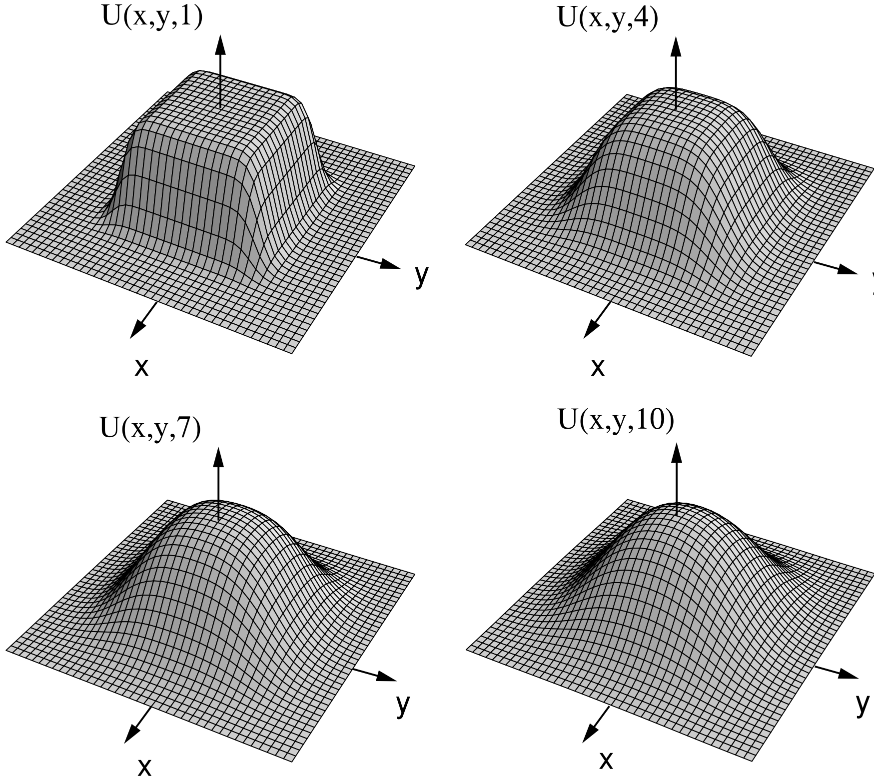

Figure 2. Graphs of the first-order approximations to the solution of problem (1) for different values of z. The graphs are obtained by using (59) for $V_1(\xi, \eta, z)$ and the relations for negative ξ and η of Section 5. $V_1(\xi, \eta, z)$ is the first-order approximation of $V(\xi, \eta, z)$ of (7), the function that represents any of the four components in (5). We observe that the solution takes a value close to 1 on the square $(x, y) \in [-1, 1] \times [-1, 1]$ and 0 everywhere else. On the sides of this square the solutions takes the value $\frac{1}{2}$ and experiences a fast transition from 0 to 1. We also observe that the larger z is, the smoother the solution is.

Observe that the Dirichlet datum at $z = 0$ is discontinuous at the boundary $\partial\Omega_L$ of a square Ω_L located in the plane $z = 0$:

$$\Omega_L \equiv \{(x, y, 0) \in \mathbb{R}^3; -1 \leq x, y \leq 1\}.$$

The set $\tilde{\Omega}$ in (71) is precisely the closed set $\bar{\Omega}$ with that contour removed:

$$\tilde{\Omega} \equiv \bar{\Omega} \backslash \partial\Omega_L.$$

We have the following uniqueness result:

THEOREM 1: *Problem (71) has at most one solution.*

Proof: Suppose that U_1 and U_2 are two solutions of (71). Then, the function $G(x, y, z) \equiv (U_1(x, y, z) - U_2(x, y, z))\, e^{-\omega z}$ verifies:

$$\begin{cases} G \in \mathcal{C}(\tilde{\Omega}) \cap \mathcal{D}^2(\Omega) & G \text{ bounded on bounded subsets of } \tilde{\Omega}, \\ \Delta G - \omega^2 G = 0 & \text{in } \Omega \\ G(x, y, 0) = 0 & \text{for } -\infty \le x, y \le \infty, \\ G(x, y, z) = o\left(\dfrac{e^{\omega r_k}}{\sqrt{\omega r_k}}\right) & \text{as } r_k \to \infty \text{ in } \Omega \text{ with } k = 1, 2, 3. \end{cases} \tag{72}$$

Consider the following auxiliary function defined on $\bar{\Omega}$:

$$V_a(x, y, z) \equiv \begin{cases} \dfrac{G(x, y, z)}{H_a(x, y, z)} & \text{if} \quad r_1^{\pm} \neq 0 \neq r_2^{\pm} \\ 0 & \text{if} \quad r_1^{\pm} = 0 \text{ or } r_2^{\pm} = 0, \end{cases}$$

with

$$\begin{aligned} H_a(x, y, z) \equiv\, & K_0\big(\omega r_1^+\big) + K_0\big(\omega r_1^-\big) + K_0\big(\omega r_2^+\big) + K_0\big(\omega r_2^-\big) \\ & + I_0(\omega r_1) + I_0(\omega r_2) + I_0(\omega r_3) + a, \\ r_1^{\pm} \equiv\, & \sqrt{(x \pm 1)^2 + z^2}, \quad r_2^{\pm} \equiv \sqrt{(y \pm 1)^2 + z^2}, \quad a > 0, \end{aligned}$$

and K_0 and I_0 being modified Bessel functions of order zero. The function $H_a(x, y, z)$ is positive in Ω, of the order $\mathcal{O}(e^{\omega r_k}/\sqrt{\omega r_k})$ as $\omega r_k \to \infty$ for $k = 1, 2, 3$ and $\mathcal{O}(\log(\omega r_k^{\pm}))$ as $\omega r_k^{\pm} \to 0$ for $k = 1, 2$ ([1, Eqs. 9.7.1 and 9.6.13]). Moreover, $H_a(x, y, z) \in \mathcal{D}^2(\Omega)$ and satisfies the equation: $\Delta H_a - \omega^2 H_a + a\omega^2 = 0$ in Ω [1, Eq. 9.6.1]). Therefore, using also that G is bounded near $\partial\Omega_L$, we have that the auxiliary function V_a is continuous in $\bar{\Omega}$ and verifies:

$$\begin{cases} \Delta V_a + \dfrac{2}{H_a} \vec{\nabla} H_a \cdot \vec{\nabla} V_a = \dfrac{a\omega^2}{H_a} V_a & \text{in } \Omega, \\ V_a(x, y, 0) = 0 & \text{for } -\infty \le x, y \le \infty, \\ \lim_{r_k \to \infty} V_a(x, y, z) = 0 & \forall\, (x, y, z) \in \bar{\Omega},\ k = 1, 2, 3. \end{cases}$$

Consider the open finite box of size R : $\Omega_R \equiv (-R, R) \times (-R, R) \times (0, R)$. At points $(x, y, z) \in \Omega_R$ where $\vec{\nabla} V_a = 0$ and $V_a \neq 0$, we have that $V_a \cdot \Delta V_a > 0$. Therefore, V_a has neither positive relative maximums nor negative relative minimums in Ω_R. Then $\mathrm{Sup}_{\Omega_R} |V_a| \le \mathrm{Sup}_{\partial\Omega_R} |V_a|$.

Using that $V_a(x, y, 0) = 0\ \forall\ (x, y) \in R^2$ and that $\lim_{r_k \to \infty} V_a(x, y, z) = 0$ for $k = 1, 2, 3$ we have that, $\forall\, \delta > 0$, there is a $R > 0$ such that $|V_a(x, y, z)| \le \delta\ \forall\ (x, y, z) \in \partial\Omega_R$. Therefore, $|V_a(x, y, z)| \le \delta\ \forall\ \delta > 0$ and every $(x, y, z) \in \Omega_R$. Taking the limit $\delta \to 0$ $(R \to \infty)$ we have that $V_a = 0$ in $\bar{\Omega}$. Therefore, $G = 0$ and $U_1 = U_2$ in Ω. ■

7. Further properties of $F(\lambda, u, v)$

We give a few further properties of the function (see (28–30))

$$F(\lambda, u, v) = \int_0^\infty \frac{re^{-\lambda r^2}\,dr}{\sqrt{r^2+u^2}(r^2+v^2)}$$
$$= e^{\lambda u^2}\int_u^\infty \frac{e^{-\lambda s^2}\,ds}{s^2+\zeta^2} \tag{73}$$
$$= \frac{\pi}{2\zeta}e^{\lambda v^2}\operatorname{erfc}\big(\zeta\sqrt{\lambda}\big) - e^{\lambda u^2}\int_0^u \frac{e^{-\lambda s^2}\,ds}{s^2+\zeta^2},$$

where $\zeta^2 = v^2 - u^2$.

The function F reduces to a complementary error function when $u = v$. We have

$$F(\lambda, u, u) = e^{\lambda u^2}\int_u^\infty \frac{e^{-\lambda s^2}}{s^2}\,ds = \frac{1}{2}\sqrt{\lambda}e^{\lambda u^2}\Gamma\left(-\frac{1}{2}, \lambda u^2\right), \tag{74}$$

where we use the incomplete gamma function defined by

$$\Gamma(a, z) = \int_z^\infty t^{a-1}e^{-t}\,dt. \tag{75}$$

By using integration by parts, we can write

$$F(\lambda, u, u) = \frac{1}{u} - \sqrt{\pi\lambda}\,e^{\lambda u^2}\operatorname{erfc}\big(u\sqrt{\lambda}\big). \tag{76}$$

A series in powers of λ follows by expanding the exponential function in the third integral in (73). This gives

$$F(\lambda, u, v) = \frac{\pi}{2\zeta}e^{\lambda v^2}\operatorname{erfc}\big(\zeta\sqrt{\lambda}\big) - e^{\lambda u^2}\sum_{n=0}^\infty \frac{(-\lambda)^n}{n!}\Phi_n(u, v), \tag{77}$$

where

$$\Phi_n(u, v) = \int_0^u \frac{s^{2n}}{s^2+\zeta^2}\,ds, \quad n = 0, 1, 2, \ldots. \tag{78}$$

We have

$$\Phi_0(u, v) = \frac{1}{\zeta}\arctan\frac{u}{\zeta}, \tag{79}$$

and the remaining Φ_n can be computed through the recursion relation

$$\Phi_{n+1}(u, v) = \frac{u^{2n+1}}{2n+1} - \zeta^2\Phi_n(u, v), \quad n = 0, 1, 2, \ldots. \tag{80}$$

To obtain a series with positive terms we expand

$$F(\lambda, u, v) = \frac{\pi}{2\zeta} e^{\lambda v^2} \operatorname{erfc}\bigl(\zeta\sqrt{\lambda}\bigr) - \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \Psi_n(u, v), \tag{81}$$

where

$$\Psi_n(u, v) = \int_0^u \frac{(u^2 - s^2)^n}{s^2 + \zeta^2}\, ds, \quad n = 0, 1, 2, \ldots. \tag{82}$$

We have $\Psi_0(u, v) = \Phi_0(u, v)$, and for the other ones we have

$$\Psi_{n+1}(u, v) = v^2 \Psi_n(u, v) - u^{2n+1} \frac{\Gamma\left(\frac{3}{2}\right) n!}{\Gamma\left(n + \frac{3}{2}\right)}, \quad n = 0, 1, 2, \ldots. \tag{83}$$

The Ψ_n are in fact hypergeometric functions. We have (see [9, p. 110])

$$\Psi_n(u, v) = \frac{u^{2n+1}}{\zeta^2} \frac{\Gamma\left(\frac{3}{2}\right) n!}{\Gamma\left(n + \frac{3}{2}\right)} {}_2F_1\left(\begin{matrix} 1, \ \frac{1}{2} \\ n + \frac{3}{2} \end{matrix} ; -\frac{u^2}{\zeta^2}\right), \tag{84}$$

or

$$\Psi_n(u, v) = \frac{u^{2n+1}}{v^2} \frac{\Gamma\left(\frac{3}{2}\right) n!}{\Gamma\left(n + \frac{3}{2}\right)} {}_2F_1\left(\begin{matrix} 1, \ n+1 \\ n + \frac{3}{2} \end{matrix} ; \frac{u^2}{v^2}\right). \tag{85}$$

An asymptotic expansion for large values of λ follows from the first integral in (73) by expanding

$$\frac{u v^2 r}{\sqrt{r^2 + u^2}(r^2 + v^2)} = \sum_{n=0}^{\infty} c_n r^{2n+1}. \tag{86}$$

We have

$$c_0 = 1, \quad c_1 = -\frac{2u^2 + v^2}{2u^2 v^2}, \quad c_2 = \frac{8u^4 + 4u^2 v^2 + 3v^4}{8u^4 v^4}. \tag{87}$$

More coefficients can be computed by using the recursion relation

$$u^2 v^2 (n+1) c_{n+1} = -\left[(n+1)u^2 + \left(n + \frac{1}{2}\right) v^2\right] c_n - \left(n + \frac{1}{2}\right) c_{n-1}, \quad n \geq 1. \tag{88}$$

By substituting the expansion in (86) into (73) the following asymptotic expansion

$$F(\lambda, u, v) \sim \frac{1}{2uv^2\lambda} \sum_{n=0}^{\infty} c_n \frac{n!}{\lambda^n}, \quad \lambda \to \infty \tag{89}$$

is obtained, which holds uniformly for $u \geq u_0$, $v \geq v_0$, where u_0 and v_0 are fixed positive numbers.

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