The role of the error function in three-dimensional singularly perturbed convection-diffusion problems with discontinuous data

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Resumen

We consider singularly perturbed convection-diffusion problems defined in three-dimensional domains, problems of parabolic type: $-\varepsilon(u_{xx}+u_{yy})+u_t+v_1u_x+v_2u_y=0$ and of elliptic type: $-\varepsilon(u_{xx}+u_{yy}+u_{zz})+v_1u_x+v_2u_y+v_3u_z=0$, where v_1, v_2 and v_2 are real constants. For every one of these two kind of problems we consider several three-dimensional domains. We also consider for all of these problems Dirichlet data discontinuous at certain regions of the boundaries of the domains. For each problem, an asymptotic approximation of the solution is obtained from an integral representation when the singular parameter $\varepsilon \to 0^+$. The solution is approximated by products of error functions, and this approximation characterizes the effect of the discontinuities on the small $\varepsilon-$ behaviour of the solution and its derivatives in the boundary layers or the internal layers.

Palabras clave: singular perturbation problem, discontinuous boundary data, asymptotic expansions, error function.

Clasificación para el Cedya 2005: EDPs

0.1. Introduction

As it has been shown recently [1]-[5], the error function plays a fundamental role in the approximation of 2D singularly perturbed convection-diffusion problems with discontinuous Dirichlet data. In the problems there analyzed it is shown that the solution is approximated by a finite combination of error functions as $\varepsilon \to 0^+$. For example, in the problem defined on the first quadrant $\Omega \equiv (0,\infty)\times(0,\infty)$ and with a discontinuous Dirichlet boundary condition at the corner point $(0,0)$ (see Figure 1) [1]:

$$\begin{cases} -\varepsilon\Delta U + \overrightarrow{v}\cdot\overrightarrow{\nabla}U = 0, & (x,y)\in\Omega, \\ U(x,0)=0, U(0,y)=1, & U\in\mathcal{C}(\tilde{\Omega})\cap\mathcal{D}^2(\Omega), \end{cases} \qquad (P)$$

where $\tilde{\Omega}\equiv\bar{\Omega}\setminus\{(0,0)\}$, the solution can be approximated, for $0<\beta<\pi/2$, by

$$U_\beta(x,y) = U_\beta^0(x,y)(1+\mathcal{O}(\sqrt{\varepsilon})) \quad \text{as } \varepsilon\to 0^+,\ r\geq r_0>0$$

with

$$U_\beta^0(x,y) \equiv \frac{1}{2}\mathrm{erfc}\left[\sqrt{\frac{r}{2\epsilon}}\sin\left(\frac{\phi-\beta}{2}\right)\right]. \qquad (1)$$

We have used the polar coordinates $x=r\sin\phi$, $y=r\cos\phi$ $(0\leq r<\infty, 0\leq\phi\leq\pi/2)$.

After the two-dimensional study performed in [1]-[2], we wonder if it is possible to extend our two-dimensional analysis to three-dimensional problems with discontinuous data and if the error function is also useful in the approximation of these kind of problems. Then, we analyze several 3D parabolic and elliptic problems defined on different domains. We only detail here a 3D parabolic problem defined on a half space $(x,y,t)\in\Omega_1\equiv(-\infty,\infty)\times(-\infty,\infty)\times(0,\infty)$ with a discontinuous initial condition over a rectangle and a 3D elliptic problem defined on the first octant $\Omega_2=(0,\infty)\times(0,\infty)\times(0,\infty)$, with a discontinuous Dirichlet data at the X and Y axes.

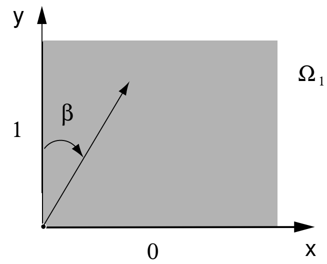


Figura 1: *Domain* Ω *and boundary conditions in problem* (P).

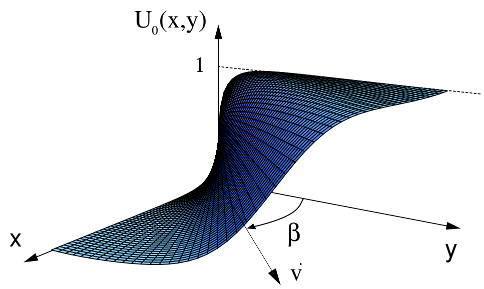


Figura 2: *Graph of the first order approximation* $U^0_{\pi/4}(x,y)$ *to the solution of problem* (P) *for* $\varepsilon = 0{,}1$ *and* $\beta = \pi/4$. *The convection vector* $\overrightarrow{v}$ *"drags"the discontinuity of the boundary condition at* $(0,0)$ *originating a parabolic layer along the characteristic defined by* $\overrightarrow{v}$ *and emanating from* $(0,0)$.

0.2. A 3D parabolic problem in a half space

We consider a problem defined in the upper half space: $(x,y,t) \in \Omega_1 \equiv (-\infty,\infty) \times (-\infty,\infty) \times (0,\infty)$, with a " rectangular source of contamination" located on the rectangle $(a,b) \times (c,d)$ (see Figure 3):

$$\begin{cases} U \in \mathcal{C}(\tilde{\Omega}_1), U_x, U_y, U_t, U_{xx}, U_{yy} \in \mathcal{C}(\Omega_1), & U \text{ bounded on bounded subsets of } \tilde{\Omega}_1, \\ -\varepsilon(U_{xx} + U_{yy}) + v_1 U_x + v_2 U_y + U_t = 0, & \text{in } \Omega_1, \\ U(x,y,0) = \chi_{(a,b)}(x)\chi_{(c,d)}(y), & \text{for } (x,y) \in \mathbb{R}^2. \end{cases} \tag{2}$$

In this formula, $a < b$, $c < d$ and the positive numbers $b-a$ and $d-c$ represent the length of the sides of the "source of contamination". Observe that the initial condition is discontinuous at $\{x = a, t = 0, c < y < d\}$; $\{x = b, t = 0, c < y < d\}$; $\{y = c, t = 0, a < x < b\}$ and $\{y = d, t = 0, a < x < b\}$. The set $\tilde{\Omega}_1$ is the closed set $\bar{\Omega}_1$ with the regions of discontinuity of the boundary data removed: $\tilde{\Omega}_1 \equiv \bar{\Omega}_1 \setminus \{\{(x,c,0),(x,d,0), a \le x \le b\} \cup \{(a,y,0),(b,y,0), c \le y \le d\}\}$.

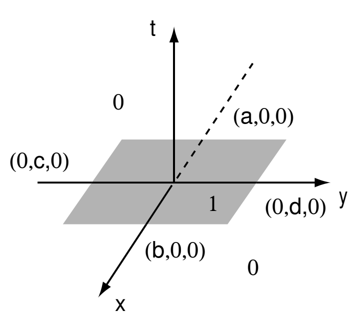


Figura 3: *Domain* Ω_1 *and initial conditions of problem (2).*

The change of variable $U(x,y,t) = e^{[v_1 x + v_2 y - t/2]/(2\epsilon)} F(x,y,t)$ in (2) yields the following problem for $F(x,y,t)$:

$$\begin{cases} F \in \mathcal{C}(\tilde{\Omega}_1), F_x, F_y, F_t, F_{xx}, F_{yy} \in \mathcal{C}(\Omega_1), & F \text{ bounded on bounded subsets of } \tilde{\Omega}_1, \\ F_{xx} + F_{yy} - \frac{1}{\epsilon} F_t = 0, & \text{in } \Omega_1, \\ F(x,y,0) = e^{-[v_1 x + v_2 y]/(2\epsilon)} \chi_{(a,b)}(x) \chi_{(c,d)}(y), & \text{for } (x,y) \in \mathbb{R}^2. \end{cases} \tag{3}$$

We add a radiation condition to (2) in order to assure the uniqueness of the solution. Hence, we consider the following problem:

$$\begin{cases} U \in \mathcal{C}(\tilde{\Omega}_1), U_x, U_y, U_t, U_{xx}, U_{yy} \in \mathcal{C}(\Omega_1), & U \text{ bounded on bounded subsets of } \tilde{\Omega}_1, \\ -\varepsilon(U_{xx} + U_{yy}) + v_1 U_x + v_2 U_y + U_t = 0, & \text{in } \Omega_1, \\ U(x,y,0) = \chi_{(a,b)}(x) \chi_{(c,d)}(y), & \text{for } (x,y) \in \mathbb{R}^2, \\ U, U_t, U_x, U_y, U_{xx}, U_{yy} = o\left(1/r\right), & \text{as } r \to \infty, \end{cases} \tag{P_1}$$

where $r \equiv \sqrt{x^2+y^2}$. This problem has at most one solution.

A solution of problem (3) may be derived by using the Fourier transform in (3) with respect to x and y. Therefore, we obtain the solution of (P_1) expressed in terms a double integral:

$$U(x,y,t) = \frac{1}{4\pi\epsilon t} e^{[v_1 x + v_2 y - t/2]/(2\epsilon)} \int_a^b e^{-(x-s)^2/(4\epsilon t)} e^{-v_1 s/(2\epsilon)} ds \int_c^d e^{-(y-u)^2/(4\epsilon t)} e^{-v_2 u/(2\epsilon)} du, \tag{4}$$

That may be evaluated exactly in terms of error functions:

$$\begin{aligned} U(x,y,t) = \frac{1}{4} & \left\{ \operatorname{erfc}\left(\frac{v_1 t + a - x}{2\sqrt{\varepsilon t}}\right) - \operatorname{erfc}\left(\frac{v_1 t + b - x}{2\sqrt{\varepsilon t}}\right) \right\} \\ \times & \left\{ \operatorname{erfc}\left(\frac{v_2 t + c - y}{2\sqrt{\varepsilon t}}\right) - \operatorname{erfc}\left(\frac{v_2 t + d - y}{2\sqrt{\varepsilon t}}\right) \right\}. \end{aligned} \tag{5}$$

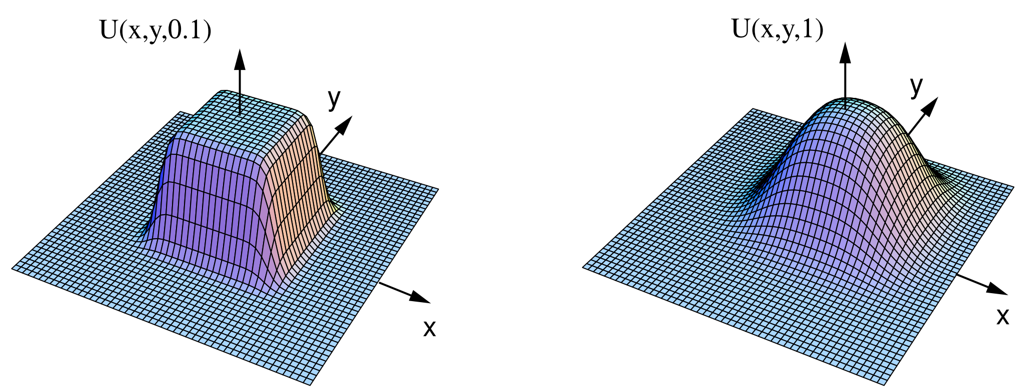


Figura 4: *Graphs of the solution of problem* (P_1) *given in (5) for* $\varepsilon = 0{,}1$, $a = -b = 1$, $c = -d = 1$, $v_1 = 0{,}5$ *and different values of* t*.*

0.3. A 3D elliptic problem in an octant

We consider the first octant of R^3 with discontinuities at one-dimensional corners of the boundary of the domain $\Omega_2 = (0,\infty) \times (0,\infty) \times (0,\infty)$, with an "infinite source of contamination" located at the plane $z = 0$ (see Figure 5) and with the convection vector $\overrightarrow{v} = (0,0,1)$:

$$\begin{cases} U \in \mathcal{C}(\tilde{\Omega}_2) \cap \mathcal{D}^2(\Omega_2), & \text{U bounded on bounded subsets of } \tilde{\Omega}_2, \\ -\varepsilon \Delta U + U_z = 0, & \text{in } \Omega_2, \\ U(x,y,0) = 1, U(0,y,z) = U(x,0,z) = 0, & \text{for } (x,y,z) \in \tilde{\Omega}_2. \end{cases} \tag{6}$$

Observe that the Dirichlet data are discontinuous at the X and Y axes. The set $\tilde{\Omega}_2$ is the closed set $\bar{\Omega}_2$ with the X and Y axes removed: $\tilde{\Omega}_2 \equiv \Omega_2 \cup \{(x,y,0); x,y > 0\} \cup \{(0,y,z); y \geq 0, z > 0\} \cup$

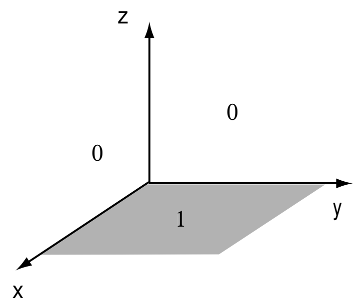


Figura 5: *Domain Ω_2 and Dirichlet conditions of problem (6).*

$\{(x, 0, z); x \geq 0, z > 0\}$.

After the change $U(x,y,z) = e^{\vec{v}\cdot\vec{r}/(2\epsilon)} F(x,y,z)$, problem (6) is transformed into the Yukawa equation for $F(x,y,z)$:

$$\begin{cases} F \in \mathcal{C}(\tilde{\Omega}_2) \cap \mathcal{D}^2(\Omega_2), & \text{F bounded on bounded subsets of } \tilde{\Omega}_2, \\ \Delta F - \frac{1}{4\epsilon^2} F = 0, & \text{in } \Omega_2, \\ F(x,y,0) = 1, F(0,y,z) = F(x,0,z) = 0, & \text{for } (x,y,z) \in \tilde{\Omega}_2. \end{cases} \tag{7}$$

This problem may have not a unique solution unless we impose a convenient condition upon $U(x,y,z)$ (or upon $F(x,y,z)$) concerning its growth at infinity. Then, we add a radiation condition to (6) and consider the following problem:

$$\begin{cases} U \in \mathcal{C}(\tilde{\Omega}_2) \cap \mathcal{D}^2(\Omega_2), & \text{U bounded on bounded subsets of } \tilde{\Omega}_2, \\ -\varepsilon \Delta U + U_z = 0, & \text{in } \Omega_2, \\ U(x,y,0) = 1, U(0,y,z) = U(x,0,z) = 0, & \text{for } (x,y,z) \in \tilde{\Omega}_2, \\ U(x,y,z) = o\left(\dfrac{\sqrt{2\epsilon}}{\sqrt{r_k}} e^{(r_k+z)/(2\epsilon)}\right), & \text{as } r_k \to \infty \text{ with } k = 1,2,3, \end{cases} \tag{P_2}$$

where $r_1 \equiv \sqrt{x^2+z^2}$, $r_2 \equiv \sqrt{y^2+z^2}$ and $r_3 \equiv \sqrt{x^2+y^2}$.

As a difference with problem (P_1), the solution of (P_2) cannot be evaluated in terms of known functions, but may be approximated in terms of error functions when $\epsilon \to 0^+$. For this purpose we define the region:

$$\begin{aligned} \Omega_2^* \equiv \Omega_2 \setminus \{ & \{(x,y,z) \in \Omega_2, 0 < x \leq x_0, 0 < z \leq z_0\} \\ & \cup \{(x,y,z) \in \Omega_2, 0 < y \leq y_0, 0 < z \leq z_0\}\}. \end{aligned}$$

The unique solution of problem (P_2) may be derived by using the Fourier sine transform with respect to x and y:

$$U(x,y,z) = \frac{e^{z/(2\epsilon)}}{\pi^2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} ds \frac{\sin(xt/(2\epsilon))}{t} \frac{\sin(ys/(2\epsilon))}{s} e^{-z\sqrt{1+t^2+s^2}/(2\epsilon)}. \tag{8}$$

After the change of variable $s \to u$ defined by $s = \sqrt{1+t^2}u$ in the $s-$integral we obtain:

$$U(x,y,z) = \frac{e^{z/(2\epsilon)}}{\pi^2} \int_{-\infty}^{\infty} \frac{\sin(xt/(2\epsilon))}{t} dt \int_{-\infty}^{\infty} \frac{\sin(y\sqrt{1+t^2}u/(2\epsilon))}{u} e^{-z\sqrt{1+t^2}\sqrt{1+u^2}/(2\epsilon)} du.$$

The integral in the u variable is just the solution of a similar two-dimensional convection-diffusion problem defined on the quarter plane $(y,z) \in (0,\infty) \times (0,\infty)$ where it is shown that:

$$\frac{e^{z/(2\epsilon)}}{\pi} \int_{-\infty}^{\infty} \frac{\sin(yu/(2\epsilon))}{u} e^{-z\sqrt{1+u^2}/(2\epsilon)} du = \text{erf}\sqrt{\frac{\sqrt{y^2+z^2}-z}{2\epsilon}} + \tilde{R}(y,z,\epsilon),$$

with $\tilde{R}(y,z,\epsilon) = \mathcal{O}(\sqrt{\epsilon})$ uniformly in y, z for $y^2+z^2 \geq r_0 > 0$:

$$|\tilde{R}(y,z,\epsilon)| \leq \frac{Cy\sqrt{\epsilon}}{(y^2+z^2)^{3/4}} e^{[z-\sqrt{y^2+z^2}]/(2\epsilon)}. \tag{9}$$

with C a positive constant independent of y, z and ϵ. Therefore, we can write

$$U(x,y,z) = U_0(x,y,z) + U_1(x,y,z), \tag{10}$$

with

$$U_0(x,y,z) \equiv \frac{e^{z/(2\epsilon)}}{\pi} \int_{-\infty}^{\infty} \frac{\sin(xt/(2\epsilon))}{t} e^{-z\sqrt{1+t^2}/(2\epsilon)} \mathrm{erf}\sqrt{\frac{\sqrt{1+t^2}}{2\epsilon}[\sqrt{y^2+z^2}-z]}dt$$

and

$$U_1(x,y,z) \equiv 2\frac{e^{z/(2\epsilon)}}{\pi} \int_0^{\infty} \frac{\sin(xt/(2\epsilon))}{t} e^{-z\sqrt{1+t^2}/(2\epsilon)} \tilde{R}(y,z,2\epsilon/\sqrt{1+t^2})dt.$$

The function U_1 admits the following bound uniformly valid in Ω_2 with $y^2+z^2 \geq r_0 > 0$:

$$|U_1(x,y,z)| \leq \frac{Cxy}{y^2+z^2} e^{[z-\sqrt{y^2+z^2}]/(2\epsilon)}. \tag{11}$$

On the other hand we write

$$U_0(x,y,z) = U_{00}(x,y,z) + U_{01}(x,y,z), \tag{12}$$

with

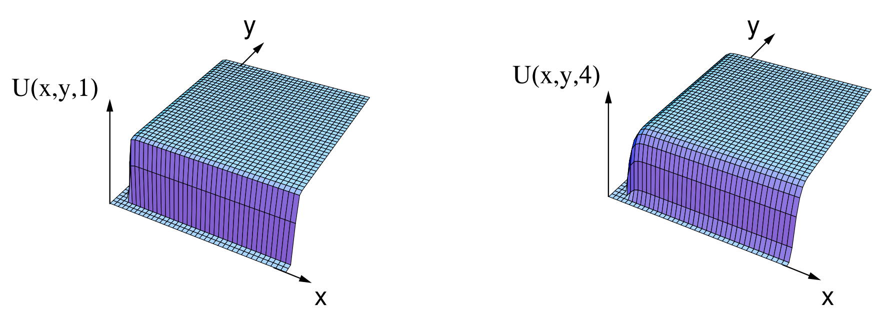
$$U_{00}(x,y,z) \equiv \mathrm{erf}\sqrt{\frac{\sqrt{z^2+y^2}-z}{2\epsilon}} \frac{e^{z/(2\epsilon)}}{\pi} \int_{-\infty}^{\infty} \frac{\sin(xt/(2\epsilon))}{t} e^{-z\sqrt{1+t^2}/(2\epsilon)}dt$$

and

$$\begin{aligned} U_{01}(x,y,z) \equiv\, & 2\frac{e^{z/(2\epsilon)}}{\pi} \int_0^{\infty} \frac{\sin(xt/(2\epsilon))}{t} e^{-z\sqrt{1+t^2}/(2\epsilon)} \\ & \times \left[\mathrm{erf}\sqrt{\frac{\sqrt{1+t^2}}{2\epsilon}[\sqrt{y^2+z^2}-z]} - \mathrm{erf}\sqrt{\frac{\sqrt{z^2+y^2}-z}{2\epsilon}} \right] dt. \end{aligned} \tag{13}$$

From here we can obtain a bound for U_{01} similar to (11) and use the solution of a similar 2D problem defined on a quarter plane to analyze U_{00}. We finally obtain that, for $(x,y,z) \in \Omega_2^*$, the solution $U(x,y,z)$ of problem (P_2) is

$$U(x,y,z) = \mathrm{erf}\sqrt{\frac{\sqrt{z^2+x^2}-z}{2\epsilon}} \mathrm{erf}\sqrt{\frac{\sqrt{z^2+y^2}-z}{2\epsilon}} \left[1 + \mathcal{O}(\sqrt{\epsilon})\right]. \tag{14}$$



(a) $z = 1$ (b) $z = 4$

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