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Asymptotic behaviour of three-dimensional singularly perturbed convection–diffusion problems with discontinuous data

José L. López [a,*], Ester Pérez Sinusía [a], Nico M. Temme [b]

[a] *Departamento de Matemática e Informática, Universidad Pública de Navarra, 31006-Pamplona, Spain*
[b] *CWI, PO Box 94079, 1090 GB Amsterdam, The Netherlands*

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Abstract

We consider three singularly perturbed convection–diffusion problems defined in three-dimensional domains: (i) a parabolic problem $-\epsilon(u_{xx}+u_{yy})+u_t+v_1u_x+v_2u_y=0$ in an octant, (ii) an elliptic problem $-\epsilon(u_{xx}+u_{yy}+u_{zz})+v_1u_x+v_2u_y+v_3u_z=0$ in an octant and (iii) the same elliptic problem in a half-space. We consider for all of these problems discontinuous boundary conditions at certain regions of the boundaries of the domains. For each problem, an asymptotic approximation of the solution is obtained from an integral representation when the singular parameter $\epsilon\to 0^+$. The solution is approximated by a product of two error functions, and this approximation characterizes the effect of the discontinuities on the small $\epsilon-$ behaviour of the solution and its derivatives in the boundary layers or the internal layers.

Keywords: Singular perturbation problem; Discontinuous boundary data; Asymptotic expansions; Error function

1. Introduction

Mathematically speaking, a singularly perturbed convection–diffusion problem is a boundary value problem of the second order in which the coefficients of the second order derivatives are

* Corresponding author.
E-mail addresses: jl.lopez@unavarra.es (J.L. López), ester.perez@unavarra.es (E. Pérez Sinusía), nico.temme@cwi.nl (N.M. Temme).

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small. In this paper we focus our attention on three-dimensional linear parabolic convection–diffusion problems of the form: find a function $u \in \mathcal{C}(\tilde{\Omega})$ and $u_x, u_y, u_t, u_{xx}, u_{yy} \in \mathcal{C}(\Omega)$ such that

$$\begin{cases} -\epsilon(u_{xx}+u_{yy})+v_1u_x+v_2u_y+u_t=0, & (x,y,t)\in\Omega\subset\mathbb{R}^3, \\ u(x,y,0)=f(x,y), \quad u(x,y,t)|_{\partial\Omega_0\times(0,\infty)}=g(x,y,t), & \text{for } (x,y,t)\in\partial\Omega. \end{cases} \tag{1}$$

In this formula, Ω_0 is a region in $\mathbb{R}^2$, $\Omega = \Omega_0 \times (0,\infty)$ and $\tilde{\Omega}$ is the closed domain $\bar{\Omega}$ with the discontinuity points of the boundary condition g or the initial condition f removed. We also consider three-dimensional linear elliptic convection–diffusion problems of the form: find a function $u \in \mathcal{C}(\tilde{\Omega}) \cap \mathcal{D}^2(\Omega)$ such that

$$\begin{cases} -\epsilon(u_{xx}+u_{yy}+u_{zz})+v_1u_x+v_2u_y+v_3u_z=0, & (x,y,z)\in\Omega\subset\mathbb{R}^3, \\ u(x,y,z)=h(x,y,z), & \text{for } (x,y,z)\in\partial\Omega. \end{cases} \tag{2}$$

Again, in this formula, $\tilde{\Omega}$ is the closed domain $\bar{\Omega}$ with the discontinuity points of the boundary condition h removed. In both formulas, ϵ is a small positive parameter, $\vec{v} = (v_1, v_2)$ or $\vec{v} = (v_1, v_2, v_3)$ is the convection vector, $f(x, y)$ is the initial data, $g(x, y, t)$ and $h(x, y, z)$ are the Dirichlet condition and $\mathcal{D}^2(\Omega)$ is the set of functions with partial derivatives up to order two defined in all points of Ω.

The location and shape of the boundary layers of u depend, among other things, on the prescribed velocity field $\vec{v}$, on the shape of the boundary $\partial\Omega$, on the existence of discontinuities in $f(x, y)$, $g(x, y, t)$ or $h(x, y, z)$ and on a possible non-smooth matching of the initial condition $f(x, y)$ and the boundary condition at $t = 0$: $g(x, y, 0)$. Usually, regular boundary layers of size $\mathcal{O}(\epsilon)$ appear on the outflow boundary, whereas parabolic boundary layers of size $\mathcal{O}(\sqrt{\epsilon})$ appear along the characteristic boundaries. For more details on the shape and nature of boundary layers see for example [3–7] and references therein.

To get the exact solution of a problem of the form (1) or (2) in terms of elementary functions is, in general, an impossible mission. Then, an approximation of the solution adapted to the singular character of this kind of problems (an asymptotic expansion) is of interest. For two-dimensional problems, there is an extensive literature devoted to the construction of approximated solutions of singular perturbation problems based on matching of asymptotic expansions. The book of Il'in [7] contains a quite exhaustive and general analysis for different equations and domains. Other important references are for example [4,8] or [14]. However, a perturbation analysis based on an expansion of the solution in powers of the perturbation parameter is very complicated when the boundary condition is discontinuous [11,19].

Techniques based on an exact representation of the solution (integral representations) are of interest for these kind of problems. For example, Hedstrom and Osterheld [6] studied the two-dimensional problem $\epsilon\Delta u - u_y = 0$ on the positive quarter plane with boundary conditions $u(x, 0) = 0$ and $u(0, y) = 1$. They obtained the first two terms of the asymptotic expansion of u for $\epsilon \to 0^+$ from a Fourier integral representation of u. The first term of this expansion is an error function. A more detailed investigation has been developed by Temme in [17]: an integral representation for u is obtained from the associated Yukawa equation and a complete asymptotic expansion of u for $\epsilon \to 0^+$ is derived from this integral representation. The same equation $\epsilon\Delta u - u_y = 0$, but in a general two-dimensional sector, is considered in [18], where an integral representation for u is obtained from the associated Yukawa equation. Different asymptotic expansions as $\epsilon \to 0^+$ are obtained depending on the angle of the sector and again the error function plays an important role in the analysis. A similar problem defined in the interior of a two-dimensional circle is discussed in [19]. In all these problems, the approximation is not valid near the discontinuities of the boundary condition. Two-dimensional problems of the form $-\epsilon\Delta u + \vec{v}\cdot\vec{\nabla}u = 0$ defined in an infinite strip or in half-infinite strip with discontinuous boundary data have been studied in [9,10]. Also, the error function seems to play a fundamental role in two-dimensional parabolic problems with discontinuous boundary data. Shagi-Di Shih has studied parabolic problems in a quarter plane with discontinuities in the Dirichlet data or in its derivatives showing that, in the singular limit, the solution is approached by error functions or primitives of error functions [15,16].

We observe that most of the singular perturbation problems with discontinuous boundary or initial data analyzed in the literature (using either matching techniques or asymptotics of integrals) are two-dimensional problems. In this paper we will shed light on the influence that the discontinuities of the boundary conditions have on the boundary or interior layers of the solution of three-dimensional parabolic or elliptic convection–diffusion problems. We want to investigate if, as in the examples mentioned in the above paragraph, the error function is also involved in the approximation of the solution. For this purpose we analyze a problem of the form (1) and two problems of the form (2). As in the references mentioned in the paragraph above, the starting point is an integral representation of the solution. As a difference with the two-dimensional case, the solution is not represented by a simple integral, but by a double integral. Then, we approximate the solution by deriving the first term of the asymptotic expansion of that double integral in the singular limit $\epsilon \to 0^+$.

In Section 2 we analyze a parabolic problem. In Sections 3 and 4 we analyze two elliptic problems. Some comments and conclusions are given in Section 5.

The problem considered in Section 4 is discussed earlier in [12], where we have used saddle point methods for a two-dimensional integral to obtain a first order approximation of the solution of the 3D problem. In that paper we have introduced a generalization of the error function to describe the behaviour of the solution in the internal boundary layers.

Throughout this paper we use the notation

$$\omega \equiv \frac{1}{2\epsilon}. \tag{3}$$

2. A parabolic problem in an octant

We consider the following parabolic convection–diffusion problem defined in the first octant: $(x, y, t) \in \Omega_1 \equiv (0,\infty)\times(0,\infty)\times(0,\infty)$, with a "rectangular source of contamination" located around the origin (see Fig. 1):

$$\begin{cases} U \in \mathcal{C}(\tilde{\Omega}_1),\ U_x, U_{xx}, U_y, U_{yy}, U_t \in \mathcal{C}(\Omega_1), \\ U \text{ bounded in bounded subsets of } \tilde{\Omega}_1, \\ -\epsilon(U_{xx}+U_{yy}) + v_1 U_x + v_2 U_y + U_t = 0 & \text{in } \Omega_1, \\ U(x,y,0) = \chi_{(0,a)}(x)\chi_{(0,b)}(y) \quad \text{and} \quad U(x,0,t) = U(0,y,t) = 0 & \text{in } \tilde{\Omega}_1 \setminus \Omega_1. \end{cases} \tag{P$_1$}$$

In this formula, (v_1, v_2) is a vector of modulus 1, the positive numbers a and b represent the length of the sides of the "source of contamination" and $\chi_{(a,b)}(x)$ represents the characteristic function of the interval (a, b):

$$\chi_{(a,b)}(x) \equiv \begin{cases} 1 & \text{if } x \in (a,b), \\ 0 & \text{if } x \notin (a,b). \end{cases}$$

Observe that the initial condition and the boundary condition do not match continuously at $\{y = t = 0, 0 < x < a\}$ and at $\{x = t = 0, 0 < y < b\}$. Moreover, the initial condition is discon-

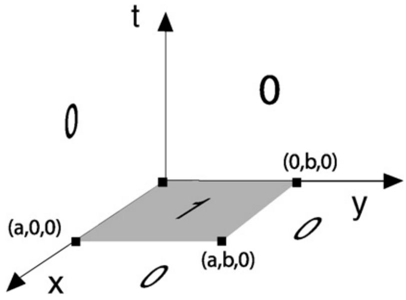


Fig. 1. Domain $\Omega_1 = (0,\infty) \times (0,\infty) \times (0,\infty)$ and boundary condition of problem (P_1).

tinuous at $\{y = b, t = 0, 0 < x < a\}$ and at $\{x = a, t = 0, 0 < y < b\}$. The set $\tilde{\Omega}_1$ is the closed set $\bar{\Omega}_1$ with the discontinuity points of the boundary data removed: $\tilde{\Omega}_1 \equiv \bar{\Omega}_1 \setminus \{\{(x,0,0),(x,b,0), 0 \leqslant x \leqslant a\} \cup \{(0,y,0),(a,y,0), 0 \leqslant y \leqslant b\}\}$.

After the change of unknown $U(x,y,t) = e^{\omega[v_1 x + v_2 y - t/2]} F(x,y,t)$, problem ($\mathrm{P}_1$) is transformed into the heat equation for $F(x,y,t)$:

$$\begin{cases} F \in \mathcal{C}(\tilde{\Omega}_1),\ F_x, F_{xx}, F_y, F_{yy}, F_t \in \mathcal{C}(\Omega_1), \\ F \text{ bounded in bounded subsets of } \tilde{\Omega}_1, \\ F_{xx} + F_{yy} - 2\omega F_t = 0 & \text{in } \Omega_1, \\ F(x,y,0) = e^{-\omega[v_1 x + v_2 y]} \chi_{(0,a)}(x) \chi_{(0,b)}(y),\ F(x,0,t) = F(0,y,t) = 0 & \text{in } \tilde{\Omega}_1 \setminus \Omega_1. \end{cases} \tag{4}$$

A solution of problem (4) (and therefore of (P_1)) may be derived by using Fourier sine transforms with respect to x and with respect to y. The result is

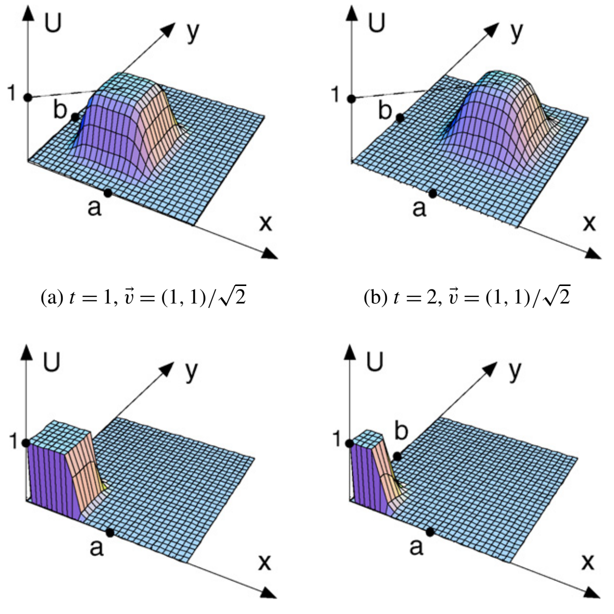
$$U(x,y,t) = \frac{\omega}{2\pi t} e^{\omega[v_1 x + v_2 y - t/2]} \int_0^a e^{-\omega v_1 s} \left[e^{-\omega(x-s)^2/(2t)} - e^{-\omega(x+s)^2/(2t)} \right] ds$$
$$\times \int_0^b e^{-\omega v_2 u} \left[e^{-\omega(y-u)^2/(2t)} - e^{-\omega(y+u)^2/(2t)} \right] du. \tag{5}$$

It is easy to check by direct substitution that this function is a solution of problem (P_1). By using the error function [1, Eq. 7.1.1]

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt, \tag{6}$$

the function $U(x,y,t)$ can be written in the form

$$U(x,y,t) = \frac{1}{4} \left\{ \operatorname{erf}\left(\frac{a + v_1 t - x}{2\sqrt{\epsilon t}} \right) - \operatorname{erf}\left(\frac{v_1 t - x}{2\sqrt{\epsilon t}} \right) \right.$$
$$\left. - e^{2\omega v_1 x} \left[\operatorname{erf}\left(\frac{a + v_1 t + x}{2\sqrt{\epsilon t}} \right) - \operatorname{erf}\left(\frac{v_1 t + x}{2\sqrt{\epsilon t}} \right) \right] \right\}$$
$$\times \left\{ \operatorname{erf}\left(\frac{b + v_2 t - y}{2\sqrt{\epsilon t}} \right) - \operatorname{erf}\left(\frac{v_2 t - y}{2\sqrt{\epsilon t}} \right) \right.$$



(a) $t = 1, \vec{v} = (1, 1)/\sqrt{2}$ (b) $t = 2, \vec{v} = (1, 1)/\sqrt{2}$

(c) $t = 1, \vec{v} = (-1, -1)/\sqrt{2}$ (d) $t = 2, \vec{v} = (-1, -1)/\sqrt{2}$

Fig. 2. Graphs of the solution (7) of problem (P_1) for $\epsilon = 0.1$, $a = b = 1$, two different values of t and two different values of the convection vector $\vec{v}$.

$$- e^{2\omega v_2 y}\left[\operatorname{erf}\left(\frac{b + v_2 t + y}{2\sqrt{\epsilon t}}\right) - \operatorname{erf}\left(\frac{v_2 t + y}{2\sqrt{\epsilon t}}\right)\right]\Bigg\}. \tag{7}$$

This solution may not be unique unless we impose a convenient condition upon $U(x, y, t)$ (or upon $F(x, y, t)$) concerning its growth at infinity. For details we refer to Appendix A.

For $v_1 > 0$ and $v_2 > 0$, the solution of this problem presents internal layers of the size $\mathcal{O}(\sqrt{\epsilon})$ along the surfaces $\{v_1 t < x < v_1 t + a,\ y = v_2 t \text{ or } y = v_2 t + b\}$ and $\{v_2 t < y < v_2 t + b,\ x = v_1 t \text{ or } x = v_1 t + a\}$. For $v_1 = 0$ and/or $v_2 = 0$, two/one of these layers are located on the boundary $x = 0$ and/or $y = 0$. For $v_1 < 0$ and/or $v_2 < 0$, the solution presents also boundary layers of size $\mathcal{O}(\epsilon)$ on the boundary $x = 0$ and/or $y = 0$. See Fig. 2.

3. An elliptic problem in an octant

We consider a singularly perturbed elliptic convection–diffusion problem defined in the first octant: $\Omega_2 = (0, \infty) \times (0, \infty) \times (0, \infty)$, with an "infinite source of contamination" located at the plane $z = 0$ (see Fig. 3):

$$\begin{cases} U \in \mathcal{C}(\tilde{\Omega}_2) \cap \mathcal{D}^2(\Omega_2), & U \text{ bounded in bounded subsets of } \tilde{\Omega}_2, \\ -\epsilon \Delta U + U_z = 0 & \text{in } \Omega_2, \\ U(x, y, 0) = 1, \quad U(0, y, z) = U(x, 0, z) = 0 & \text{in } \tilde{\Omega}_2 \setminus \Omega_2. \end{cases} \tag{P$_2$}$$

Observe that the Dirichlet data are discontinuous at the X and Y axes. The set $\tilde{\Omega}_2$ is the set $\bar{\Omega}_2$ with the X and Y axes removed: $\tilde{\Omega}_2 \equiv \bar{\Omega}_2 \setminus \{(x, 0, 0), (0, x, 0); x \geqslant 0\}$.

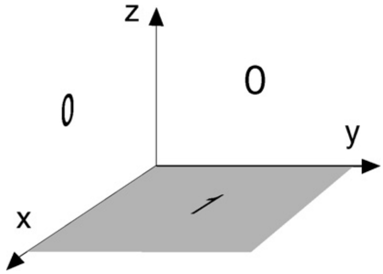


Fig. 3. Domain $\Omega_2 = (0,\infty)\times(0,\infty)\times(0,\infty)$ and boundary conditions of problem (P_2).

After the change of the unknown $U = e^{\omega z}F$, problem (P_2) is transformed into the Yukawa equation for $F(x,y,z)$:

$$\begin{cases} F \in \mathcal{C}(\tilde{\Omega}_2)\cap\mathcal{D}^2(\Omega_2), & F \text{ bounded in bounded subsets of } \tilde{\Omega}_2, \\ \Delta F - \omega^2 F = 0 & \text{in } \Omega_2, \\ F(x,y,0)=1, \quad F(0,y,z)=F(x,0,z)=0 & \text{in } \tilde{\Omega}_2\setminus\Omega_2. \end{cases} \tag{8}$$

We will obtain a solution of problem (8) and therefore of problem (P_2) below, but this solution may not be unique unless we impose a convenient condition upon $U(x,y,z)$ (or upon $F(x,y,z)$) concerning its growth at infinity. For details we refer to Appendix B.

The unique solution of problem (P_2) can be derived by using Fourier sine transforms with respect to x and y:

$$U(x,y,z) = \frac{e^{\omega z}}{\pi^2}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} ds\, \frac{\sin(\omega x t)}{t}\frac{\sin(\omega y s)}{s} e^{-\omega z\sqrt{1+t^2+s^2}}.$$

It is easy to check by direct substitution that this function is a solution of problem (P_2). After the change of variable $s\to u$ defined by $s = u\sqrt{1+t^2}$ in the s-integral we obtain:

$$U(x,y,z) = \frac{e^{\omega z}}{\pi^2}\int_{-\infty}^{\infty} \frac{\sin(\omega x t)}{t}\,dt \int_{-\infty}^{\infty} \frac{\sin(\omega y u\sqrt{1+t^2})}{u} e^{-\omega z\sqrt{1+t^2}\sqrt{1+u^2}}\,du.$$

The integral in the u variable is just the solution of a similar two-dimensional convection–diffusion problem defined on a quarter plane $(x,y)\in(0,\infty)\times(0,\infty)$ and studied in [9,17]. It is shown there [9, Theorem 1] that:

$$\frac{e^{\omega z}}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\omega y u)}{u} e^{-\omega z\sqrt{1+u^2}}\,du = \operatorname{erf}\sqrt{\omega\big[\sqrt{y^2+z^2}-z\big]} + \tilde{R}(y,z,\omega),$$

with

$$\big|\tilde{R}(y,z,\omega)\big| \leqslant \frac{Cy}{\sqrt{\omega}(y^2+z^2)^{3/4}} e^{\omega[z-\sqrt{y^2+z^2}]}, \quad \text{for } y,z,\omega>0, \tag{9}$$

and C is a positive constant independent of y, z and ω. (In the following, we denote by C any positive constant independent of x, y, z and ω.) This bound means that $\tilde{R}/\operatorname{erf}\sqrt{\omega[\sqrt{y^2+z^2}-z]}$ is exponentially small away from the plane $y=0$ and at least $\mathcal{O}(\omega^{-1/2})$ away from the X axis. In any case, we can write

$$U(x,y,z) = U_0(x,y,z) + U_1(x,y,z), \tag{10}$$

with

$$U_0(x,y,z) \equiv \frac{e^{\omega z}}{\pi} \int_{-\infty}^{\infty} \frac{\sin(\omega x t)}{t} e^{-\omega z\sqrt{1+t^2}} \operatorname{erf}\sqrt{\omega\sqrt{1+t^2}\big[\sqrt{y^2+z^2}-z\big]}\, dt$$

and

$$U_1(x,y,z) \equiv 2\frac{e^{\omega z}}{\pi} \int_{0}^{\infty} \frac{\sin(\omega x t)}{t} e^{-\omega z\sqrt{1+t^2}} \tilde{R}\big(y,z,\omega\sqrt{1+t^2}\big)\, dt.$$

Now we use the bound (9) in this integral with ω replaced by $\omega\sqrt{1+t^2}$. (Observe that it is a bound uniformly valid for $t \in (0,\infty)$.) We perform also the change of variable $t \to u$ defined by $1+t^2 = (u+1)^2$ in this integral and use the bound $|\sin(\omega x t)/t| \leqslant \omega x$ $\forall \omega, x, t > 0$ to obtain

$$\big|U_1(x,y,z)\big| \leqslant \frac{Cxy}{y^2+z^2} e^{\omega[z-\sqrt{y^2+z^2}]}. \tag{11}$$

On the other hand, we write

$$U_0(x,y,z) = U_{00}(x,y,z) + U_{01}(x,y,z), \tag{12}$$

with

$$U_{00}(x,y,z) \equiv \operatorname{erf}\sqrt{\omega\big[\sqrt{z^2+y^2}-z\big]}\, \frac{e^{\omega z}}{\pi} \int_{-\infty}^{\infty} \frac{\sin(\omega x t)}{t} e^{-\omega z\sqrt{1+t^2}}\, dt$$

and

$$\begin{aligned} U_{01}(x,y,z) \equiv{}& 2\frac{e^{\omega z}}{\pi} \int_{0}^{\infty} \frac{\sin(\omega x t)}{t} e^{-\omega z\sqrt{1+t^2}} \\ &\times \Big[\operatorname{erf}\sqrt{\omega\sqrt{1+t^2}\big[\sqrt{y^2+z^2}-z\big]} - \operatorname{erf}\sqrt{\omega\big[\sqrt{z^2+y^2}-z\big]}\Big]\, dt. \end{aligned} \tag{13}$$

From Lagrange's formula for the remainder of the Taylor expansion of the function $\operatorname{erf}\sqrt{\omega\sqrt{1+t^2}}$ at $t^2 = 0$,

$$\begin{aligned} &\Big|\operatorname{erf}\sqrt{\omega\sqrt{1+t^2}\big[\sqrt{y^2+z^2}-z\big]} - \operatorname{erf}\sqrt{\omega\big[\sqrt{z^2+y^2}-z\big]}\Big| \\ &\quad \leqslant C\sqrt{\omega\big[\sqrt{z^2+y^2}-z\big]}\, e^{\omega[z-\sqrt{z^2+y^2}]} t^2, \quad t \in [0,\infty). \end{aligned}$$

Introducing this bound and the bound $|\sin(\omega x t)/t| \leqslant \omega x$ for $\omega, x, t \in (0,\infty)$ in the right-hand side of (13) and performing the change of variable $t \to u$ defined by $\sqrt{1+t^2} = u$ we find

$$\big|U_{01}(x,y,z)\big| \leqslant \frac{Cx}{z^{3/2}} \sqrt{\sqrt{z^2+y^2}-z}\, e^{\omega[z-\sqrt{z^2+y^2}]}. \tag{14}$$

On the other hand, using the integral representation of the error function (6) in (13) and interchanging the orders of integration, we obtain

$$U_{01}(x,y,z) = \frac{4e^{\omega z}}{\pi^{3/2}} \int_{B}^{\infty} e^{-u^2}\, du \int_{A(u)}^{\infty} \frac{\sin(\omega x t)}{t} e^{-\omega z\sqrt{1+t^2}}\, dt, \tag{15}$$

where

$$B \equiv \sqrt{\omega\big[\sqrt{z^2+y^2}-z\big]} \quad \text{and} \quad A(u) \equiv \sqrt{\frac{u^4}{[\sqrt{z^2+y^2}-z]^2}-1}.$$

After the change of variable in the t-integral: $t \to t/(\omega x)$ and splitting the integration interval at the points $t = n\pi$, $n \in \mathbb{Z}$, we find

$$\int_{A(u)}^{\infty} \frac{\sin(\omega x t)}{t} e^{-\omega z\sqrt{1+t^2}}\,dt = \int_{\omega x A(u)}^{n_0\pi} \frac{\sin t}{t} e^{-\omega z\sqrt{1+(t/\omega x)^2}}\,dt + \sum_{n=n_0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{\sin t}{t} e^{-\omega z\sqrt{1+(t/\omega x)^2}}\,dt, \tag{16}$$

where

$$n_0 \equiv \left\lfloor \frac{\omega x A(u)}{\pi} + 1 \right\rfloor.$$

The first integral in the right-hand side of (16) may be bounded by

$$\left| \int_{\omega x A(u)}^{n_0\pi} \frac{\sin t}{t} e^{-\omega z\sqrt{1+(t/\omega x)^2}}\,dt \right| \leqslant \pi\left(n_0 - \frac{\omega x A(u)}{\pi}\right) e^{-\omega z} \leqslant \pi e^{-\omega z}.$$

On the one hand, the integrands in every integral of the sum in the right-hand side of (16) have a constant sign. On the other hand, the function $f(t) \equiv e^{-\omega z\sqrt{1+(t/\omega x)^2}}$ is a decreasing function of t. Using these facts it is easy to see that that sum may be bounded in the form

$$\left| \sum_{n=n_0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{\sin t}{t} e^{-\omega z\sqrt{1+(t/\omega x)^2}}\,dt \right| \leqslant \sum_{n=n_0}^{\infty} (-1)^n a_n f_n,$$

where

$$a_n \equiv \int_{n\pi}^{(n+1)\pi} \frac{|\sin t|}{t}\,dt \quad \text{and} \quad f_n \equiv \begin{cases} f(n\pi) & \text{if } n \text{ even}, \\ f((n+1)\pi) & \text{if } n \text{ odd}. \end{cases}$$

Using the bound $f_n \leqslant e^{-\omega z}$ $\forall n \in \mathbb{N}$ and the fact that a_n and f_n are positive and decreasing functions of n, we find

$$\left| \int_{A(u)}^{\infty} \frac{\sin(\omega x t)}{t} e^{-\omega z\sqrt{1+t^2}}\,dt \right| \leqslant C e^{-\omega z}.$$

Introducing this bound in the right-hand side of (15) we find

$$\big|U_{01}(x,y,z)\big| \leqslant C \operatorname{erfc}\sqrt{\omega\big[\sqrt{z^2+y^2}-z\big]}. \tag{17}$$

On the other hand, from [9, Theorem 1] we have

$$U_{00}(x,y,z) = \operatorname{erf}\sqrt{\omega\big[\sqrt{z^2+y^2}-z\big]}\Big\{\operatorname{erf}\sqrt{\omega\big[\sqrt{z^2+x^2}-z\big]} + \tilde{R}(x,z,\omega)\Big\}, \tag{18}$$

with

$$\left|\tilde{R}(x,z,\omega)\right| \leqslant \frac{Cx}{\sqrt{\omega}(x^2+z^2)^{3/4}} e^{\omega[z-\sqrt{z^2+x^2}]}. \tag{19}$$

Then, from (10), (12) and (18) we have

$$U(x,y,z) = \operatorname{erf}\sqrt{\omega\left[\sqrt{z^2+x^2}-z\right]}\operatorname{erf}\sqrt{\omega\left[\sqrt{z^2+y^2}-z\right]}\left[1+R(x,y,z,\omega)\right]$$

with

$$R(x,y,z,\omega) \equiv \frac{\tilde{R}(x,z,\omega)}{\operatorname{erf}\sqrt{\omega[\sqrt{z^2+x^2}-z]}} + \frac{U_1(x,y,z)+U_{01}(x,y,z)}{\operatorname{erf}\sqrt{\omega[\sqrt{z^2+x^2}-z]}\operatorname{erf}\sqrt{\omega[\sqrt{z^2+y^2}-z]}}.$$

Define the region:

$$\begin{aligned}\Omega_2^* \equiv \Omega_2 \setminus \Big\{&\big\{(x,y,z)\in\Omega_2 \text{ with } 0<z\leqslant z_0 \text{ and } 0<x\leqslant x_0\big\}\\ &\cup\big\{(x,y,z)\in\Omega_2 \text{ with } 0<z\leqslant z_0 \text{ and } 0<y\leqslant y_0\big\}\Big\},\end{aligned} \tag{20}$$

with $x_0, y_0, z_0 > 0$. This region is the open region Ω_2 indented around the X and Y axes (the X and Y axes are the points of discontinuity of the boundary condition).

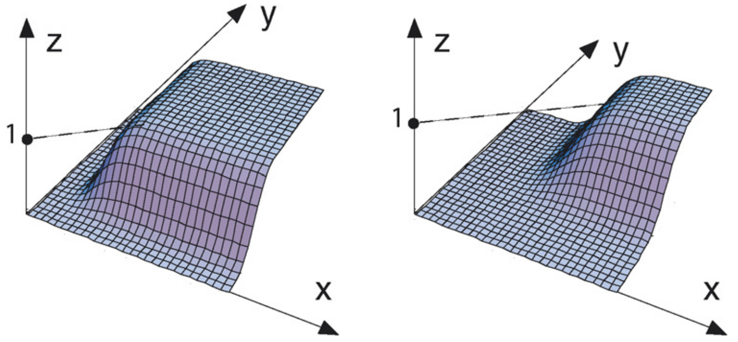
From (11), (14), (17) and (19) we see that $R(x,y,z,\omega)$ is exponentially small in Ω_2^* when $\omega\to\infty$ unless $\omega[\sqrt{z^2+y^2}-z]=\mathcal{O}(1)$ or $\omega[\sqrt{z^2+x^2}-z]=\mathcal{O}(1)$. In the first case, if $\omega[\sqrt{z^2+y^2}-z]=\mathcal{O}(1)$, then $y/z=\mathcal{O}(\omega^{-1/2})$. In the second case, if $\omega[\sqrt{z^2+x^2}-z]=\mathcal{O}(1)$, then $x/z=\mathcal{O}(\omega^{-1/2})$. In any case, $R(x,y,z,\omega)=\mathcal{O}(\omega^{-1/2})$ uniformly in $(x,y,z)\in\Omega_2^*$ and therefore,

$$U(x,y,z) = \operatorname{erf}\sqrt{\omega\left[\sqrt{z^2+x^2}-z\right]}\operatorname{erf}\sqrt{\omega\left[\sqrt{z^2+y^2}-z\right]}\left[1+\mathcal{O}\left(\omega^{-1/2}\right)\right] \tag{21}$$

in Ω_2^*.

A slight generalization of problem (P_2) with a more general unitary convection vector $\vec{v}=(a,b,c)$ is the following:

$$\begin{cases} U\in\mathcal{C}(\tilde{\Omega}_2)\cap\mathcal{D}^2(\Omega_2), & U \text{ bounded in bounded subsets of } \tilde{\Omega}_2,\\ -\varepsilon\Delta U+\vec{v}\cdot\vec{\nabla}U=0 & \text{in } \Omega_2,\\ U(x,y,0)=1,\quad U(0,y,z)=U(x,0,z)=0 & \text{in } \tilde{\Omega}_2\setminus\Omega_2. \end{cases} \tag{P_2'}$$



(a) $z=1$ (b) $z=3$

Fig. 4. Graphs of the approximation (22) for the solution of problem (P_2') with $\epsilon=0.1$, $\vec{v}=(1,1,1)/\sqrt{3}$ and two different values of z.

It can be shown in a similar way that, for $(x,y,z) \in \Omega_2^*$ the solution $U(x,y,z)$ of this more general problem is

$$U(x,y,z) = \frac{1}{4}\operatorname{erfc}\left[\sqrt{2\omega\sqrt{\left(a^2+c^2\right)\left(x^2+z^2\right)}}\sin\left(\frac{\arctan(z/x)-\arctan(c/a)}{2}\right)\right] \times \operatorname{erfc}\left[\sqrt{2\omega\sqrt{\left(b^2+c^2\right)\left(y^2+z^2\right)}}\sin\left(\frac{\arctan(z/y)-\arctan(c/b)}{2}\right)\right] \times \left[1+\mathcal{O}\left(\omega^{-1/2}\right)\right]. \tag{22}$$

Observe that, for $c > 0$ and $a, b \geqslant 0$, the solution of this problem has singular (internal or boundary) layers along the planes $\{cx = az, cy > bz\}$ and $\{cy = bz, cx > az\}$ of size $\mathcal{O}(\sqrt{\epsilon})$. For $c > 0$ and $a < 0$ and/or $b < 0$, the solution of this problem has boundary layers along the planes $\{x = 0, cy > bz\}$ and/or $\{y = 0, cx > az\}$ of size $\mathcal{O}(\epsilon)$. See Fig. 4.

4. An elliptic problem in the half-space $z > 0$

We consider a singularly perturbed elliptic convection–diffusion problem defined in the half-space $z > 0$: $\Omega_3 = (-\infty,\infty) \times (-\infty,\infty) \times (0,\infty)$, with a "finite source of contamination" located at the plane $z = 0$ (see Fig. 5):

$$\begin{cases} U \in \mathcal{C}(\tilde{\Omega}_3) \cap \mathcal{D}^2(\Omega_3), & U \text{ bounded in bounded subsets of } \tilde{\Omega}_3, \\ -\epsilon\Delta U + U_z = 0 & \text{in } \Omega_3, \\ U(x,y,0) = \chi_{(-1,1)}(x)\chi_{(-1,1)}(y), & \text{for } (x,y) \in (-\infty,\infty) \times (-\infty,\infty). \end{cases} \tag{P$_3$}$$

Observe that the Dirichlet data are discontinuous at the lines $\{x = \pm 1, -1 \leqslant y \leqslant 1\}$ and $\{y = \pm 1, -1 \leqslant x \leqslant 1\}$. The set $\tilde{\Omega}_3$ is precisely the closed set $\bar{\Omega}_3$ with these lines removed: $\tilde{\Omega}_3 \equiv \bar{\Omega}_3 \setminus \{\{(x,\pm 1,0); -1 \leqslant x \leqslant 1\} \cup \{(\pm 1,y,0); -1 \leqslant y \leqslant 1\}\}$.

After the change of the unknown $U = e^{\omega z}F$, problem (P$_3$) is transformed into the Yukawa equation for $F(x,y,z)$:

$$\begin{cases} F \in \mathcal{C}(\tilde{\Omega}_3) \cap \mathcal{D}^2(\Omega_3), & F \text{ bounded in bounded subsets of } \tilde{\Omega}_3, \\ \Delta F - \omega^2 F = 0 & \text{in } \Omega_3, \\ F(x,y,0) = \chi_{(-1,1)}(x)\chi_{(-1,1)}(y), & \text{for } (x,y) \in (-\infty,\infty) \times (-\infty,\infty). \end{cases} \tag{23}$$

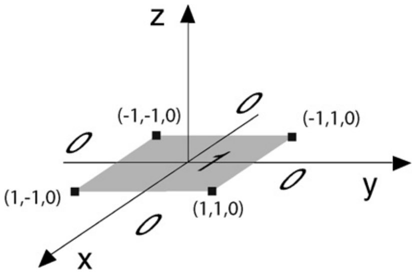


Fig. 5. Domain Ω_3 and boundary condition of problem (P$_3$).

The solution of problem (P_3) can be derived by using Fourier transforms with respect to x and y:

$$U(x,y,z)=\frac{e^{\omega z}}{\pi^2}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}ds\,\frac{\sin(\omega t)}{t}\frac{\sin(\omega s)}{s}e^{i\omega xt+i\omega ys-\omega z\sqrt{1+t^2+s^2}}.$$

It is easy to check by direct substitution that this function is a solution of problem (P_3).

Using the identity $e^{iz}=\cos(z)+i\sin(z)$ and deleting vanishing integrals with odd integrands we have

$$U(x,y,z)=\frac{e^{\omega z}}{\pi^2}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}ds\,\frac{\sin(\omega t)\cos(\omega xt)}{t}\frac{\sin(\omega s)\cos(\omega ys)}{s}e^{-\omega z\sqrt{1+t^2+s^2}}.$$

Using the formula $\sin a\cos b=\frac{1}{2}[\sin(a-b)+\sin(a+b)]$ we have

$$\begin{aligned}U(x,y,z)=\frac{1}{4}\big[&V(x+1,y+1,z)+V(x+1,1-y,z)\\&+V(1-x,y+1,z)+V(1-x,1-y,z)\big],\end{aligned}\tag{24}$$

where $V(x,y,z)$ is the solution of problem (P_2). As in (21) we have

$$V(x,y,z)=\operatorname{erf}\sqrt{\omega\big(\sqrt{z^2+x^2}-z\big)}\operatorname{erf}\sqrt{\omega\big(\sqrt{z^2+y^2}-z\big)}\left[1+\mathcal{O}\left(\frac{1}{\sqrt{\omega}}\right)\right].$$

In this case x and y can assume negative values, and the arguments of the error functions are interpreted as

$$\sqrt{\omega\big[\sqrt{z^2+x^2}-z\big]}=x\sqrt{\frac{\omega}{\sqrt{z^2+x^2}+z}},\qquad \sqrt{\omega\big[\sqrt{z^2+y^2}-z\big]}=y\sqrt{\frac{\omega}{\sqrt{z^2+y^2}+z}}.$$

Then,

$$\begin{aligned}U(x,y,z)=\frac{1}{4}&\big[\operatorname{erf}\big(\zeta(x+1,z)\big)+\operatorname{erf}\big(\zeta(1-x,z)\big)\big]\\&\times\big[\operatorname{erf}\big(\zeta(y+1,z)\big)+\operatorname{erf}\big(\zeta(1-y,z)\big)\big]\big[1+\mathcal{O}\big(\omega^{-1/2}\big)\big]\end{aligned}\tag{25}$$

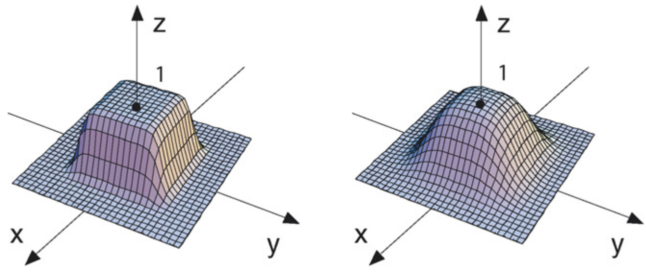
in Ω_3^* with

$$\zeta(u,z)=u\sqrt{\frac{\omega}{\sqrt{u^2+z^2}+z}}$$

and Ω_3^* is the set Ω_3 indented around the discontinuity lines of the boundary condition:

$$\Omega_3^*\equiv\Omega_3\setminus\big\{(x,y,z),(y,x,z);\ -x_0\leqslant x\pm 1\leqslant x_0,\ -1\leqslant y\leqslant 1,\ 0\leqslant z\leqslant z_0\big\},$$
$$x_0,z_0>0.$$

The function $U(x,y,z)$ is represented in Fig. 6.



(a) $z = 1$ (b) $z = 4$

Fig. 6. Graphs of the approximation (25) of the solution of problem (P_3) for two different values of z and $\epsilon = 0.1$.

The solution of problem (23) and therefore of problem (P_3), will be unique when we impose a convenient condition upon $U(x, y, z)$ concerning its growth at infinity. We add a radiation condition to (P_3) and consider the following problem:

$$\begin{cases} U \in \mathcal{C}(\tilde{\Omega}_3) \cap \mathcal{D}^2(\Omega_3), & U \text{ bounded in bounded subsets of } \tilde{\Omega}_3, \\ -\epsilon \Delta U + U_z = 0 & \text{in } \Omega_3, \\ U(x, y, 0) = \chi_{(-1,1)}(x)\chi_{(-1,1)}(y), & \text{for } (x, y) \in (-\infty, \infty) \times (-\infty, \infty), \\ U(x, y, z) = o\left(\dfrac{e^{\omega(r_k+z)}}{\sqrt{\omega r_k}}\right) & \text{as } r_k \to \infty \text{ with } k = 1, 2, 3, \end{cases} \tag{P$_3$}$$

where $r_1 \equiv \sqrt{x^2 + z^2}$, $r_2 \equiv \sqrt{y^2 + z^2}$ and $r_3 \equiv \sqrt{x^2 + y^2}$.

Then, problem (P_3) has at most one solution. The proof is similar to the one of problem (P_2) given in Appendix B.

5. Conclusions

The singularly perturbed three-dimensional convection–diffusion problems (P_1)–(P_3) have been supplied with discontinuous boundary or initial conditions. For every problem we have obtained an integral representation of its unique solution as a starting point for an asymptotic analysis. An asymptotic approximation has been obtained in the singular limit $\epsilon \to 0^+$ valid away from the discontinuities of the boundary or initial conditions. The solution of the parabolic problem (P_1) is given exactly in terms of a product of error functions. The solutions of the elliptic problems (P_2) and (P_3) cannot be given exactly in terms of known functions, but may be approximated by a product of error functions.

These approximations show that the main contribution from the data's discontinuities to the shape of the solution on the singular layers is contained in a product of two error functions. Each one of these error functions is precisely the asymptotic approximation of the solution of a similar two-dimensional problem [9]. This product of error functions reproduces approximately the behaviour of the solution on the interior layers of size $\mathcal{O}(\sqrt{\epsilon})$, on the characteristic layers of size $\mathcal{O}(\sqrt{\epsilon})$ or on the outflow layers of size $\mathcal{O}(\epsilon)$.

We suspect that, as in the problems analyzed here, the error function plays a fundamental role in the approximation of the solution of more general singularly perturbed convection–diffusion problems with discontinuities in the boundary conditions (problems defined over more general domains and by more general coefficients). This will be the subject of further investigations. Then, the asymptotic approximation of the solutions of problems (P_1)–(P_3) presented here may give a qualitative idea about the behaviour of the solutions of more realistic convection–diffusion problems with discontinuous boundary or initial conditions. This should help in the development of suitable numerical methods for those problems [20, p. 6]. For a similar discussion with a parabolic problem see [2,13].

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Appendix A

To prove that $U(x,y,t)$ given in (5) is the unique solution of (P_1) we impose a convenient condition upon $U(x,y,t)$ concerning its growth at infinity. By adding a radiation condition to (P_1) we consider the following problem:

$$\begin{cases} U\in\mathcal{C}(\tilde{\Omega}_1),\ U_x,U_{xx},U_y,U_{yy},U_t\in\mathcal{C}(\Omega_1), & \\ U \text{ bounded in bounded subsets of } \tilde{\Omega}_1, & \\ -\epsilon(U_{xx}+U_{yy})+v_1U_x+v_2U_y+U_t=0 & \text{in } \Omega_1, \\ U(x,y,0)=\chi_{(0,a)}(x)\chi_{(0,b)}(y),\quad U(x,0,t)=U(0,y,t)=0 & \text{in } \tilde{\Omega}_1\setminus\Omega_1, \\ U,U_t,U_x,U_y,U_{xx},U_{yy}=o(1/r) & \text{as } r\to\infty, \end{cases} \tag{P_1}$$

where $r\equiv\sqrt{x^2+y^2}$.

The solution of problem (P_1) is unique. In fact, suppose that U_1 and U_1' are solutions of (P_1). Then, $W\equiv e^{\omega[t/2-v_1x-v_2y]}(U_1-U_1')$ is a solution of

$$\begin{cases} W\in\mathcal{C}(\tilde{\Omega}_1),\ W_x,W_{xx},W_y,W_{yy},W_t\in\mathcal{C}(\Omega_1), & \\ W \text{ bounded in bounded subsets of } \tilde{\Omega}_1, & \\ -\epsilon(W_{xx}+W_{yy})+W_t=0 & \text{in } \Omega_1, \\ W(x,y,0)=W(x,0,t)=W(0,y,t)=0 & \text{in } \tilde{\Omega}_1\setminus\Omega_1, \\ W,W_t,W_x,W_y,W_{xx},W_{yy}=o(1/r) & \text{as } r\to\infty. \end{cases}$$

Define the function

$$H(t)\equiv\frac{1}{2}\int_0^\infty dx\int_0^\infty dy\,W^2(x,y,t),\qquad t\geqslant 0.$$

The function $W(x,y,t)$ is continuous in Ω_1 and bounded in $\tilde{\Omega}_1$ for bounded t. The discontinuity points of $W(x,y,t)$ at $t=0$ are a set of measure 0 in $\mathbb{R}^2$. Therefore, $H(t)$ is continuous for $t\geqslant 0$ with $H(0)=0$. Using the differential equation, integrating by parts and using the boundary and asymptotic behaviour of W we obtain

$$\frac{d}{dt}H(t)=-\epsilon\int_0^\infty dx\int_0^\infty dy\big(W_x^2+W_y^2\big)\leqslant 0,\qquad t>0.$$

On the other hand, from the definition of $H(t)$, $H(t)\geqslant 0$. Therefore, $H(t)=0$ and then $W(x,y,t)=0$ in Ω_1 and $U_1(x,y,t)=U_1'(x,y,t)$ in Ω_1.

Appendix B

We add a radiation condition to (P_2) and consider the following problem:

$$\begin{cases} U \in \mathcal{C}(\tilde{\Omega}_2) \cap \mathcal{D}^2(\Omega_2), & U \text{ bounded in bounded subsets of } \tilde{\Omega}_2, \\ -\epsilon \Delta U + U_z = 0 & \text{in } \Omega_2, \\ U(x,y,0) = 1, \quad U(0,y,z) = U(x,0,z) = 0 & \text{in } \tilde{\Omega}_2 \setminus \Omega_2, \\ U(x,y,z) = o\left(\dfrac{e^{\omega(r_k+z)}}{\sqrt{\omega r_k}}\right) & \text{as } r_k \to \infty \text{ with } k = 1,2,3, \end{cases} \tag{P$_2$}$$

and $r_1 \equiv \sqrt{x^2+z^2}$, $r_2 \equiv \sqrt{y^2+z^2}$ and $r_3 \equiv \sqrt{x^2+y^2}$. Then, problem ($P_2$) has at most one solution.

To show this, suppose that U_2 and U_2' are two solutions of (P_2). Then, the function $G(x,y,z) \equiv (U_2(x,y,z) - U_2'(x,y,z))e^{-\omega z}$ verifies:

$$\begin{cases} G \in \mathcal{C}(\tilde{\Omega}_2) \cap \mathcal{D}^2(\Omega_2), & G \text{ bounded in bounded subsets of } \tilde{\Omega}_2, \\ \Delta G - \omega^2 G = 0 & \text{in } \Omega_2, \\ G(x,y,0) = G(x,0,z) = G(0,y,z) = 0 & \text{in } \tilde{\Omega}_2 \setminus \Omega_2, \\ G(x,y,z) = o\left(\dfrac{e^{\omega r_k}}{\sqrt{\omega r_k}}\right) & \text{as } r_k \to \infty \text{ with } k = 1,2,3. \end{cases} \tag{B.1}$$

Consider the following auxiliary function defined on $\bar{\Omega}_2$:

$$V_a(x,y,z) \equiv \begin{cases} \frac{G(x,y,z)}{H_a(x,y,z)} & \text{if } r_1 \neq 0 \neq r_2, \\ 0 & \text{if } r_1 = 0 \text{ or } r_2 = 0, \end{cases}$$

with

$$H_a(x,y,z) \equiv K_0(\omega r_1) + K_0(\omega r_2) + I_0(\omega r_1) + I_0(\omega r_2) + I_0(\omega r_3) + a, \quad a > 0,$$

K_0 and I_0 being modified Bessel functions of order zero. The function $H_a(x,y,z)$ is positive in $\bar{\Omega}_2$, of the order $\mathcal{O}(e^{\omega r_k}/\sqrt{\omega r_k})$ as $\omega r_k \to \infty$ for $k = 1,2,3$ and $\mathcal{O}(\log(\omega r_k))$ as $\omega r_k \to 0$ for $k = 1,2$ [1, Eqs. 9.7.1 and 9.6.13]. Moreover, $H_a(x,y,z) \in \mathcal{C}(\tilde{\Omega}_2) \cap \mathcal{D}^2(\Omega_2)$ and satisfies the equation: $\Delta H_a - \omega^2 H_a + a\omega^2 = 0$ in Ω_2 [1, Eq. 9.6.1]. Therefore, the auxiliary function V_a is continuous in $\bar{\Omega}_2$ and verifies:

$$\begin{cases} \Delta V_a + \dfrac{2}{H_a} \vec{\nabla} H_a \cdot \vec{\nabla} V_a = \dfrac{a\omega^2}{H_a} V_a & \text{in } \Omega_2, \\ V_a(x,y,0) = V_a(x,0,z) = V_a(0,y,z) = 0 & \forall (x,y,z) \in \bar{\Omega}_2, \\ \lim\limits_{r_k \to \infty} V_a(x,y,z) = 0 & \forall (x,y,z) \in \bar{\Omega}_2, \ k = 1,2,3. \end{cases}$$

Consider the open finite box of side R: $\Omega_R \equiv (0,R) \times (0,R) \times (0,R)$. At points $(x,y,z) \in \Omega_R$ where $\vec{\nabla} V_a = 0$ and $V_a \neq 0$, we have that $V_a \cdot \Delta V_a > 0$. Therefore, V_a has not positive relative maximums neither negative relative minimums in Ω_R. Then $\operatorname{Sup}_{\Omega_R}|V_a| \leqslant \operatorname{Sup}_{\partial\Omega_R}|V_a|$.

Using that $V_a(x,y,0) = V_a(x,0,z) = V_a(0,y,z) = 0 \quad \forall (x,y,z) \in \bar{\Omega}_2$ and that $\lim_{r_k \to \infty} V_a(x,y,z) = 0$ for $k = 1,2,3$ we have that, $\forall \delta > 0$, there is a $R > 0$ such that $|V_a(x,y,z)| \leqslant \delta \ \forall (x,y,z) \in \partial\Omega_R$. Therefore, $|V_a(x,y,z)| \leqslant \delta \ \forall \delta > 0$ and every $(x,y,z) \in \Omega_R$. Taking the limit $\delta \to 0$ ($R \to \infty$) we have that $V_a = 0$ in $\bar{\Omega}_2$. Therefore, $G = 0$ and $U_2 = U_2'$ in Ω_2.

In a similar way we can prove that problem (P_2') has a unique solution (approximated by (22)) if we add the radiation condition:

$$U(x, y, z) = o\left(\frac{e^{\omega(r_k+ax+by+cz)}}{\sqrt{\omega r_k}}\right) \quad \text{as } r_k \to \infty \text{ with } k = 1, 2, 3.$$

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