Approximations of Stieltjes transforms for large values of the transformation parameter

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ABSTRACT

Asymptotic expansions of Stieltjes and generalized Stieltjes transforms of functions having an asymptotic expansion in negative integer powers of their variable have been exhaustively investigated by J. P. McClure and R. Wong. In this paper we obtain asymptotic expansions of more general Stieljes transforms: $\int_0^\infty f(t)/(t^s+z)^\rho dt$ for large z, and $\int_0^\infty f(t)/((t^s+z)^\rho(t^s+w)^\sigma)dt$ for large z and w. Error bounds are obtained at any order of the approximation for a large family of integrands.

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1. Introduction

The generalized Stieltjes transform of a locally integrable function $f(t)$ on $[0,\infty)$ is defined by the integral [[17], chap. 8]

$$S(\rho;z)\equiv\int_0^\infty\frac{f(t)}{(t+z)^\rho}dt,$$

where z is a complex variable in the cut plane $|\arg(z)|<\pi$ and $\rho>0$. If $f(t)\sim\mathcal{O}(t^{-\alpha})$ as $t\to\infty$, then $\alpha+\rho>1$ is required. The standard Stieltjes transform corresponds with $\rho=1$.

When

$$f(t)\sim\sum_{k=0}^\infty a_k t^{-k-\alpha},\qquad t\to\infty,$$

where $0 < \alpha \leq 1$ and $\{a_k,\ k = 0, ..., \infty\}$ is a sequence of complex numbers, asymptotic expansions of $S(1; z)$ and, in general, of $S(\rho; z)$ for large values of z have been derived by R. Wong. An asymptotic expansion of $S(1; z)$ is obtained by using the distributional approach [[16], chap. 6], whereas Mellin transforms techniques are used in [12] to derive an asymptotic expansion of $S(\rho; z)$.

These expansions have been used in [10] and [11] to obtain uniform and nonuniform asymptotic expansions of symmetric standard elliptic integrals for real values of their parameters.

On the other hand, mathematical calculations in Quantum Mechanics and in Quantum Field Theory require the computation or, at least, the approximation of integrals of the form

$$S_s(\rho; z) \equiv \int_0^\infty \frac{f(t)}{(t^s + z)^\rho} dt, \qquad S_s(\rho; z, w) \equiv \int_0^\infty \frac{f(t)}{(t^s + z)^\rho (t^s + w)^\sigma} dt, \tag{1}$$

where s is a positive integer and

$$f(t) = \sum_{k=K}^{n-1} a_k t^{-k-\alpha} + f_n(t), \tag{2}$$

where $0 < \alpha \leq 1$, $K \in \mathbb{Z}$, $a_k, k = K, K + 1, K + 2, ...$ is a sequence of complex numbers and $f_n(t) = \mathcal{O}(t^{-n-\alpha})$ as $t \to \infty$. This kind of integrals appears in one-loop calculations of physical observables and effective actions in Quantum Field Theory, where $f(t)$ is a rational function [[7], chap. 8, sec. 4.2], [[8], chap. 10, sec. 8]. In particular, as has been established recently, the determination of the effective Chern-Simons coupling constant requires the calculation of integrals of the form (1), where z and/or w are large real parameters [2], [3], [4]. In general, the regularization techniques used to define the quantum theories require the introduction of a large parameter (regularizator) and then, the parameters z and/or w in (1) are large [[7], chap. 8, sec. 1], [[8], chap. 7, sec. 5]. On the other hand, the first integral in (1) for $s = 2$ and $\rho = 1/2$ is nothing but the Glasser transform of $f(t)$ [6], [[17], chap. 27].

In section 2, we derive asymptotic expansion of (1) for large z using a generalization of the distributional technique of McClure and Wong [[16], chap. 6].

This paper is a strong revised version of [9]. In that paper, after a change of variable, we replaced t^s in (1) by t and $f(t)$ by $f(t^{1/s})$. Then, we introduced a complicated generalization of McClure and Wongs distributional theory [[16], chap. 6] replacing t by $t^{1/s}$ in (2) in order to apply that theory to the integrals in (1). In this paper we just show that McClure and Wong's distributional theory can be applied directly to the integrals in (1) for any positive integer s (the original theory is formulated only for $s = 1$). Therefore, in this paper, we offer simpler expansions than in [9] by means of much simpler proofs.

In the remaining of the paper empty sums must be understood as zero.

2. Asymptotic expansions

Theorem 1. *Let $f(t)$ be a locally integrable function on $[0,\infty)$ which satisfies (2) with $0<\alpha<1$. Then, for $z\in\mathcal{C}\setminus\mathbb{R}^-$, $\rho>0$, $\alpha+K+s\rho>1$ and $n=1,2,3,...$,*

$$\int_0^\infty \frac{f(t)}{(t^s+z)^\rho}dt=\sum_{k=K}^{n-1}a_k\frac{\Gamma((1-\alpha-k)/s)\Gamma((k+\alpha-1)/s+\rho)}{s\Gamma(\rho)z^{(\alpha+k-1)/s+\rho}}+ \\ \sum_{k=0}^{\lfloor(n-1)/s\rfloor}\binom{-\rho}{k}\frac{M[f;ks+1]}{z^{k+\rho}}+R_n(\rho,s;z), \tag{3}$$

where $M[f;k+1]$ denotes de Mellin transform of $f(t)$ at $w=k+1$: $\int_0^\infty t^{w-1}f(t)dt$ or its analytic continuation at that point. The remainder term is defined by

$$R_n(\rho,s;z)\equiv\frac{(-1)^n}{z^\rho}\int_0^\infty f_{n,n}(t)\frac{d^n}{dt^n}\left[\frac{1}{(t^s/z+1)^\rho}\right]dt, \tag{4}$$

where

$$f_{n,n}(t)\equiv\frac{(-1)^{n-1}}{(n-1)!}\int_t^\infty(u-t)^{n-1}f_n(u)du. \tag{5}$$

Proof. Consider the tempered distributions $\mathbf{f}$, $\mathbf{t}_+^{-\mathbf{k}-\mathbf{s}}$, $\delta^{(\mathbf{k})}$ and $\mathbf{f_n}$ acting over functions $h(t)\in\mathcal{C}^{(\infty)}$ [[16], chap. 6]:

$$<\mathbf{f},h>=\int_0^\infty f(t)h(t)dt, \qquad <\mathbf{f_n},h>=(-1)^n\int_0^\infty f_{n,n}(t)h^{(n)}(t)dt,$$

$$<\mathbf{t}_+^{\mathbf{k}-\alpha},h>=\int_0^\infty t^{k-\alpha}h(t)dt, \qquad <\mathbf{t}_+^{-\mathbf{k}-\alpha},h>=\frac{1}{(\alpha)_k}\int_0^\infty t^{-\alpha}h^{(k)}(t)dt \tag{6}$$

for $k=0,1,2,...$ From [[16], chap 6, lemma 1] we have that these four distributions are related by the equality:

$$\mathbf{f}=\sum_{k=K}^{n-1}a_k\mathbf{t}_+^{-\mathbf{k}-\alpha}+\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}M[f;k+1]\delta^{(\mathbf{k})}+\mathbf{f_n}, \tag{7}$$

where $\delta^{(k)}$ is the k-derivative of the delta distribution at the origin: $<\delta^{(k)},h>=(-1)^kh^{(k)}(0)$. The third integral in (6) is indeed the Mellin transform of $h(t)$, $M[h;w]$ at $w=k+1-\alpha$: $<\mathbf{t}_+^{\mathbf{k}-\alpha},h>=M[h,k+1-\alpha]$. Moreover, by integration by parts it may be proved that the last integral in (6) is the analytic continuation of $M[h;w]$ to the point $w=1-k-\alpha$.

On the other hand $S_s(\rho;z)$, may be written as

$$S_s(\rho;z)=\int_0^\infty\frac{f(t)}{(t^s+z)^\rho}dt=\frac{1}{z^\rho}\int_0^\infty\frac{f(t)}{(t^s/z+1)^\rho}dt.$$

Therefore, applying (7) to the function $h(t)=(t^s/z+1)^{-\rho}$, and using the above formulas and [[14], p. 298, eq. 24] we obtain (3)-(4). □

Theorem 2. *Let $f(t)$ be a locally integrable function on $[0,\infty)$ which satisfies (2) with $\alpha=1$. Then, for $\rho>0$, $z\in\mathbb{C}\setminus\mathbb{R}^-$, $1+K+s\rho>1$ and $n=1,2,3,\ldots$,*

$$\int_0^\infty \frac{f(t)}{(t^s+z)^\rho}dt=\sum_{k=0}^{\lfloor n/s-1\rfloor}\sum_{j=1}^{s-1}a_{ks+j}\frac{\Gamma(-k-j/s)\Gamma(k+j/s+\rho)}{s\Gamma(\rho)z^{k+j/s+\rho}}+$$
$$\sum_{k=0}^{\lfloor (n-1)/s\rfloor}\binom{-\rho}{k}\frac{1}{z^{k+\rho}}\left[c_{ks+1}+\frac{a_{ks}}{s}\left(\psi(k+\rho)+\psi(k+1)+\log z-s\psi(ks+1)-s\gamma\right)\right]$$
$$+\sum_{k=K}^{-1}a_k\frac{\Gamma(-k/s)\Gamma(k/s+\rho)}{s\Gamma(\rho)z^{k/s+\rho}}+R_n(\rho,s;z), \tag{8}$$

where c_k are given by

$$c_{n+1}\equiv\lim_{w\to n}\left[M[f;w+1]+\frac{a_n}{w-n}\right]+a_n(\gamma+\psi(n+1)), \tag{9}$$

γ is the Euler constant and ψ the digamma function. The remainder term $R_n(\rho,s;z)$ is given by (4).

Proof. It is similar to the proof of theorem 1. But now, the second line of (6) is replaced by

$$<\mathbf{t}_+^{\mathbf{k}},h>=\int_0^\infty t^k h(t)dt, \qquad <\mathbf{t}_+^{-\mathbf{k}-\mathbf{1}},h>=-\frac{1}{k!}\int_0^\infty h^{(k+1)}(t)\log t dt \tag{10}$$

for $k=0,1,2,\ldots$. As in the preceding proof, $<\mathbf{t}_+^{\mathbf{k}},h>=M[h;k+1]$. But now, $<\mathbf{t}_+^{-\mathbf{k}-\mathbf{1}},h>$ is related to $M[h;-k]$ by means of a more sophisticated formula:

$$<\mathbf{t}_+^{-\mathbf{k}-\mathbf{1}},h>=\lim_{w\to -k}\left[M[h;w]-\frac{h^k(0)}{k!(w+k)}\right]-\frac{h^k(0)}{k!}\left(\psi(k+1)+\gamma\right),$$

where $M[h;w]$ represents indeed the analytic continuation of $\int_0^\infty t^{w-1}h(t)dt$ and may be derived by integration by parts. Using this last formula and the first of (10) for $h(t)=(t^s/z+1)^{-\rho}$ we obtain:

$$<\mathbf{t}_+^{\mathbf{k}-\mathbf{1}},h>=\frac{\Gamma(k/s)\Gamma(-k/s+\rho)}{z^{-k/s}s\Gamma(\rho)},$$

$$<\mathbf{t}_+^{-\mathbf{ks}-\mathbf{1}},h>=\frac{1}{z^k s}\binom{-\rho}{k}\left[\psi(k+\rho)+\psi(k+1)+\log z-s\psi(ks+1)-s\gamma\right]$$

and

$$<\mathbf{t}_+^{-\mathbf{ks}-\mathbf{j}-\mathbf{1}},h>=\frac{\Gamma(-k-j/s)}{z^{k+j/s}s\Gamma(\rho)}\Gamma(k+j/s+\rho), \quad j=1,2,\ldots,s-1.$$

To complete this proof, formula (7) must also be replaced by [[16], chap 6, lemma 2]:

$$\mathbf{f} = \sum_{k=K}^{n-1} a_k \mathbf{t}_+^{-\mathbf{k}-\mathbf{1}} + \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} c_{k+1} \delta^{(\mathbf{k})} + \mathbf{f_n},$$

where c_k are given in (9). □

Theorem 3. *Let $f(t)$ be as in theorem 1. Then, for $az, bz \in \mathbb{C} \setminus \mathbb{R}^-$, $\rho, \sigma > 0$, $\alpha + K + s(\rho+\sigma) > 1$ and $n = 1, 2, 3, ...$,*

$$\int_0^\infty \frac{f(t)}{(t^s+az)^\rho (t^s+bz)^\sigma} dt = \sum_{k=K}^{n-1} \frac{A_k}{z^{(k+\alpha-1)/s+\rho+\sigma}} + \sum_{k=0}^{\lfloor (n-1)/s \rfloor} \frac{M[f; ks+1] B_k}{z^{k+\rho+\sigma}} + R_n(\rho, \sigma, s; z). \tag{11}$$

Here, the coefficients A_k are defined by

$$A_k \equiv a_k \frac{\Gamma\left((1-\alpha-k)/s\right) \Gamma\left(\rho+\sigma-(1-\alpha-k)/s\right)}{s\Gamma\left(\rho+\sigma\right) a^{\rho+(k+\alpha-1)/s} b^\sigma} F\left(\begin{array}{c} (1-\alpha-k)/s, \sigma \\ \rho+\sigma \end{array} \middle| 1 - \frac{a}{b}\right),$$

where $F\left(\begin{array}{c} \gamma, \beta \\ \delta \end{array} \middle| z\right)$ is the Gauss hypergeometric function, and coefficients B_k are given by:

$$B_k \equiv \frac{(-1)^k (\rho+\sigma)_k}{k! a^{k+\rho} b^\sigma} F\left(\begin{array}{c} -k, \sigma \\ \rho+\sigma \end{array} \middle| 1 - \frac{a}{b}\right). \tag{12}$$

The remainder term is given by

$$R_n(\rho, \sigma, s; z) \equiv \frac{(-1)^n}{z^{\rho+\sigma}} \int_0^\infty f_{n,n}(t) \frac{d^n}{dt^n} \left[\frac{1}{(t^s/z+a)^\rho (t^s/z+b)^\sigma}\right] dt, \tag{13}$$

where $f_{n,n}(t)$ is defined in (5).

Proof. The same proof as in theorem 1, but replacing $h(t) = (t^s/z+1)^{-\rho}$ by $h(t) = (t^s/z+a)^{-\rho}(t^s/z+b)^{-\sigma}$ and using [[14], p. 303, eq. 24]. □

Theorem 4. *Let $f(t)$ be as in theorem 2. Then, for $az, bz \in \mathbb{C} \setminus \mathbb{R}^-$, $\rho, \sigma > 0$, $K + 1 + s(\rho+\sigma) > 1$ and $n = 1, 2, 3, ...$,*

$$\int_0^\infty \frac{f(t)}{(t^s+az)^\rho (t^s+bz)^\sigma} dt = \sum_{k=0}^{\lfloor (n-1)/s \rfloor} \frac{B_k \left(c_{ks+1} - a_{ks}(\gamma + \psi(ks+1) - \log z)\right)}{z^{k+\rho+\sigma}} + \sum_{k=K}^{n-1} \frac{D_k}{z^{k/s+\rho+\sigma}} + R_n(\rho, \sigma, s; z), \tag{14}$$

where c_{ks+1} are defined in (9), B_k are given in (12) and the coefficients D_k are defined by

$$D_{-k} \equiv a_{-k}\frac{\Gamma\left((k/s)\,\Gamma\left(\rho+\sigma-k/s\right)\right.}{s\Gamma\left(\rho+\sigma\right)a^{\rho-k/s}b^{\sigma}}F\left(\begin{matrix}k/s,\sigma\\ \rho+\sigma\end{matrix}\middle|\,1-\frac{a}{b}\right),$$

$$D_{ks+j} \equiv \frac{a_{ks+j}}{sa^{\rho+k+j/s}b^{\sigma}\Gamma(\rho+\sigma)}\Gamma(-k-j/s)\Gamma(k+j/s+\rho+\sigma)F\left(\begin{matrix}-k-j/s,\sigma\\ \rho+\sigma\end{matrix}\middle|\,1-\frac{a}{b}\right),$$

for $j=1,\ldots,s-1$, and

$$D_{ks} \equiv \frac{a_{ks}(\rho+\sigma)_k(-1)^k}{sk!a^{\rho+k}b^{\sigma}}\left[F'\left(\begin{matrix}-k,\sigma\\ \rho+\sigma\end{matrix}\middle|\,1-\frac{a}{b}\right)+F\left(\begin{matrix}-k,\sigma\\ \rho+\sigma\end{matrix}\middle|\,1-\frac{a}{b}\right)\times\right.$$
$$\left.(\psi(k+1)-\psi(\rho+\sigma+k)+\log a)\right].$$

Here, $F'\left(\begin{matrix}\gamma,\beta\\ \delta\end{matrix}\middle|\,z\right)$ is the derivative of the Gauss hypergeometric function with respect to the parameter γ. The remainder term $R_n(\rho,\sigma,s;z)$ is given in (13).

Proof. The proof is the same as in theorem 2, but replacing $h(t)=(t^s/z+1)^{-\rho}$ by $h(t)=(t^s/z+a)^{-\rho}(t^s/z+b)^{-\sigma}$. □

At this moment, expansions (3), (8), (11) and (14) are only formal asymptotic expansions for large z. In the following theorem we show that these expansions are in fact asymptotic expansions for large z.

Theorem 5. *In the region of validity of the expansions (3), (8), (11) and (14), the remainder terms $R_n(\rho,s;z)$ and $R_n(\rho,\sigma,s;z)$ in these expansions satisfy,*

$$|R_n(\rho,s;z)| \le \frac{C_n}{|z|^{(n+\alpha-1)/s+\rho}}, \qquad |R_n(\rho,\sigma,s;z)| \le \frac{C_n}{|z|^{(n+\alpha-1)/s+\rho+\sigma}} \tag{15}$$

if $0<\alpha<1$ and

$$|R_n(\rho,s;z)| \le \frac{C_n\log|z|}{|z|^{n/s+\rho}}, \qquad |R_n(\rho,\sigma,s;z)| \le \frac{C_n\log|z|}{|z|^{n/s+\rho+\sigma}} \tag{16}$$

if $\alpha=1$, where the constants C_n are independent of $|z|$.

Proof. From [[9], theorem 5] we have that, for $0<\alpha<1$, $|f_{n,n}(t)|\le C_{1,n}t^{-\alpha}$ $\forall$ $t\in[0,\infty)$, where $C_{1,n}$ is any positive constant. On the other hand,

$$\frac{d^n}{dt^n}\left[\frac{1}{(t^s/z+1)^{\rho}}\right]=\frac{1}{z^{n/s}}\frac{d^n}{dt^n}\left[\frac{1}{(t^s+1)^{\rho}}\right] \tag{17}$$

and

$$\frac{d^n}{dt^n}\left[\frac{1}{(t^s/z+a)^{\rho}(t^s/z+b)^{\sigma}}\right]=\frac{1}{z^{n/s}}\frac{d^n}{dt^n}\left[\frac{1}{(t^s+a)^{\rho}(t^s+b)^{\sigma}}\right]. \tag{18}$$

Then, from (4) we obtain

$$|R_n(\rho,s;z)| \le \frac{C_{1,n}}{|z|^{n/s+\rho}} \int_0^\infty t^{-\alpha} \left| \frac{d^n}{dt^n} \left[\frac{1}{(t^s+1)^\rho} \right] \right| dt,$$

and, from (13)

$$|R_n(\rho,\sigma,s;z)| \le \frac{C_{1,n}}{|z|^{n/s+\rho+\sigma}} \int_0^\infty t^{-\alpha} \left| \frac{d^n}{dt^n} \left[\frac{1}{(t^s+a)^\rho (t^s+b)^\sigma} \right] \right| dt,$$

which provides (15) with the obvious definition of C_n in each case.

For $\alpha = 1$, from [[9], theorem 5] we have $|f_{n,n}(t)| \le C_{2,n} t^{-\alpha}$ $\forall\, t \in [t_0,\infty)$ and $|f_{n,n}(t)| \le C_{3,n}(|\log t| + 1)$ $\forall\, t \in [0,t_0]$, where t_0, $C_{2,n}$ and $C_{3,n}$ are certain positive constants. Dividing the integration interval $[0,\infty)$ in the definition (4) of $R_n(\rho,s;z)$ at the point t_0 and using (17) and these bounds in each of the intervals $[0,t_0]$ and $[t_0,\infty)$, we obtain the first bound in (16). Using the above mentioned argument and (18) in (13), we obtain the second bound in (16). □

The previous theorem does not offer an accurate bound for the remainder in the expansions. Accurate error bounds are obtained in the following propositions for the expansions in theorems 1 and 2 if the bound $|f_n(t)| \le c_n t^{-n-\alpha}$ holds $\forall\, t \in [0,\infty)$.

Proposition 1. *If, for $0 < \alpha < 1$, the remainder $f_n(t)$ in the expansion (2) satisfies the bound $|f_n| \le c_n t^{-n-\alpha}$ $\forall\, t \in [0,\infty)$ for some positive c_n. Then, the remainder $R_n(\rho,s;z)$ in the expansion (3) satisfies the bound*

$$|R_n(\rho,s;z)| \le \frac{c_n \pi C M(\alpha,r,\rho)}{|\sin(\pi\alpha)|\Gamma(n+\alpha)|z|^{(n+\alpha-1)/s+\rho}}, \tag{19}$$

where $0 < r < \min\{\left|\Im(z^{1/s})\right|, \left|\Im((-z)^{1/s})\right|\}$. The constant C is a bound for $(w+1)^{s\rho}/(w^s+1)^\rho$ in its analyticity region as a function of w, W_1, given in Fig 1(a). The remainder $R_n(\rho,\sigma,s;z)$ in the expansion (11) satisfies the bound

$$|R_n(\rho,\sigma,s;z)| \le \frac{c_n \pi \tilde{C} M(\alpha,\tilde{r},\rho+\sigma)}{|\sin(\pi\alpha)|\Gamma(n+\alpha)|z|^{(n+\alpha-1)/s+\rho+\sigma}}, \tag{20}$$

where $0 < \tilde{r} < \min\{\left|\Im(z^{1/s})\right|, \left|\Im((-az)^{1/s})\right|, \left|\Im((-bz)^{1/s})\right|\}$. The constant $\tilde{C}$ is a bound for $(w+1)^{s(\rho+\sigma)}/((w^s+a)^\rho(w^s+b)^\sigma)$ in its analyticity region as a function of w, W_2, given in Fig 1(b), and

$$M(\alpha,r,\rho) = \frac{(1-r)^{1-\alpha-\rho s}}{r^n} n!(\rho s)_{(\alpha-1)} - \frac{r}{n+1}(\rho s)_{(\alpha+n)} F\left(\begin{matrix} 1, \alpha+n+\rho s \\ 2+n \end{matrix} \middle| r \right). \tag{21}$$

Proof. Introducing the bound $|f_n| \le c_n t^{-n-\alpha}$ in the definition (5) of $f_{n,n}(t)$ we have

$$|f_{n,n}(t)| \le \frac{c_n \Gamma(\alpha)}{\Gamma(n+\alpha) t^\alpha} \qquad \forall\, t \in [0,\infty). \tag{22}$$

On the other hand we write

$$\frac{1}{(t^s/z+1)^\rho}=\frac{1}{(tz^{-1/s}+1)^{s\rho}}\phi(tz^{-1/s}),\quad \phi(w)\equiv\frac{(w+1)^{s\rho}}{(w^s+1)^\rho}.$$

Then

$$\left|\frac{d^n}{dt^n}\left[\frac{1}{(t^s/z+1)^\rho}\right]\right|\le\frac{1}{|z|^{n/s}}\sum_{j=0}^{n}\binom{n}{j}\frac{(s\rho)_j}{(tz^{-1/s}+1)^{s\rho+j}}|\phi^{(n-j)}(tz^{-1/s})|. \tag{23}$$

Using the Cauchy formula for the derivative of an analytic function we obtain

$$|\phi^{(n-j)}(tz^{-1/s})|\le C\frac{(n-j)!}{r^{n-j}}.$$

Formula (19) follows after introducing this bound in (23) and using this last bound in the definition (4) of $R_n(\rho,s;z)$.

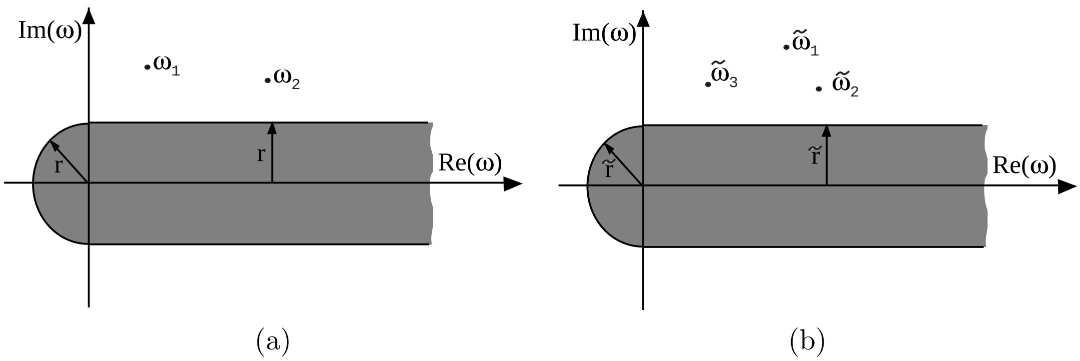

Figure 1. *(a) Region W_1 considered in (19) in proposition 1, where $\omega_1=-z^{1/s}$ and $\omega_2=(-z)^{1/s}$. (b) Region W_2 considered in (20) in proposition 1, where $\tilde\omega_1=-z^{1/s}$, $\tilde\omega_2=(-az)^{1/s}$ and $\tilde\omega_3=(-bz)^{1/s}$.*

To obtain (20), we write

$$\frac{1}{(t^s/z+a)^\rho(t^s/z+b)^\sigma}=\frac{\tilde\phi(tz^{-1/s})}{(tz^{-1/s}+1)^{s(\rho+\sigma)}},\quad \tilde\phi(w)\equiv\frac{(w+1)^{s(\rho+\sigma)}}{(w^s+a)^\rho(w^s+b)^\sigma}.$$

Then

$$\left|\frac{d^n}{dt^n}\left[\frac{1}{(t^s/z+a)^\rho(t^s/z+b)^\sigma}\right]\right|\le\frac{1}{|z|^{n/s}}\sum_{j=0}^{n}\binom{n}{j}\frac{(s(\rho+\sigma))_j}{(tz^{-1/s}+1)^{s(\rho+\sigma)+j}}|\tilde\phi^{(n-j)}(tz^{-1/s})|. \tag{24}$$

Using the Cauchy formula for the derivative of an analytic function we obtain

$$|\tilde{\phi}^{(n-j)}(tz^{-1/s})| \leq \tilde{C}\frac{(n-j)!}{\tilde{r}^{n-j}}.$$

Formula (20) follows after introducing this bound in (24) and using this last bound in the definition (13) of $R_n(\rho,\sigma,s;z)$. $\square$

Proposition 2. *Suppose that, for $\alpha=1$, each remainder $f_n(t)$ in the expansion (2) satisfies the bound $|f_n|\leq c_n t^{-n-1}$ $\forall\ t\in(0,\infty)$ for some positive c_n. Then,*

The remainder $R_n(\rho,s;z)$ in the expansion (8) satisfies the bound

$$|R_n(\rho,s;z)| \leq \frac{\bar{c}_n\pi C M(1/2,r,\rho)}{\Gamma(n+1/2)|z|^{(n-1/2)/s+\rho}}, \tag{25}$$

where $\bar{c}_n\equiv \mathrm{Max}\{c_n, c_{n-1}+|a_{n-1}|\}$, and r, C and $M(1/2,r,\rho)$ are given in proposition 1. It also satisfies the bound

$$\begin{aligned}|R_n(\rho,s;z)| \leq &\frac{C}{|z|^{n/s+\rho}}\left\{\frac{1}{(n-1)!}\left[(c_{n-1}+|a_{n-1}|)\,\varepsilon+c_n\right]M(1,r,\rho)\right.\\ &\left.+\frac{c_n}{n!}\sum_{j=0}^{n}\binom{n}{j}\frac{(n-j)!(s\rho)_j}{r^{n-j}}\Theta_j(z^{-1/s},\varepsilon,\rho)\right\},\end{aligned} \tag{26}$$

where ε is an arbitrary positive number, and

$$\Theta_j(x,\varepsilon,\rho)\equiv\begin{cases}\dfrac{2^{1-j-s\rho}}{s\rho+j-1}-\log(\varepsilon|x|) & \text{if}\quad \varepsilon|x|<1\\ \dfrac{1}{s\rho+j-1}\left(1+\varepsilon|x|\right)^{1-j-s\rho} & \text{if}\quad \varepsilon|x|\geq 1.\end{cases} \tag{27}$$

For enough small x and fixed n, the optimum value for ε is given by

$$\varepsilon=\frac{c_n}{n(c_{n-1}+|a_{n-1}|)}. \tag{28}$$

The remainder $R_n(\rho,\sigma,s;z)$ in the expansion (14) satisfies the bound

$$|R_n(\rho,\sigma,s;z)| \leq \frac{\bar{c}_n\pi\tilde{C}M(1/2,\tilde{r},\rho+\sigma)}{\Gamma(n+1/2)|z|^{(n-1/2)/s+\rho+\sigma}}, \tag{29}$$

where $\tilde{r}$, $\tilde{C}$ are given in proposition 1. It also satisfies

$$\begin{aligned}|R_n(\rho,\sigma,s;z)| \leq &\frac{\tilde{C}}{|z|^{n/s+\rho+\sigma}}\left\{\frac{1}{(n-1)!}\left[(c_{n-1}+|a_{n-1}|)\,\varepsilon+c_n\right]M(1,\tilde{r},\rho+\sigma)\right.\\ &\left.+\frac{c_n}{n!}\sum_{j=0}^{n}\binom{n}{j}\frac{(n-j)!(s(\rho+\sigma))_j}{\tilde{r}^{n-j}}\Theta_j(z^{-1/s},\varepsilon,\rho+\sigma)\right\}\end{aligned} \tag{30}$$

Proof. From [[9], proposition 2] we have

$$|f_{n,n}(t)| \leq \frac{1}{(n-1)!}\left[(c_{n-1}+|a_{n-1}|)\log\left(\frac{\varepsilon}{t}\right)+\frac{c_n}{\varepsilon}\right] \qquad \forall\ t\in[0,\varepsilon], \tag{31}$$

where $\varepsilon > 0$ is a fixed point, and

$$|f_{n,n}(t)| \leq \frac{c_n}{n!}\frac{1}{t} \qquad \forall \quad t\in[0,\infty). \tag{32}$$

We divide the integral in the right hand side of (4) at the point $t=\varepsilon>0$. On the one hand we use the bound (32) for $f_{n,n}(t)$ in the integral over $[\varepsilon,\infty)$ and the bound (31) in the integral over $[0,\varepsilon]$. On the other hand we use again the bound for $d^n(1/(t^s/z+1)^\rho)/dt^n$ given in (23). Then, we obtain (26) after straightforward computations. Similarly, we use $d^n(1/((t^s/z+a)^\rho(t^s/z+b)^\sigma))/dt^n$ given in (24) to obtain (30). For large t and fixed n, this bound takes its optimum value for ϵ given in (28) [[9], proposition 2].

For deriving (25)-(29), we use that $f_n(t)$ satisfies the bound required in proposition 1 with $\alpha=1/2$ and c_n replaced by $\bar{c}_n$ [[9], proposition 2]. □

3. Numerical experiments

3.1. The tadpole in the theory of the scalar field in $3+1$ dimensions

The mass renormalization of the scalar field in $3+1$ dimensions regularized by means of high derivatives [[5], chap. 4, sec. 4] requires the calculation of the integral

$$I_{s,\rho}(m,\Lambda)\equiv\frac{2s}{m^4}\int_0^\infty\frac{p^3dp}{(p^2+m^2)(p^{2s}+\Lambda^{2s})^\rho},$$

where m is the bare mass of the scalar field, Λ is the regulator parameter and the parameters $\rho>0$ and $s\in\mathbb{N}$ verify $s\rho>1$. Physical observables are defined for large values of the regulator parameter and then, an approximation of the integral for large values of Λ is required. By means of a simple change of variable, this integral reads

$$I_{s,\rho}(m,\Lambda)\equiv\frac{s}{m^4}\int_0^\infty\frac{tdt}{(t+m^2)(t^s+\Lambda^{2s})^\rho}.$$

Therefore, up to a factor, it has the form considered in theorem 2 with $z\equiv\Lambda^{2s}$ and

$$f(t)\equiv\frac{t}{t+m^2}=\sum_{k=-1}^{n-1}\frac{(-m^2)^{k+1}}{t^{k+1}}+f_n(t), \quad t\to\infty. \tag{33}$$

Then, the asymptotic expansion of $f(t)$ for large t has the form considered in theorem 2 with $K=-1$ and $a_k=(-m^2)^{k+1}$. Then, applying theorem 2 we have

$$I_{s,\rho}(m,\Lambda) = \frac{s}{m^4}\left\{\sum_{k=0}^{\lfloor n/s-1\rfloor}\sum_{j=1}^{s-1}(-m^2)^{ks+j+1}\frac{\Gamma(-k-j/s)\Gamma(k+j/s+\rho)}{s\Gamma(\rho)\Lambda^{2s(k+j/s+\rho)}}+\right.$$
$$\sum_{k=0}^{\lfloor (n-1)/s\rfloor}\binom{-\rho}{k}\frac{1}{\Lambda^{2s(k+\rho)}}\left[c_{ks+1}+(-m^2)^{ks+1}\left(\frac{\psi(k+\rho)+\psi(k+1)}{s}+\log\Lambda^2-\psi(ks+1)-\gamma\right)\right]$$
$$\left.+\sum_{k=K}^{-1}(-m^2)^{k+1}\frac{\Gamma(-k/s)\Gamma(k/s+\rho)}{s\Gamma(\rho)\Lambda^{2s(k/s+\rho)}}+R_n(\rho,s;\Lambda)\right\}, \tag{34}$$

where, using (9), c_k are given by

$$c_{k+1}=(-m^2)^{k+1}\left(\gamma+\psi(k+1)-\log m^2\right).$$

On the other hand, the function (33) verifies the error test and then $|f_n(t)|\leq m^{2(n+1)}t^{-n-1}$. Therefore, we apply proposition 2 to obtain two different bounds, (25) and (26) where $c_{n-1}=0$ and $c_n=m^{2(n+1)}$. These bounds show that the expansion (34) is convergent for $m<\Lambda$ if $\rho\geq 1$ and for $m\leq\Lambda$ if $\rho<1$.

Λ^2	$I_{2,2}(1,\Lambda)$	2nd order approx.	Relative error	Rel. er. bound	4th order approx.	Relative error	Rel. er. bound
5^4	86177.578	86127.772	0.0006	0.04	86177.7	1.977e-6	0.004
10^4	1488.239	1488.2217	1.2e-5	6.8e-5	1488.239	1.8e-9	3.6e-6
15^4	134.05817	134.058	1.2e-6	6.e-5	134.05817	3.6e-11	6.3e-8
20^4	24.114217	24.1142	2.3e-7	1.e-5	24.114217	2.2e-12	2.e-9
25^4	6.356105	6.356105	6.5e-8	3.e-6	6.356105	4.5e-13	3.9e-10

Table 1: *Second, third and sixth columns represent $10^9 I_{2,2}(1,\Lambda)$, approximation (34) for $n=2$ and approximation (34) for $n=4$ respectively. Fourth and seventh columns represent the absolute value of the respective relative errors in (34). Fifth and last columns represent the respective error bounds given by Min$\{$(25),(26)$\}$.*

3.2. The third symmetric elliptic integral with two parameters large

The third symmetric standard elliptic integral is defined by [[15], chap 12]

$$R_J(x,y,z,p)=\frac{3}{2}\int_0^\infty\frac{dt}{\sqrt{(t+x)(t+y)(t+z)}(t+p)},$$

where the parameters x, y, z and p are nonnegative. The integral $(2/3)R_J(x,az,bz,p)$ with z large (and $|az|\leq|bz|$) has the form considered in theorem 3 with $s=1$, $\rho=\sigma=$

$\alpha = 1/2$, $K = 1$ and

$$f(t) = \frac{1}{\sqrt{t+x}(t+p)} = \sum_{k=1}^{n-1} \frac{a_k}{t^{k+1/2}} + f_n^J(t).$$

Therefore, the asymptotic expansion of $(2/3)R_J(x, az, bz, p)$ for large z follows from eq. (11) in theorem 3. Coefficients a_k are trivially given by,

$$a_k = (-1)^{k-1} \sum_{j=0}^{k} \frac{(1/2)_j}{j!} x^j p^{k-j-1}. \tag{35}$$

The Mellin transform $M[f; k+1]$ in formula (11) can be obtained from [[14], p. 303, eq. 24]. Therefore, applying theorem 3 we obtain

$$\begin{aligned} R_J(x, az, bz, p) =& \frac{3}{2} \left\{ \sum_{k=1}^{n-1} \frac{A_k(a,b)}{z^{k+1/2}} + \sum_{k=0}^{n-1} \frac{B_k(a,b) x^{k+1/2} k! \Gamma(1/2-k)}{p\sqrt{\pi} z^{k+1}} \right. \\ & \left. \times F\left(\begin{matrix} k+1, 1 \\ 3/2 \end{matrix} \middle| 1 - \frac{x}{p} \right) + R_n^J((x, az, bz, p) \right\}, \end{aligned} \tag{36}$$

where the coefficients $A_k(a,b)$ and $B_k(a,b)$ are given by

$$A_k(a,b) = a_k \frac{\Gamma(1/2-k)\Gamma(1/2+k)}{a^k b^{1/2}} F\left(\begin{matrix} 1/2-k, 1/2 \\ 1 \end{matrix} \middle| 1 - \frac{a}{b} \right),$$

$$B_k(a,b) = \frac{2(-1)^k}{k! a^{k+1/2} b^{1/2}} F\left(\begin{matrix} -k, 1/2 \\ 1 \end{matrix} \middle| 1 - \frac{a}{b} \right).$$

On the other hand, the function $f(t)$ satisfies the condition $f_n(t) \le c_n t^{-n-1/2}$ of proposition 1 with $c_n = |a_n|$. Then, for $x, p, az, bz \in \mathbb{C} \setminus \mathbb{R}^-$ and $n = 1, 2, 3, \ldots$, the bound (20) holds for $R_J(x, az, bz, p)$ setting $s = 1$, $\rho = \sigma = \alpha = 1/2$ and $c_n = |a_n|$ given in (35). Therefore,

$$|R_n^J(x, az, bz, p)| \le \frac{|a_n| \pi \tilde{C} M(1/2, \tilde{r}, 1)}{\Gamma(n+1/2)|z|^{n+1/2}}, \tag{37}$$

where $0 < \tilde{r} < \min\{|\Im(z)|, |\Im((-az))|, |\Im((-bz))|\}$, and $\tilde{C}$ and $M(1/2, \tilde{r}, 1)$ are given in proposition 1.

z	$R_J(1, az, bz, 2)$	2nd order approx.	Relative error	Rel. er. bound	3rd order approx.	Relative error	Rel. er. bound
10	0.0896732 - 0.1176762i	0.0680162 - 0.1261349i	0.157	0.9	0.0868656595 - 0.1146834i	0.0277	0.3
50	0.01901547 -0.0327173i	0.0185433 -0.0328573i	0.01	0.06	0.0190047 -0.0327039i	0.0005	0.004
100	0.0094934 -0.01776i	0.0094057 -0.0177839i	0.004	0.02	0.0094925 -0.017759i	8.e-5	6.5e-4
500	0.00186473 -0.003971997i	0.001863 -0.00397239i	4.e-4	1.8e-3	0.0018647 -0.00397199i	1.4e-6	1.e-5
1000	0.00092517 -0.00203988i	0.00092487 -0.0020399i	1.e-4	6.e-4	0.00092517 -0.00203988i	2.3e-7	1.8e-6

Table 3. *Numerical example of the approximation (36). Second, third and sixth columns represent* $R_J(1, az, bz, 2)$ *for* $a = e^{i\pi/4}$ *and* $b = e^{i\pi/2}$*, approximation (36) for* $n = 2$ *and approximation (36) for* $n = 3$ *respectively. Fourth and seventh columns represent the respective relative errors in (36). Fifth and last columns represent the respective error bounds given by eq. (37).*

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