Asymptotic approximations for a singularly perturbed convection-diffusion problem with discontinuous data in a sector

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Abstract

We consider a singularly perturbed convection-diffusion problem, $-\epsilon \bigtriangleup u + \vec{v} \cdot \vec{\nabla} u = 0$ defined on a sector $A \equiv \{r, \phi \,| r \geq 0, 0 \leq \phi \leq \alpha\}$ with discontinuous Dirichlet conditions $u(r,0) = 0$ and $u(r,\alpha) = 1$. An asymptotic expansion of $u(r,\phi)$ is obtained from an integral representation when: a) the singular parameter $\epsilon \to 0^+$ (with fixed r) and b) $r \to 0^+$ (with fixed ϵ). In the singular limit, the solution u is approximated by error functions and exponential functions. Near the discontinuity of the boundary condition $r = 0$, the solution u of the problem is approximated by a linear function of the polar angle ϕ.

Introduction

The solution of a singularly perturbed convection-diffusion problem usually presents boundary and/or interior layers. The location and shape of these layers depend, among other things, on the discontinuities of the boundary condition. An "a priori" knowledge of the location of the internal or boundary layers is quite useful to design numerical methods for this kind of problems. This information may be obtained from an asymptotic expansion of the solution [5], [7]. There is an extensive literature devoted to the construction of approximated solutions of singular perturbation problems based on matching of asymptotic expansions (see for example [2], [5], [6] or [7] for a historical survey on the subject). But a perturbative analysis based on an expansion of the solution in powers of the perturbation parameter does not always work for discontinuous Dirichlet boundary conditions [10]. This is so, because the coefficients of the expansion contain derivatives of the boundary condition, whereas the solution of the elliptic problem is smooth inside the domain.

Some particular problems with discontinuous Dirichlet data have been already considered in the literature. For example, Hedstrom and Osterheld [3] studied the problem $\epsilon\Delta u - \partial_y u = 0$ on the positive quarter plane with boundary conditions $u(x,0) = 0$ and $u(0,y) = 1$. They obtained the first two terms of the asymptotic expansion of u for $\epsilon \to 0^+$ from a Fourier integral representation of u. The first term of this expansion is an error function. A more detailed investigation has been developed by Temme in [8]: an integral representation for u is obtained from the associated Helmholtz equation and a complete asymptotic expansion of u for $\epsilon \to 0^+$ is derived from this integral representation. The same equation $\epsilon\Delta u - \partial_y u = 0$, but in a generic sector, is considered in [9], where an integral representation for u is obtained from the associated Helmholtz equation. Different asymptotic expansions for $\epsilon \to 0^+$ are obtained depending on the angle of the sector and again the error function plays an important role in the analysis. A similar problem defined in the interior of a circle is analyzed in [10]. In all these problems, the approximation is not valid near the discontinuities of the boundary condition.

In this paper we try to shed light on the influence that the discontinuities of the boundary condition have on the boundary or interior layers of the solution of convection-diffusion problems. For that purpose, we analyze the problem considered by Temme on a sector but for a general convection vector $\vec{v}$, not only $\vec{v} = (0, 1)$. This problem displays boundary or interior layers. As in the references mentioned in the above two paragraphs, the starting point to analyze the problem is an integral representation for the solution. We approximate the solution by deriving asymptotic expansions from this integral, not only in the singular limit $\epsilon \to 0^+$, but also near the discontinuity ($r \to 0^+$), where r represents the distance to the discontinuities. Then, we approximate the solution on the whole domain, including the neighborhood of the discontinuity point $r = 0$.

In section 2 we obtain an integral representation for the solution. In section 3 we derive an asymptotic expansion of the solution for $\epsilon \to 0^+$ whereas in section 4 we derive an asymptotic expansion for $r \to 0^+$. Some comments are postponed to section 5.

The problem and an exact solution

We consider the problem:

$$\left\{ \begin{array}{l} -\epsilon \triangle U + \vec{v} \cdot \vec{\nabla} U = 0 \ \text{ in } \ \Omega \equiv \{ r, \ \phi \mid r > 0, \ 0 < \phi < \alpha \} \\ U(r, 0) = 0, \quad U(r, \alpha) = 1, \ |\lim_{r\to\infty} U(r, \phi)| < \infty, \ \ U \in \mathcal{C}(\bar{\Omega} \setminus \{(0, \phi)\}) \bigcap \mathcal{D}^2(\Omega) \end{array} \right. \qquad (P)$$

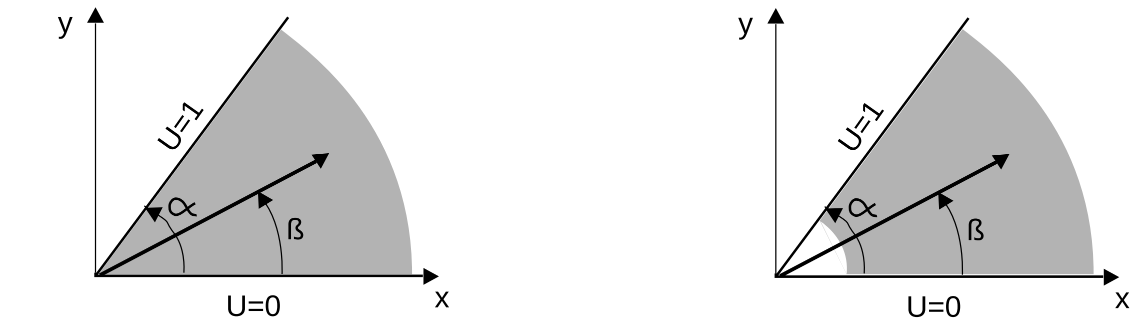
where $\vec{v} \equiv |\vec{v}|(\cos\beta, \sin\beta)$, $0 \le \beta < 2\pi$ is a constant vector, $\epsilon > 0$ is a small parameter and $0 < \alpha < 2\pi$. (Observe the discontinuous Dirichlet condition at the corner of the sector, see fig. 1 (a)). We use the polar coordinates $x = r\cos\phi$, $y = r\sin\phi$ with $0 < r < +\infty$ and $0 \le \phi \le \alpha$.

After the change of the dependent variable $U(r, \phi) = 1 - F(r, \phi) \exp\left(\vec{v} \cdot \vec{r}/(2\epsilon)\right)$, where $\vec{r} \equiv (r\cos\phi, r\sin\phi)$, the problem (P) is transformed in the Helmholtz equation for $F(r, \phi)$:

$$\left\{ \begin{array}{l} \triangle F - w^2 F = 0 \ \text{ in } \ \Omega \\ F(r, 0) = e^{-wr\cos\beta}, \ F(r, \alpha) = 0 \\ |\lim_{r\to\infty} F(r, \phi) \exp\left(wr\cos(\beta - \phi)\right)| < \infty, \ F \in \mathcal{C}(\bar{\Omega} \setminus \{(0, \phi)\}) \bigcap \mathcal{D}^2(\Omega) \end{array} \right.$$

where $w \equiv |\vec{v}|/(2\epsilon)$.

In the following proposition we obtain the solution of the problem (P) by means of an integral representation.



(a) (b)

Figure 1. (a) Domain Ω of problem (P). (b) Indented region Ω^* in theorem 1.

Proposition 1. *Write $w \equiv |\overrightarrow{v}|/(2\epsilon)$. Then, for $(r,\phi) \in \Omega$ the solution $U_\beta(r,\phi)$ of (P) reads*

1. If $0 < \beta < \alpha < 2\pi$:

$$U_\beta(r,\phi) = \chi_{(\beta,\alpha]}(\phi) + \frac{1}{2}\delta_{\phi,\beta} - I(r,\phi),$$

2. If $0 < \alpha < \beta < \alpha + \pi$:

$$\begin{aligned} U_\beta(r,\phi) = \; & \sum_{k=1}^{\left[\frac{\beta+\phi}{2\alpha}\right]} e^{-wr(\cos(\beta+\phi-2k\alpha)-\cos(\beta-\phi))} - \sum_{k=1}^{\left[\frac{\beta-\phi}{2\alpha}\right]} e^{-wr(\cos(\beta-\phi-2k\alpha)-\cos(\beta-\phi))} + \\ & \frac{1}{2}e^{-wr(1-\cos(\beta-\phi))}\left(\delta_{\frac{\beta-\phi}{2\alpha},\left[\frac{\beta-\phi}{2\alpha}\right]} - \delta_{\frac{\beta+\phi}{2\alpha},\left[\frac{\beta+\phi}{2\alpha}\right]}\right) - I(r,\phi), \end{aligned}$$

3. If $0 < \alpha + \pi \leq \beta < 2\pi$:

$$\begin{aligned} U_\beta(r,\phi) = \; & 1 + \sum_{k=1}^{\left[\frac{2\pi+\phi-\beta}{2\alpha}\right]} e^{-wr(\cos(-\phi+\beta+2k\alpha)-\cos(\beta-\phi))} - \sum_{k=0}^{\left[\frac{2\pi-\phi-\beta}{2\alpha}\right]} e^{-wr(\cos(\phi+\beta+2k\alpha)-\cos(\beta-\phi))} + \\ & \frac{1}{2}e^{-wr(1-\cos(\beta-\phi))}\left(\delta_{\frac{2\pi-\phi-\beta}{2\alpha},\left[\frac{2\pi-\phi-\beta}{2\alpha}\right]} - \delta_{\frac{2\pi+\phi-\beta}{2\alpha},\left[\frac{2\pi+\phi-\beta}{2\alpha}\right]}\right) - I_{\beta-2\pi}(r,\phi), \end{aligned}$$

4. If $\beta = \alpha$:

$$U_\beta(r,\phi) = -I(r,\phi),$$

5. If $\beta = 0$:

$$U_\beta(r,\phi) = 1 - I(r,\phi)$$

where the function $I(r,\phi)$ is defined as follows:

$$I(r,\phi) \equiv \frac{e^{wr\cos(\beta-\phi)}}{2\alpha}\int_{-\infty}^{\infty} e^{-wr\cosh t}\frac{\sin(\mu\phi)}{\cosh(\mu(t-i\beta))-\cos(\mu\phi)}dt$$

and

$$\mu \equiv \pi/\alpha.$$

The solution of (P) can not be written in terms of known functions. But, for $\epsilon \to 0^+$ and r away from 0, we can approximate $U_\beta(r,\phi)$ by an error function and elementary functions plus an asymptotic expansion in powers of ϵ. For $r \to 0$ (and $\epsilon \geq \epsilon_0 > 0$), we can approximate $U_\beta(r,\phi)$ by an asymptotic expansion in powers of r. This is the subject of the two following sections.

Asymptotic expansion of $U(r,\phi)$ in the singular limit

In this section we denote by Ω^* the sector shaped domain indented at the point $(0,\phi)$ (see fig. 1 (b)):

$$\Omega^* \equiv \{(r,\phi),\ 0 \leq \phi \leq \alpha,\ 0 < r_0 < r < \infty\}$$

Theorem 1. *Write $w \equiv |\vec{v}|/(2\epsilon)$. Then, for $(r,\phi) \in \Omega^*$, the solution $U_\beta(r,\phi)$ of (P) given in proposition 1 reads*

$$U_\beta(r,\phi) = U_\beta^0(r,\phi) + \frac{1}{2\pi\sqrt{2w}} U_\beta^1(r,\phi) + R(r,\phi), \tag{1}$$

where

1. If $0 < \beta < \alpha < 2\pi$:

$$U_\beta^0(r,\phi) = \chi_{(\beta,\alpha]}(\phi) + \frac{1}{2}\delta_{\phi,\beta} + \frac{1}{2}\text{sign}(\beta-\phi)\,\text{erfc}(\sqrt{wr(1-\cos(\beta-\phi))}), \tag{2}$$

2. If $0 < \alpha < \beta < \alpha + \pi$:

$$\begin{aligned} U_\beta(r,\phi) = & \sum_{k=1}^{\left[\frac{\beta+\phi}{2\alpha}\right]} e^{-wr(\cos(\beta+\phi-2k\alpha)-\cos(\beta-\phi))} - \sum_{k=1}^{\left[\frac{\beta-\phi}{2\alpha}\right]} e^{-wr(\cos(\beta-\phi-2k\alpha)-\cos(\beta-\phi))} + \\ & \frac{1}{2} e^{-wr(1-\cos(\beta-\phi))}\left(\delta_{\frac{\beta-\phi}{2\alpha},\left[\frac{\beta-\phi}{2\alpha}\right]} - \delta_{\frac{\beta+\phi}{2\alpha},\left[\frac{\beta+\phi}{2\alpha}\right]}\right), \end{aligned} \tag{3}$$

3. If $0 < \alpha + \pi \leq \beta < 2\pi$:

$$\begin{aligned} U_\beta(r,\phi) = \ & 1 + \sum_{k=1}^{\left[\frac{2\pi+\phi-\beta}{2\alpha}\right]} e^{-wr(\cos(-\phi+\beta+2k\alpha)-\cos(\beta-\phi))} - \sum_{k=0}^{\left[\frac{2\pi-\phi-\beta}{2\alpha}\right]} e^{-wr(\cos(\phi+\beta+2k\alpha)-\cos(\beta-\phi))} + \\ & \frac{1}{2} e^{-wr(1-\cos(\beta-\phi))}\left(\delta_{\frac{2\pi-\phi-\beta}{2\alpha},\left[\frac{2\pi-\phi-\beta}{2\alpha}\right]} - \delta_{\frac{2\pi+\phi-\beta}{2\alpha},\left[\frac{2\pi+\phi-\beta}{2\alpha}\right]}\right), \end{aligned} \tag{4}$$

4. If $\beta = \alpha$:

$$U_\beta^0(r,\phi) = \text{erfc}(\sqrt{wr(1-\cos(\alpha-\phi))}), \tag{5}$$

5. If $\beta = 0$:

$$U_\beta^0(r,\phi) = 1 - \text{erfc}\sqrt{wr(1-\cos\phi)} \tag{6}$$

The function $U_\beta^1(r,\phi)$ has an asymptotic expansion in powers of w^{-1}:

$$U_\beta^1(r,\phi) = \sum_{k=0}^{n-1} \frac{T_k(r,\phi)}{(2w)^k} + R_n(r,\phi), \tag{7}$$

where empty sums must be understood as zero. The coefficients $T_k(r,\phi)$ *are regular functions of* r *and* ϕ *in* Ω^*.

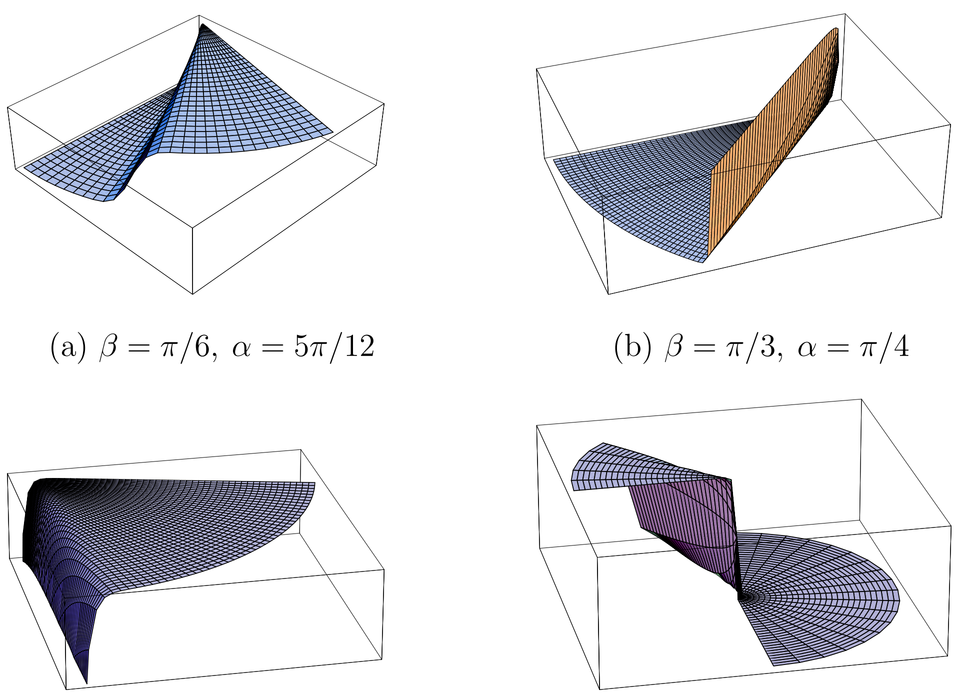
The remainder $R_n(r,\phi)$ *satisfies a bound of the form*

$$|R_n(r,\phi)| \le M\frac{\Gamma(n+1/2)}{(2wdr)^n}e^{-wr(1-\cos(\beta-\phi))} \tag{8}$$

for some positive constants M *and* d *and for some* $c>0$ *the function* $R(r,\phi)$ *satisfies*

$$\begin{cases} |R(r,\phi)| \le ce^{wr(\cos(\beta-\phi)-1)} & \text{if } \alpha<\beta \\ R(r,\phi)=0 & \text{if } \alpha>\beta. \end{cases} \tag{9}$$

Remark 1. From (1), (7), (8) and (9) we see that $U_\beta(r,\phi) = U_\beta^0(r,\phi) + \mathcal{O}(\sqrt{\epsilon})$ as $\epsilon \to 0^+$ away from the point $r=0$. Then, the first order approximation to the solution of (P) is a linear combination of error functions and elementary functions. The error functions in (2), (5) or (6) exhibit an interior layer of width $\mathcal{O}(\sqrt{\epsilon})$ and parabolic level lines of equation $r(1-\cos(\beta-\phi)) =$ constant. When $\overrightarrow{v}$ is not inside the sector, the exponential functions in (3) and (4) exhibits boundary layers of width $\mathcal{O}(\epsilon)$ (see fig 4).



(a) $\beta=\pi/6,\ \alpha=5\pi/12$ (b) $\beta=\pi/3,\ \alpha=\pi/4$

(c) $\beta=2\pi/3,\ \alpha=5\pi/12$ (d) $\beta=7\pi/6,\ \alpha=3\pi/2$

Figure 4. Graphic of the first order approximation, $U_\beta^0(r,\phi)$, to the solution of the problem (P) for different values of α and β and $\epsilon=0.1$.

Asymptotic expansion of $U(r,\phi)$ near the discontinuity

The asymptotic expansion (7) breaks down when $r \to 0^+$. Then, formulas in theorem 1 become senseless. The asymptotic approximation of $U_\beta(r,\phi)$ near the point $r=0$ requires a completely different analysis given in the following theorem.

Theorem 2. *Write $w \equiv |\overrightarrow{v}|/(2\epsilon)$. Then, for $(r,\phi)\in\Omega$, the solution $U_\beta(r,\phi)$ of (P) reads*

$$U_\beta(r,\phi) = \frac{\phi}{\alpha} + \frac{wr}{\alpha} e^{-wr(1-\cos(\beta-\phi))} U^2_\beta(r,\phi) \tag{10}$$

where $U^2_\beta(r,\phi) = \mathcal{O}(1)$ when $wr \to 0^+$. Moreover for $n = 1,2,3,...$, $U^2_\beta(r,\phi)$ has a convergent expansion in powers of wr:

$$\begin{aligned} U^2_\beta(r,\phi) \equiv \quad & \frac{T_0(\phi,\beta)}{rw}\left[1 - e^{wr(1-\cos(\beta-\phi))}\right] + \sum_{k=1}^{n-1} \frac{(-1)^k}{k!}\left[T_k(\phi,\beta) - \right. \\ & \left. V_k(\phi,\beta)\log(rw)\right](rw)^{k-1} + R_n(r,\phi), \end{aligned} \tag{11}$$

where empty sums must be understood as zero and sign$(0)=0$. Coefficients $T_k(\phi,\beta)$ and $V_k(\phi,\beta)$ are regular functions of ϕ and β in Ω.

The remainder term $R_n(r,\phi)$ verifies a bound of the form

$$|R_n(r,\phi)| \le \frac{M}{d^n n!}[n(2+d) + |\log(rw)|](rw)^{n-1} \tag{12}$$

for some positive constants M and d.

Remark 2. From (10), (11) and (12) we see that

$$U_\beta(r,\phi) = \frac{\phi}{\alpha} + \mathcal{O}\left(\frac{r}{\epsilon}\right) \quad \text{when} \quad r/\epsilon \to 0^+.$$

The discontinuity of the inflow boundary condition is smoothed inside the domain by a linear function of the angle ϕ (see fig. 5).

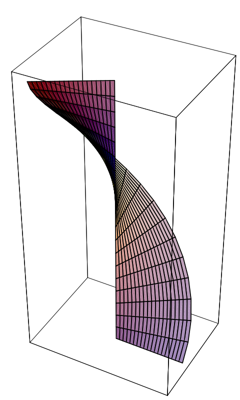


Figure 5. Graphic of the first order approximation, $\frac{\phi}{\alpha}$ to the solution of the problem (P) near the discontinuity point of the boundary condition $r=0$ for $\epsilon = 0.1$ and $\alpha = 7\pi/6$.

Conclusions

The singularly perturbed convection-diffusion problem (P) has been defined on a sector by means of a Dirichlet boundary condition with a discontinuity located at the corner of the domain. We have obtained an integral representation of the solution susceptible of an asymptotic analysis. Then, two complementary asymptotic expansions of the solution have been obtained. One expansion is valid in the singular limit $\epsilon \to 0^+$ and away from the discontinuity $r = 0$. The other one is valid near the discontinuity point $r = 0$.

These two asymptotic expansions are derived from two quite different asymptotic analysis. Whereas the asymptotic expansion in the singular limit is obtained from a classical uniform method, the asymptotic expansion near the discontinuity is derived by means of a distributional approach. Two quite different asymptotic principles match into the same problem.

The asymptotic expansion in the singular limit shows that the main contribution from the data's discontinuities to the shape of the solution on the boundary layers is contained in a certain combination of error functions, exponential functions and step functions. This combination is necessary to approach the behaviour of the solution on the interior layer of width $\mathcal{O}(\sqrt{\epsilon})$ or on the boundary layer of width $\mathcal{O}(\epsilon)$. On the other hand, the asymptotic expansion near the discontinuities shows that the discontinuity on the boundary is smoothed inside the domain by means of a simply linear function of the polar angle.

We suspect that, as in the problem analyzed here, the error function plays a fundamental role in the approximation of the solution of many singularly perturbed convection-diffusion problems with discontinuities in the boundary conditions (problems defined over more general domains and by more general coefficients). This will be the subject of further investigations. Then, the asymptotic expansions of the solutions of (P) presented here may give a qualitative idea about the behaviour of the solutions of more realistic convection-diffusion problems with discontinuous Dirichlet conditions. This should help in the development of suitable numerical methods for those problems.

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