Asymptotic Expansions of Generalized Stieltjes Transforms of Algebraically Decaying Functions

By José L. López and Chelo Ferreira

Asymptotic expansions of Stieltjes and generalized Stieltjes transforms of functions having an asymptotic expansion in negative integer powers of their variable have been exhaustively investigated by R. Wong. In this article, we extend this analysis to Stieltjes and generalized Stieltjes transforms of functions having an asymptotic expansion in negative rational powers of their variable. Distributional approach is used to derive asymptotic expansions of the Stieltjes and generalized Stieltjes transforms of this kind of functions for large values of the parameter(s) of the transformation. Error bounds are obtained at any order of the approximation for a large family of integrands. The asymptotic approximation of an integral involved in the calculation of the mass renormalization of the quantum scalar field and of the third symmetric elliptic integral are given as illustrations.

1. Introduction

The generalized Stieltjes transform of a locally integrable function $f(t)$ on $[0, \infty)$ is defined by the integral ([1], Ch. 8)

$$S(\rho; z) \cong \int_0^\infty \frac{f(t)}{(t+z)^\rho}\, dt,$$

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where z is a complex variable in the cut plane $|\arg(z)| < \pi$ and $\rho > 0$. If $f(t) \sim \mathcal{O}(t^{-\alpha})$ as $t \to \infty$, then $\alpha + \rho > 1$ is required. The standard Stieltjes transform corresponds with $\rho = 1$.

When

$$f(t) \sim \sum_{k=0}^{\infty} a_k t^{-k-\alpha}, \qquad t \to \infty,$$

where $0 < \alpha \leq 1$ and $\{a_k, k = 0, \ldots, \infty\}$ is a sequence of complex numbers, asymptotic expansions of $S(1; z)$ and, in general, of $S(\rho; z)$ for large values of z have been derived by R. Wong. An asymptotic expansion of $S(1; z)$ is obtained by using the distributional approach ([2], Ch. 6); whereas, Mellin transforms techniques are used in [3] to derive an asymptotic expansion of $S(\rho; z)$.

These expansions have been used in [4] and [5] to obtain uniform and nonuniform asymptotic expansions of symmetric standard elliptic integrals for real values of their parameters.

On the other hand, mathematical calculations in quantum mechanics and in quantum field theory require the computation or, at least, the approximation of integrals of the form

$$\int_0^{\infty} \frac{f(t)}{(t^s + z)^{\rho}} dt, \qquad \int_0^{\infty} \frac{f(t)}{(t^s + z)^{\rho}(t^s + w)^{\sigma}} dt, \tag{1}$$

where s is a positive integer and

$$f(t) = \sum_{k=K}^{n-1} a_k t^{-k-\alpha} + f_n(t), \tag{2}$$

where $0 < \alpha \leq 1$, $K \in \mathbb{Z}$ and $f_n(t) = \mathcal{O}(t^{-n-\alpha})$ as $t \to \infty$. This kind of integral appears in one-loop calculations of physical observables and effective actions in quantum field theory, where $f(t)$ is a rational function ([6], Ch. 8, Sec. 4.2), ([7], Ch. 10, Sec. 8). In particular, as has been established recently, the determination of the effective Chern–Simons coupling constant requires the calculation of integrals of the form (1), where z and/or w are large real parameters [8–10]. In general, the regularization techniques used to define the quantum theories require the introduction of a large parameter (regularizator) and then, the parameters z and/or w in (1) are large ([6], Ch. 8, Sec. 1), ([7], Ch. 7, Sec. 5). On the other hand, the first integral in (1) for $s = 2$ and $\rho = 1/2$ is nothing but the Glasser transform of $f(t)$, [11], ([1], Ch. 27).

Asymptotic expansions of integrals (1) cannot be derived directly from Wong's methods when $s \geq 2$. The purpose of this paper is to generalize Wong's distributional method for Stieltjes transforms to the case $s \geq 2$ to obtain a technique valid for the integrals (1) with $f(t)$ verifying (2). By means of a simple change of variable and an obvious change of notation, we rewrite these integrals in the form

$$\int_0^\infty \frac{f(t)}{(t+z)^\rho}dt, \qquad \int_0^\infty \frac{f(t)}{(t+z)^\rho(t+w)^\sigma}dt, \tag{3}$$

where ρ, $\sigma > 0$, $|\text{Arg}(z)| < \pi$ and $|\text{Arg}(w)| < \pi$. In the first integral $\alpha + \rho > 0$ and in the second one $\alpha + \rho + \sigma > 0$. In both integrals $f(t)$ is a locally integrable function on $[0, \infty)$ that satisfies

$$f(t) = \sum_{k=K}^{n-1} a_k t^{-k/s-\alpha} + f_n(t), \tag{4}$$

where $s \in \mathbb{N}, 0 < \alpha \leq 1/s, K \in \mathbb{Z}, \{a_k, k = 0, \ldots, \infty\}$ is a sequence of complex numbers and $f_n(t) = \mathcal{O}(t^{-n/s-\alpha})$ when $t \to \infty$.

In Section 2, we generalize Wong's method for Stieltjes transforms to integrals of the form (3) with $f(t)$ verifying (4) by using the distributional approach ([2], Ch. 6). In Section 3, we show the asymptotic character of the expansions obtained in Section 2 and study error bounds for the remainders. The asymptotic expansion with error bounds of an integral from quantum field theory and of the third standard symmetric elliptic integral are shown as illustrations in Section 4. A brief summary and a few comments are postponed to Section 5.

2. Distributional approach

In the following, we use the notation introduced in [2]. In particular:

DEFINITION 1. *We denote by $\mathcal{S}$ the space of rapidly decreasing functions (infinitely differentiable functions $\varphi(t)$ defined on $[0, \infty)$ that, together with their derivatives, approach zero more rapidly than any power of t^{-1} as $t \to \infty$).*

DEFINITION 2. *We denote by $< \Lambda, \varphi >$ the image of a tempered distribution Λ (a continuous linear functional defined over $\mathcal{S}$) acting over a function $\varphi \in \mathcal{S}$. Recall that we can associate to any locally integrable function $g(t)$ on $[0, \infty)$ with finite algebraic growth at infinity; i.e., $g(t) = \mathcal{O}(t^\mu)$, $\mu \geq 0$, a tempered distribution Λ_g defined by*

$$< \Lambda_g, \varphi > \equiv \int_0^\infty g(t)\varphi(t)dt.$$

Because $f(t)$ in (3) is a locally integrable function on $[0, \infty)$, it defines a distribution

$$< \mathbf{f}, \varphi > \equiv \int_0^\infty f(t)\varphi(t)dt.$$

The distributions associated with $t^{-k-\alpha}$, $k = 0, 1, 2, \ldots, n-1$ are given by ([2], Ch. 5)

$$< \mathbf{t^{-k-\alpha}}, \varphi > \equiv \frac{1}{(\alpha)_k} \int_0^\infty t^{-\alpha}\varphi^{(k)}(t)dt$$

if $0 < \alpha < 1$ and

$$< \mathbf{t^{-k-1}}, \varphi > \equiv -\frac{1}{k!} \int_0^\infty \log(t)\varphi^{(k+1)}(t)dt.$$

To assign a distribution to the function $f_n(t)$ introduced in (4), we first define recursively the kth integral $f_{n,k}(t)$ of $f_n(t)$ by $f_{n,0}(t) \equiv f_n(t)$ and

$$f_{n,k+1}(t) \equiv -\int_t^\infty f_{n,k}(u)du = \frac{(-1)^{k+1}}{k!} \int_t^\infty (u-t)^k f_n(u)du. \tag{5}$$

For $0 < \alpha < 1/s$, it is trivial to show that $f_{n,n/s}(t)$ is bounded on $[0, T]$ for any $T>0$ and is $\mathcal{O}(t^{-\alpha})$ as $t \to \infty$. For $\alpha = 1/s$ we have $f_{n,n/s}(t) = \mathcal{O}(t^{-1/s})$ as $t \to \infty$ and $f_{n,n/s}(t) = \mathcal{O}[\log(t)]$ as $t \to 0^+$. Therefore, for $0 < \alpha \leq 1/s$, we can define the distribution associated to $f_n(t)$ by

$$< \mathbf{f_n}, \varphi > \; \equiv (-1)^{n/s} < \mathbf{f_{n,n/s}}, \varphi^{(n/s)} > \equiv (-1)^{n/s} \int_0^\infty f_{n,n/s}(t)\varphi^{(n/s)}(t)dt.$$

We recall now the definition of the Mellin transform.

DEFINITION 3. *For a locally integrable function $f(t)$ on $(0, \infty)$, we denote by $M[f;z]$ the Mellin transform of $f(t)$ or its analytic continuation. It is defined by*

$$M[f;z] \equiv \int_0^\infty t^{z-1}f(t)dt$$

when the integral converges.

Once we have assigned a distribution to each function involved in the identity (4), we are interested in finding a relation (if any) between these distributions. In fact, this relation is established in the following two lemmas.

LEMMA 1. *For $0 < \alpha < 1/s, s \in \mathbb{N}, n \geq K+1,$ and $n = s, 2s, 3s, \ldots,$ the identity*

$$\mathbf{f} = \sum_{k=K}^{n-1} a_k \mathbf{t^{-k/s-\alpha}} + \sum_{k=0}^{n/s-1} \frac{(-1)^k}{k!} M[f;k+1]\delta^{(\mathbf{k})} + \mathbf{f_n} \tag{6}$$

holds for any rapidly decreasing function $\varphi \in \mathscr{S}$, where δ *is the delta distribution in the origin.*

Proof: Let $f_0(t) \cong f(t) - \sum_{k=K}^{-1} a_k t^{-k/s-\alpha}$ (empty sums must be understood as zero). Then, for $n = 0, s, 2s, \ldots$,

$$f_{n+s}(t) = f_n(t) - \sum_{k=n}^{n+s-1} \frac{a_k}{t^{k/s+\alpha}}$$

and

$$f_{n+s,n/s}(t) = f_{n,n/s}(t) - (-1)^{n/s} \sum_{k=0}^{s-1} \frac{a_{n+k}}{(\alpha + k/s)_{n/s}} \frac{1}{t^{k/s+\alpha}}.$$

Multiplying this expression by $\varphi^{(n/s)}(t)$, integrating by parts and defining

$$H_n \equiv (-1)^{n/s} < \mathbf{f_{n,n/s}}, \varphi^{(n/s)} >, \tag{7}$$

it follows that

$$H_{n+s} = H_n - (-1)^{n/s} f_{n+s,n/s+1}(t)\varphi^{(n/s)}(t)\Big|_0^\infty - \sum_{k=0}^{s-1} a_{n+k} < \mathbf{t}^{-\alpha-(\mathbf{n+k})/\mathbf{s}}, \varphi >.$$

Now, from (4) and using that $f(t)$ is locally integrable in $t = 0$, we obtain $f_{n+s,n/s+1}(t) = \mathscr{O}(t^{-\alpha})$ as $t \to \infty$ and $f_{n+s,n/s+1}(t) = \mathscr{O}(1)$ as $t \to 0$. Therefore,

$$H_{n+s} = H_n + f_{n+s,n/s+1}(0) < \delta^{(\mathbf{n/s})}, \varphi > - \sum_{k=0}^{s-1} a_{n+k} < \mathbf{t}^{-\alpha-(\mathbf{n+k})/\mathbf{s}}, \varphi >. \tag{8}$$

From the definition (5) of $f_{n,j}(t)$ and ([2], Lemma 7, Ch. 3], we have

$$f_{n+s,n/s+1}(0) = -\frac{(-1)^{n/s}}{(n/s)!} M[f; n/s + 1].$$

Finally, applying the identity (8) n/s times, using this last identity and $H_0 \equiv < \mathbf{f}, \varphi > - \sum_{k=K}^{-1} a_k < \mathbf{t}^{-\mathbf{k/s}-\alpha}, \varphi >$, we obtain (6). ■

LEMMA 2. *For* $\alpha = 1/s, s \in \mathbb{N}, n \geq K + 1$ *and* $n = s, 2s, 3s, \ldots$, *the identity*

$$\mathbf{f} = \sum_{k=K}^{n-1} a_k \mathbf{t}^{-(\mathbf{k+1})/\mathbf{s}} + \sum_{k=0}^{n/s-1} d_{(k+1)s} \delta^{(\mathbf{k})} + \mathbf{f_n} \tag{9}$$

holds for any rapidly decreasing function $\varphi \in \mathscr{S}$, *where, for* $n = 0, s, 2s, \ldots$,

$$d_{n+s} = \frac{(-1)^{n/s}}{(n/s)!} \left[\int_0^1 t^{n/s} f_n(t) dt + \int_1^\infty t^{n/s} f_{n+s}(t) dt + \sum_{k=0}^{s-2} \frac{(n/s)! a_{n+k}}{[(k+1)/s - 1]_{n/s+1}} \right.$$

$$+\sum_{k=1}^{n/s}\sum_{j=n}^{n+s-1}\frac{(n/s-k+2)_{k-1}a_j}{[(j+1)/s-k]_k}\Bigg] \tag{10}$$

$$=\frac{(-1)^{n/s}}{(n/s)!}\Bigg\{\lim_{z\to n/s}\left[M[f;z+1]+\frac{a_{n+s-1}}{z-n/s}\right]$$
$$+\sum_{k=0}^{s-2}\left[\frac{(n/s)!}{[(k+1)/s-1]_{n/s+1}}-\frac{1}{(k+1)/s-1}\right]a_{n+k}$$
$$+\sum_{k=1}^{n/s}\sum_{j=n}^{n+s-1}\frac{(n/s-k+2)_{k-1}a_j}{((j+1)/s-k)_k}\Bigg\}. \tag{11}$$

Proof: Let $f_0(t)\equiv f(t)-\sum_{k=K}^{-1}a_kt^{-(k+1)/s}$. Then, for $n=0, s, 2s, \ldots,$

$$f_{n+s}(t)=f_n(t)-\sum_{k=n}^{n+s-1}\frac{a_k}{t^{(k+1)/s}}$$

and

$$f_{n+s,n/s}(t)=f_{n,n/s}(t)-(-1)^{n/s}\sum_{k=0}^{s-1}\frac{a_{n+k}}{[(k+1)/s]_{n/s}}\frac{1}{t^{(k+1)/s}}. \tag{12}$$

From this it follows, by integration, that

$$\int_0^t f_{n,n/s}(u)du=f_{n+s,n/s+1}(t)+\frac{(-1)^{n/s}a_{n+s-1}}{(n/s)!}\log(t)-(-1)^{n/s}$$
$$\times\sum_{k=0}^{s-2}\frac{a_{n+k}t^{1-(k+1)/s}}{[(k+1)/s-1]_{n/s+1}}+d_{n+s}, \tag{13}$$

where we have defined the integration constant

$$d_{n+s}\equiv-\lim_{t\to 0}\left[f_{n+s,n/s+1}(t)+(-1)^{n/s}\frac{a_{n+s-1}}{(n/s)!}\log(t)\right]. \tag{14}$$

Multiplying (12) by $\varphi^{(n/s)}(t)$, integrating by parts and defining again H_n as in (7) it follows that

$$H_{n+s}=H_n-\left[(-1)^{n/s}f_{n+s,n/s+1}(t)+\frac{a_{n+s-1}}{(n/s)!}\log(t)\right]\varphi^{(n/s)}(t)\Bigg|_0^\infty$$
$$-\sum_{k=0}^{s-1}a_{n+k}<\mathbf{t}^{-(\mathbf{n}+\mathbf{k}+\mathbf{1})/\mathbf{s}},\varphi>.$$

Now, from (4) (with $\alpha = 1/s$) and using that $f(t)$ is locally integrable in $t = 0$, we obtain that $f_{n+s,\, n/s+1}(t) + (-1)^{n/s} a_{n+s-1}\log(t)/(n/s)!$ is $\mathcal{O}[\log(t)]$ as $t \to \infty$ and $\mathcal{O}(1)$ as $t \to 0$ (its limit in $t = 0$ is $-\, d_{n+s}$). Therefore,

$$H_{n+s} = H_n - d_{n+s} < \delta^{(\mathbf{n/s})}, \varphi > \; - \sum_{k=0}^{s-1} a_{n+k} < \mathbf{t}^{-(\mathbf{n+k+1})/\mathbf{s}}, \varphi > .$$

If we apply this identity n/s times and use $H_0 \equiv <\mathbf{f}, \varphi> - \sum_{k=-K}^{-1} a_k < \mathbf{t}^{-(\mathbf{k+1})/\mathbf{s}}, \varphi>$, then we obtain (9), but with d_{n+s} given in (14). It remains to show that d_{n+s} may be also expressed by the more tractable expressions (10) or (11). Setting $t = 1$ in (13) and using the recurrent definition (5) of $f_{n,j}(t)$ we obtain

$$d_{n+s} = \int_0^1 f_{n,n/s}(t)dt + \int_1^\infty f_{n+s,n/s}(t)dt + (-1)^{n/s}\sum_{k=0}^{s-2}\frac{a_{n+k}}{[(k+1)/s-1]_{n/s+1}}. \quad (15)$$

On the other hand, for $k = 1, 2, 3, \ldots, n/s$ and $n = s, 2s, 3s, \ldots$,

$$f_{n+s,k}(1) = f_{n,k}(1) - (-1)^k \sum_{j=n}^{n+s-1} \frac{a_j}{((j+1)/s-k)_k}$$

and, by integrating by parts n/s times and taking into account the asymptotic properties of $f_n(t)$ in $t = 0$ and $t = \infty$, we can check that, for $n = s, 2s, 3s, \ldots$,

$$\int_0^1 t^{n/s} f_n(t)dt = -\sum_{k=1}^{n/s}(-1)^k\left(\frac{n}{s}-k+2\right)_{k-1} f_{n,k}(1) + (-1)^{n/s}(n/s)!\int_0^1 f_{n,n/s}(t)dt$$

and

$$\begin{aligned}\int_1^\infty t^{\frac{n}{s}} f_{n+s}(t)dt &= \sum_{k=1}^{n/s}(-1)^k\left(\frac{n}{s}-k+2\right)_{k-1} f_{n+s,k}(1) + (-1)^{\frac{n}{s}}\left(\frac{n}{s}\right)! \\ &\quad \times \int_1^\infty f_{n+s,n/s}(t)dt.\end{aligned}$$

Introducing the three last identities in (15) we obtain (10). Using the definition (4) of $f_n(t)$ and its asymptotic properties in $t = 0$ and $t = \infty$, we obtain

$$\int_0^1 t^{n/s} f_n(t)dt + \int_1^\infty t^{n/s} f_{n+s}(t)dt = \lim_{z\to n/s}\left[M[f; z+1] + \sum_{k=n}^{n+s-1}\frac{a_k}{z+1-(k+1)/s}\right]$$

and (11) follows. ∎

To apply Lemmas 1 and 2 to the first integral in (3), we choose a specific function in $\mathscr{S}$,

$$\varphi_\eta(t) \equiv \frac{e^{-\eta t}}{(t+z)^\rho} \in \mathscr{S},$$

where ρ, $\eta > 0$ and $z \notin \mathbb{R}^-$. We will also need the following lemma.

LEMMA 3. *Let $f(t)$ satisfy (4). Then, for $k = 0, 1, 2, \ldots$ and $n = s, 2s, 3s, \ldots$, the following identities hold,*

$$\lim_{\eta\to 0} < \mathbf{f}, \varphi_\eta >= \int_0^\infty \frac{f(t)}{(t+z)^\rho}dt \qquad \text{for } \alpha+\rho+K/s > 1,$$

$$\lim_{\eta\to 0} < \delta, \varphi_\eta^{(k)} >= \frac{(-1)^k(\rho)_k}{z^{k+\rho}},$$

where $(\rho)_k$ denotes the Pochhammer symbol,

$$\lim_{\eta\to 0} < \mathbf{t}^{-\nu}, \varphi_\eta^{(k)} >= \frac{(-1)^k\Gamma(k+\rho+\nu-1)\Gamma(1-\nu)}{\Gamma(\rho)z^{k+\rho+\nu-1}} \qquad \text{for } 1-\rho<\nu<1,$$

$$\lim_{\eta\to 0} < \mathbf{log(t)}, \varphi_\eta^{(k+1)} >= \frac{(-1)^{k+1}}{z^{k+\rho}}(\rho)_k[\log(z)-\gamma-\psi(k+\rho)],$$

where γ is the Euler constant and ψ the digamma function and

$$\lim_{\eta\to 0} < \mathbf{f_{n,n/s}}, \varphi_\eta^{(n/s)} >= (-1)^{n/s}(\rho)_{n/s}\int_0^\infty \frac{f_{n,n/s}(t)}{(t+z)^{n/s+\rho}}dt \qquad \text{for } 1-\rho<\alpha<1.$$

Proof: The first identity is trivial by using the dominated convergence theorem. The second one follows after a simple computation. On the other hand,

$$< \mathbf{t}^{-\nu}, \varphi_\eta^{(k)} >= (-1)^k\sum_{j=0}^k \binom{k}{j}\eta^j(\rho)_{k-j}\int_0^\infty \frac{e^{-\eta t}}{t^\nu(t+z)^{k+\rho-j}}dt.$$

For $1-\rho<\nu<1$, the integrand of each integral on the right-hand side of the above equation is absolutely dominated by the integrable function $t^{-\nu}|t+z|^{j-k-\rho}$ $\forall\ \eta$, $t \geq 0$ and, hence, is finite. Therefore, using the dominated convergence theorem and after straightforward operations, we obtain the third identity. On the other hand,

$$< \mathbf{log(t)}, \varphi_\eta^{(k+1)} >= (-1)^{k+1}\sum_{j=0}^{k+1}\binom{k+1}{j}\eta^j(\rho)_{k+1-j}\int_0^\infty \frac{e^{-\eta t}\log(t)}{(t+z)^{k+\rho+1-j}}dt.$$

For $j \leq k$ or $j = k+1$ and $\rho > 1$, each integrand in the right-hand side of the above equation is absolutely dominated by the integrable function

$\log(t)|t+z|^{j-k-\rho-1}$ $\forall$ η, $t \geq 0$ and, therefore, finite. For $j = k+1$ and $\rho \leq 1$, we divide the interval $[0, \infty)$ in the above integrals at the point $t = 1$. In the interval $[0, 1]$, the integral is finite for $\eta \geq 0$. In the interval $[1, \infty]$, we perform the change of variable $\eta t = u$ and divide again the resulting u-interval $[\eta, \infty)$ at the point u_0 such that $|\eta z + u_0| = 1$ (assume $\eta|z| \leq 1$ and $\eta \leq |1+z|^{-1}$). In the u-interval $[\eta, u_0]$, we use the bound $|u+\eta z|^\rho \geq |u+\eta z|$, and in the u-interval $[u_0, \infty]$, we use the bound $|u+\eta z|^\rho \geq 1$. After straightforward operations, we observe that the integral on the t-interval $[1,\infty]$ is $\mathcal{O}[\eta^{\rho-1}\log^2(\eta)]$ as $\eta \to 0$. Therefore,

$$\lim_{\eta\to 0} < \mathbf{log(t)}, \varphi_\eta^{(k+1)} >= (-1)^{k+1}(\rho)_{k+1}\int_0^\infty \frac{\log(t)}{(t+z)^{k+\rho+1}}dt.$$

Using now formula ([12], p. 489, Eq. 7) we obtain the fourth identity. The fifth identity follows from the dominated convergence theorem, the local integrability of $f_{n,n/s}(t)$ on $[0, \infty]$ and the behavior $f_{n,n/s}(t) = \mathcal{O}(t^{-\alpha})$ as $t \to \infty$. ■

To apply Lemmas 1 and 2 to the second integral in (3), we must choose another particular function of $\mathcal{S}$,

$$\bar{\varphi}_\eta(t) \equiv \frac{e^{-\eta t}}{(t+az)^\rho(t+bz)^\sigma} \in \mathcal{S},$$

where $az, bz, \notin \mathbb{R}^-$ and ρ, σ, $\eta > 0$. We will also need the following lemma.

LEMMA 4. *Let $f(t)$ satisfy (4). Then, for $k = 0, 1, 2, \ldots$ and $n = s, 2s, 3s, \ldots$, the following identities hold,*

$$\lim_{\eta\to 0} < \mathbf{f}, \bar{\varphi}_\eta >= \int_0^\infty \frac{f(t)}{(t+az)^\rho(t+bz)^\sigma}dt \qquad \text{for } \alpha+\rho+\sigma+K/s > 1.$$

$$\lim_{\eta\to 0} <\delta, \bar{\varphi}_\eta^{(k)} >= \frac{(-1)^k}{z^{k+\rho+\sigma}}\sum_{j=0}^k \binom{k}{j}\frac{(\rho)_j(\sigma)_{k-j}}{a^{\rho+j}b^{\sigma+k-j}};$$

$$\lim_{\eta\to 0} < \mathbf{t}^{-\nu}, \bar{\varphi}_\eta^{(k)} >= \frac{(-1)^k\Gamma(1-\nu)\Gamma(k+\rho+\sigma+\nu-1)}{\Gamma(k+\rho+\sigma)z^{k+\rho+\sigma+\nu-1}}$$

$$\times\sum_{j=0}^k \binom{k}{j}\frac{(\rho)_j(\sigma)_{k-j}}{a^{\rho+j+\nu-1}b^{\sigma+k-j}}F\left(\begin{matrix}1-\nu, k+\sigma-j\\ k+\rho+\sigma\end{matrix}\middle| 1-\frac{a}{b}\right)$$

$$\text{for} \quad 1-\rho-\sigma<\nu<1,$$

where

$$F\left(\begin{matrix}\gamma, \beta\\ \delta\end{matrix}\middle| z\right)$$

is the Gauss hypergeometric function,

$$\lim_{\eta\to 0} < \mathbf{log(t)}, \bar{\varphi}_\eta^{(k+1)} >= \frac{(-1)^{k+1}}{(k+\rho+\sigma)z^{k+\rho+\sigma}} \sum_{j=0}^{k+1} \binom{k+1}{j} \frac{(\rho)_j(\sigma)_{k+1-j}}{a^{\rho+j-1}b^{\sigma+k+1-j}}$$

$$\times\left[(\log(az) - \gamma - \psi(k+\rho+\sigma))F\left(\begin{matrix}1, k+1+\sigma-j\\ k+1+\rho+\sigma\end{matrix}\middle| 1-\frac{a}{b}\right)\right.$$

$$\left.+F'\left(\begin{matrix}1, k+1+\sigma-j\\ k+1+\rho+\sigma\end{matrix}\middle| 1-\frac{a}{b}\right)\right],$$

where

$$F'\left(\begin{matrix}\gamma, \beta\\ \delta\end{matrix}\middle| z\right)$$

is the derivative of the Gauss hypergeometric function with respect to the parameter γ *and*

$$\lim_{\eta\to 0} < \mathbf{f_{n,n/s}}, \bar{\varphi}_\eta^{(n/s)} >= (-1)^{n/s} \sum_{j=0}^{n/s} \binom{n/s}{j} \int_0^\infty \frac{(\rho)_j(\sigma)_{n/s-j} f_{n,n/s}(t)}{(t+az)^{j+\rho}(t+bz)^{n/s-j+\sigma}} dt$$

for $1-\rho-\sigma<\alpha<1$.

Proof: The proof of the first, second, and last equalities is similar to the proof of the corresponding equalities in Lemma 3. The proof of the third equality is also similar, but considering the integrable function $t^{-\nu}|t+az|^{-i-\rho}|t+bz|^{i-j-\sigma}$ with $i \le j = 0, 1, 2, \ldots, k$ instead of $t^{-\nu}|t+z|^{j-k-\rho}$ and using formula ([12], p. 303, Eq. 24). The proof of the fourth equality is similar to the proof of the fourth equality in Lemma 3 using the bound $|t+az|^{-\rho}|t+bz|^{-\sigma} \le |t+az|^{-\rho-\sigma} + |t+bz|^{-\rho-\sigma}$ and using the derivative with respect to α of formula ([12], p. 303, Eq. 24) instead of ([12], p. 489, Eq. 7). ∎

With these preparations, we are able now to obtain asymptotic expansions of the integrals (3) for large z. This is achieved in the following theorems.

THEOREM 1. *Let $f(t)$ be a locally integrable function on $[0,\infty]$ which satisfies (4) with $0<\alpha<1/s$. Then, for $\rho>0$, $z\in\mathbb{C}\setminus\mathbb{R}^-$, $\alpha+\rho+K/s>1$, and $n=s, 2s, 3s, \ldots$*

$$\int_0^\infty \frac{f(t)}{(t+z)^\rho} dt = \sum_{k=K}^{-1} a_k \frac{\Gamma(\rho+\alpha+k/s-1)\Gamma(1-\alpha-k/s)}{\Gamma(\rho)z^{\rho+\alpha+k/s-1}} + \sum_{k=0}^{n/s-1} \frac{(-1)^k}{z^{k+\rho}}$$

$$\times\left[\sum_{j=0}^{s-1}\frac{\pi\Gamma(k+j/s+\rho+\alpha-1)a_{sk+j}z^{1-\alpha-j/s}}{\Gamma(k+j/s+\alpha)\Gamma(\rho)\sin(\pi(j/s+\alpha))}+\frac{(\rho)_k M[f;k+1]}{k!}\right]$$
$$+R_{n,s}(\rho;z),\tag{16}$$

where the remainder term satisfies

$$R_{n,s}(\rho;z)\equiv(\rho)_{n/s}\int_0^\infty\frac{f_{n,n/s}(t)dt}{(t+z)^{n/s+\rho}},\tag{17}$$

empty sums must be understood as zero and $f_{n,n/s}(t)$ *is defined in* (5).

Proof: It follows from Lemmas 1 and 3 when using the reflection formula of the gamma function and formula

$$< t^{-k/s-\alpha},\varphi_\eta >=\frac{1}{(\nu)_{[k/s+\alpha]}}< t^{-\nu},\varphi_\eta^{([k/s+\alpha])} > \qquad \text{if } k/s+\alpha\notin\mathbb{N}\tag{18}$$

with $\nu = k/s + \alpha - [k/s + \alpha]$.

■

THEOREM 2. *Let* $f(t)$ *be a locally integrable function on* $[0,\infty]$ *which satisfies* (4) *with* $\alpha = 1/s$. *Then, for* $\rho > 0$, $z\in\mathbb{C}\setminus\mathbb{R}^-$, $\rho + (1 + K)/s > 1$ *and* $n = s$, $2s$, $3s$, $\ldots$,

$$\int_0^\infty\frac{f(t)}{(t+z)^\rho}dt=\sum_{k=K}^{-1}a_k\frac{\Gamma(\rho+(1+k)/s-1)\Gamma(1-(k+1)/s)}{\Gamma(\rho)z^{\rho+(1+k)/s-1}}$$
$$+\sum_{k=0}^{n/s-1}\left\{a_{s(k+1)-1}\frac{(-1)^k(\rho)_k}{k!z^{k+\rho}}[\log(z)-\gamma-\psi(k+\rho)]\right.$$
$$+\frac{(-1)^k}{\Gamma(\rho)}\sum_{j=0}^{s-2}a_{sk+j}\frac{\Gamma(\rho+k+(j+1)/s-1)\Gamma(1-(j+1)/s)}{((j+1)/s)_k z^{k+\rho+(j+1)/s-1}}$$
$$\left.+d_{s(k+1)}\frac{(\rho)_k}{z^{k+\rho}}\right\}+R_{n,s}(\rho;z),\tag{19}$$

where, for $k = 0, 1, 2, \ldots$, *the coefficients* $d_{s(k+1)}$ *are given by* (10), (11) *or*

$$d_{n+s}=\frac{(-1)^{n/s}}{(n/s)!}\left\{\lim_{T\to\infty}\left[\int_0^T t^{n/s}f(t)dt+\sum_{k=K}^{n+s-2}\frac{a_kT^{(n-k-1)/s+1}}{(k-n+1)/s-1}\right.\right.$$
$$\left.-a_{n+s-1}\log(T)\right]+\sum_{k=0}^{s-2}\left(\frac{(n/s)!a_{n+k}}{[(k+1)/s-1]_{n/s+1}}+\frac{a_{n+k}}{1-(k+1)/s}\right)$$
$$\left.+\sum_{k=1}^{n/s}\sum_{j=n}^{n+s-1}\frac{(n/s-k+2)_{k-1}a_j}{((j+1)/s-k)_k}\right\},\tag{20}$$

empty sums being understood as zero. The remainder term $R_{n,s}(\rho; z)$ is given by (17).

Proof: From Lemmas 2 and 3 and formula

$$< t^{-(k+1)/s}, \varphi_\eta >= \frac{-1}{[(k+1)/s-1)!} < \log t, \varphi_\eta^{[(k+1)/s]} >$$

$$\text{if} \qquad (k+1)/s \in \mathbb{N}.$$

or formula (18) with $\alpha = 1/s$ if $(k+1)/s \notin \mathbb{N}$, we immediately obtain formulas (17) and (19), but with the coefficient $d_{s(k+1)}$ given in formulas (10) or (11). Introducing

$$f_n(t) = f(t) - \sum_{k=K}^{n-1} \frac{a_k}{t^{(k+1)/s}}$$

in the integrands on the right-hand side of (10) and after simple manipulations we obtain (20). ■

THEOREM 3. *Let $f(t)$ be as in Theorem 1. Then, for $az, bz \in \mathbb{C} \setminus \mathbb{R}^-$, $\rho, \sigma > 0$, $\alpha + \rho + \sigma + K/s > 1$ and $n = s, 2s, 3s, \ldots$,*

$$\int_0^\infty \frac{f(t)}{(t+az)^\rho (t+bz)^\sigma} dt = \sum_{k=0}^{n/s-1} \frac{(-1)^k}{z^{k+\rho+\sigma}} \left[\sum_{j=0}^{s-1} \frac{B_{k,j}}{z^{\alpha+j/s-1}} + C_k \right] + \sum_{k=K}^{-1} \frac{A_k}{z^{\rho+\sigma+\alpha+k/s-1}} + R_{n,s}(\rho, \sigma; z), \tag{21}$$

where the coefficients A_k, $B_{k,j}$ and C_k are defined by

$$A_k \equiv a_k \frac{\Gamma(1-\alpha-k/s)\Gamma(\rho+\sigma+\alpha+k/s-1)}{\Gamma(\rho+\sigma)a^{\rho+\alpha+k/s-1}b^\sigma} F\left(\begin{matrix} 1-\alpha-k/s, \sigma \\ \rho+\sigma \end{matrix} \middle| 1-\frac{a}{b} \right),$$

$$B_{k,j} \equiv \frac{\pi a_{sk+j}}{\sin((\alpha+j/s)\pi)} \frac{\Gamma(k+\rho+\sigma+\alpha+j/s-1)}{\Gamma(\alpha+k+j/s)\Gamma(k+\rho+\sigma)} \times \sum_{l=0}^{k} \binom{k}{l} \frac{(\rho)_l(\sigma)_{k-l}}{a^{\rho+l+\alpha+j/s-1}b^{\sigma+k-l}} F\left(\begin{matrix} 1-\alpha-j/s, k+\sigma-l \\ k+\rho+\sigma \end{matrix} \middle| 1-\frac{a}{b} \right)$$

and

$$C_k \equiv \sum_{k=0}^{n/s-1} \frac{M[f; k+1]}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(\rho)_j(\sigma)_{k-j}}{a^{\rho+j}b^{k+\sigma-j}},$$

empty sums must be understood as zero and the remainder term satisfies

$$R_{n,s}(\rho,\sigma;z) \equiv \sum_{j=0}^{n/s} \binom{n/s}{j} (\rho)_j(\sigma)_{n/s-j} \int_0^\infty \frac{f_{n,n/s}(t)dt}{(t+az)^{j+\rho}(t+bz)^{n/s+\sigma-j}}, \tag{22}$$

where $f_{n,n/s}(t)$ *is defined in* (5).

Proof: The proof is similar to the proof of Theorem 1, but using Lemma 4 instead of Lemma 3. ■

THEOREM 4. *Let* $f(t)$ *be as in Theorem 2. Then, for* $az, bz \in \mathbb{C}\setminus\mathbb{R}^-$, $\rho, \sigma > 0$, $\rho + \sigma + (1+K)/s > 1$ *and* $n = s, 2s, 3s, \ldots,$

$$\int_0^\infty \frac{f(t)}{(t+az)^\rho(t+bz)^\sigma}dt = \sum_{k=K}^{-1} \frac{A_k}{z^{\rho+\sigma+(1+k)/s-1}} + \sum_{k=0}^{n/s-1} \frac{(-1)^k}{z^{k+\rho+\sigma}} \times \left[B_k[\log(az) - \gamma - \psi(k+\rho+\sigma)] + B'_k + \sum_{j=0}^{s-2} \frac{C_{k,j}}{z^{(j+1)/s-1}} + D_k\right] + R_{n,s}(\rho,\sigma;z), \tag{23}$$

where empty sums must be understood as zero,

$$A_k \equiv a_k \frac{\Gamma(1-(k+1)/s)\Gamma(\rho+\sigma+(1+k)/s-1)}{\Gamma(\rho+\sigma)a^{\rho+(1+k)/s-1}b^\sigma} F\left(\begin{matrix}1-(k+1)/s, \sigma\\ \rho+\sigma\end{matrix}\middle| 1-\frac{a}{b}\right),$$

$$B_k \equiv \frac{a_{s(k+1)-1}}{k!(k+\rho+\sigma)} \sum_{l=0}^{k+1} \binom{k+1}{l} \frac{(\rho)_l(\sigma)_{k+1-l}}{a^{\rho+l-1}b^{\sigma+k+1-l}} \times F\left(\begin{matrix}1, k+1+\sigma-l\\ k+1+\rho+\sigma\end{matrix}\middle| 1-\frac{a}{b}\right),$$

$$B'_k \equiv \frac{a_{s(k+1)-1}}{k!(k+\rho+\sigma)} \sum_{l=0}^{k+1} \binom{k+1}{l} \frac{(\rho)_l(\sigma)_{k+1-l}}{a^{\rho+l-1}b^{\sigma+k+1-l}} \times F'\left(\begin{matrix}1, k+1+\sigma-l\\ k+1+\rho+\sigma\end{matrix}\middle| 1-\frac{a}{b}\right),$$

$$C_{k,j} \equiv a_{sk+j} \frac{\Gamma(1-(j+1)/s)\Gamma(k+\rho+\sigma+(j+1)/s-1)}{((j+1)/s)_k\Gamma(k+\rho+\sigma)} \times \sum_{l=0}^{k} \binom{k}{l} \frac{(\rho)_l(\sigma)_{k-l}}{a^{\rho+l+(j+1)/s-1}b^{\sigma+k-l}} F\left(\begin{matrix}1-(j+1)/s, k+\sigma-l\\ k+\rho+\sigma\end{matrix}\middle| 1-\frac{a}{b}\right)$$

and

$$D_k \equiv (-1)^k d_{s(k+1)} \sum_{j=0}^{k} \binom{k}{j} \frac{(\rho)_j(\sigma)_{k-j}}{a^{\rho+j} b^{k+\sigma-j}},$$

where $d_{s(k+1)}$ *is given in* (10), (11), *or* (20). *The remainder term* $R_{n,s}(\rho, \sigma; z)$ *is given in* (22).

Proof: The proof is similar to the proof of Theorem 2, but using Lemma 4 instead of Lemma 3. ∎

3. Error bounds

In the following theorem, we show that the expansions (16), (19), (21), and (23) given in Theorems 1–4, respectively, are, in fact, asymptotic expansions for large z.

THEOREM 5. *In the region of validity of the expansions* (16), (19), (21), *and* (23), *the remainder terms* $R_{n,s}(\rho; z)$ *and* $R_{n,s}(\rho,\sigma; z)$ *in these expansions satisfy,*

$$\left|R_{n,s}(\rho; z)\right| \le \frac{C_n}{|z|^{n/s+\alpha+\rho-1}}, \qquad \left|R_{n,s}(\rho, \sigma; z)\right| \le \frac{C_n}{|z|^{n/s+\alpha+\rho+\sigma-1}} \tag{24}$$

if $s > 1$ *or* $0 < \alpha < 1$ *and*

$$|R_{n,s}(\rho; z)| \le \frac{C_n \log|z|}{|z|^{n+\rho}}, \qquad |R_{n,s}(\rho, \sigma; z)| \le \frac{C_n \log|z|}{|z|^{n+\rho+\sigma}} \tag{25}$$

if $s = \alpha = 1$, *where the constants* C_n *are independent of* $|z|$ (*it may depend on the remaining parameters of the problem*).

Proof: On the one hand, $f_n(t) = \mathcal{O}(t^{-n/s-\alpha})$ for $t \to \infty$ (with $0 < \alpha \le 1/s$) then, there is a certain $t_0 \in (0, \infty)$ and a constant $C_{1,n}$ such that $|f_n(t)| \le C_{1,n}\, t^{-n/s-\alpha}$ $\forall\, t \in [t_0, \infty]$. Then, introducing this bound in the definition (5) of $f_{n,n/s}(t)$ we obtain the bound $|f_{n,n/s}(t)| \le C_{2,n} t^{-\alpha}$ $\forall\, t \in [t_0, \infty]$, where $C_{2,n}$ is a certain positive constant and $0 < \alpha \le 1/s$. On the other hand, $f_{n,n/s}(t)$ is bounded on any compact interval in $[0, \infty]$ for $0 < \alpha < 1$ and $f_{n,n/s}(t)$ is bounded on any compact interval in $(0, \infty)$ and $\mathcal{O}(\log t)$ as $t \to 0^+$ for $\alpha = s = 1$. Then, $\forall\, t \in [0, t_0]$, $|f_{n,n/s}(t)| \le C_{3,n} t^{-\alpha}$ for $0 < \alpha < 1$ and $|f_{n,n}(t)| \le C_{3,n}$ $(|\log t| + 1)$ for $\alpha = s = 1$, where $C_{3,n}$ is a certain positive constant.

If we divide the integration interval $[0, \infty]$ in the definition (17) of $R_{n,s}(\rho; z)$ at the point t_0 and introduce these bounds in each of the intervals $[0, t_0]$ and $[t_0, \infty]$, we obtain the first bounds in (24) and (25).

Using the inequality $|t+az|^{-\rho}|t+bz|^{-\sigma} \le |t+az|^{-\rho-\sigma} + |t+bz|^{-\rho-\sigma}$ in (22) and the above mentioned argument, we obtain the second bounds in (24) and (25).■

The bounds (24) and (25) are not useful for numerical computations unless we can calculate the constants C_n in terms of the dates of the problem [ρ, σ, a, b, Arg(z) and $f(t)$]. The following two propositions show that, if the bound $|f_n(t)| \le C_{1,n} t^{-n/s-\alpha}$ holds $\forall\, t \in [0, \infty]$ and not only for $t \in [t_0, \infty]$ then, the constants C_n may be calculated in terms of $C_{1,n}$.

PROPOSITION 1. *If, for $s > 1$ or $0 < \alpha < 1$, the remainder $f_n(t)$ in the expansion* (4) *of the function $f(t)$ satisfies the bound $|f_n(t)| \le c_n t^{-n/s-\alpha}$ $\forall\, t \in [0,\infty]$ for some positive constant c_n then, the remainders $R_{n,s}(\rho; z)$ and $R_{n,s}(\rho,\sigma; z)$ in the expansions* (16), (19), (21), *and* (23) *satisfy*

$$|R_{n,s}(\rho;z)| \le \frac{c_n \pi \Gamma(\frac{n}{s}+\rho+\alpha-1) F\left[\begin{matrix} 1-\alpha, \frac{n}{s}+\alpha+\rho-1 \\ (\frac{n}{s}+\rho+1)/2 \end{matrix}\middle| \sin^2\left(\frac{\mathrm{Arg}(z)}{2}\right)\right]}{\Gamma(\frac{n}{s}+\alpha)\Gamma(\rho)|\sin(\pi\alpha)||z|^{\frac{n}{s}+\rho+\alpha-1}} \tag{26}$$

and

$$|R_{n,s}(\rho,\sigma;z)| \le \frac{c_n \pi \Gamma(\frac{n}{s}+\rho+\sigma+\alpha-1) F\left[\begin{matrix} 1-\alpha, \frac{n}{s}+\alpha+\rho+\sigma-1 \\ (\frac{n}{s}+\rho+\sigma+1)/2 \end{matrix}\middle| \frac{1}{2}\left(1-\frac{r}{|cz|}\right)\right]}{\Gamma(\frac{n}{s}+\alpha)\Gamma(\rho+\sigma)|\sin(\pi\alpha)||cz|^{\frac{n}{s}+\rho+\sigma+\alpha-1}}, \tag{27}$$

where $c \equiv Min\{|a|,|b|\}$ and $r \equiv Min\{Re(az), Re(bz)\}$.

Proof: Introducing the bound $|f_n(t)| \le c_n t^{-n/s-\alpha}$ in the definition (5) of $f_{n,n/s}(t)$ we obtain

$$|f_{n,n/s}(t)| \le \frac{c_n \Gamma(\alpha)}{\Gamma(n/s+\alpha)t^\alpha} \qquad \forall\, t \in [0,\infty].$$

Introducing this bound in the definition (17) of $R_{n,s}(\rho;z)$ and using the duplication formula of the gamma function and ([12], p. 309, Eq. 7) we obtain the first bound. The second bound is obtained by using the inequalities $|t+az|^2$ $|t+bz|^2 \ge t^2 + 2rt + |cz|^2$ in the definition (22) of $R_{n,s}(\rho,\sigma; z)$, formula ([12], p. 309, Eq. 7) and the equality

$$\sum_{k=0}^{n/s} \binom{n/s}{k} (\rho)_k (\sigma)_{n/s-k} = (\rho+\sigma)_{n/s}. \tag{28}$$

■

PROPOSITION 2. *If, for* $s = \alpha = 1$, *each remainder* $f_n(t)$ *in the expansion* (4) *of the function* $f(t)$ satisfies the bound $|f_n(t)| \leq c_n t^{-n-1}$ $\forall\ t \in [0, \infty]$ *for some positive constant* c_n *then, the remainder* $R_{n,1}(\rho;z)$ *in the expansion* (19) *satisfies*

$$|R_{n,1}(\rho;z)| \leq \frac{\bar{c}_n \pi \Gamma(n+\rho-1/2) F\left[\begin{matrix} 1/2, n+\rho-1/2 \\ (n+\rho+1)/2 \end{matrix}\middle| \sin^2\left(\frac{\mathrm{Arg}(z)}{2}\right)\right]}{\Gamma(n+1/2)\Gamma(\rho)|z|^{n+\rho-1/2}} \equiv \mathbf{R}_n^{(1)}(\rho;z), \tag{29}$$

where $\bar{c}_n \equiv Max\{c_n, c_{n-1} + |a_{n-1}|\}$ *and*

$$|R_{n,1}(\rho;z)| \leq \frac{(\rho)_n}{|z|^{n+\rho}} \left\{ \frac{\epsilon(c_{n-1}+|a_{n-1}|)+c_n}{(n-1)!\Theta(z)^{n+\rho}} + \frac{c_n}{n!}\left|1+\frac{\epsilon}{z}\right|^{-n-\rho} \left[\log|z| \right.\right.$$
$$+ \frac{(n+\rho)\{[2\epsilon + \mathrm{Re}(z) + |\mathrm{Re}(z)|](|z|^{-1}-1) + (|\mathrm{Re}(z)| - \mathrm{Re}(z)]\log|z|\}}{2(n+\rho+1)|z+\epsilon|} F_1$$
$$\left.\left. + \frac{4\epsilon + \mathrm{Re}(z) + |\mathrm{Re}(z)| - 2\epsilon|z|}{2\epsilon(n+\rho+1)|z|} F_0 + \frac{2|\epsilon+z|F_{-1}}{\epsilon[(n+\rho)^2-1]|z|} \right] \right\} \equiv \mathbf{R}_n^2(\rho;z), \tag{30}$$

where ϵ *is an arbitrary positive number,*

$$F_a \equiv F\left[\begin{matrix} 2-a, n+\rho+a \\ (n+\rho+3)/2 \end{matrix}\middle| \sin^2\left(\frac{\mathrm{Arg}(z+\epsilon)}{2}\right)\right]$$

and

$$\Theta(z) \equiv \begin{cases} 1 & \text{if} \quad \mathrm{Re}(z) \geq 0 \\ |\sin(\mathrm{Arg}(z)| & \text{if} \quad \epsilon \geq -\mathrm{Re}(z) > 0 \\ |1+\epsilon/z| & \text{if} \quad -\mathrm{Re}(z) > \epsilon > 0. \end{cases}$$

For large z and fixed n, the optimum value for ϵ *is approximately given by*

$$\epsilon^2 = \frac{c_n}{n(c_{n-1}+|a_{n-1}|} \left\{ \frac{2F_{-1}}{(n+\rho)^2-1} + \frac{[\mathrm{Re}(z)+|\mathrm{Re}(z)|]F_0}{2(n+\rho+1)|z|} \right\}. \tag{31}$$

The remainder $R_{n,1}(\rho,\sigma;z)$ *in expansion* (23) *satisfies*

$$|R_{n,1}(\rho,\sigma;z)| \leq \mathbf{R}_n^{(i)}(\rho+\sigma;az) + \mathbf{R}_n^{(i)}(\rho+\sigma;bz) \tag{32}$$

for $i = 1, 2$. If a, b, and z are positive real numbers, then

$$|R_{n,1}(\rho;z)| \leq \{n\epsilon(c_{n-1} + |a_{n-1}|) + c_n[S_n(z,\epsilon,\rho) + T_n(z,\epsilon,\rho)]\} \frac{(\rho)_n}{n! z^{n+\rho}}, \quad (33)$$

where ϵ is again an arbitrary positive number,

$$S_n(z,\epsilon,\rho) = \text{Min}\left\{\frac{nz[(\epsilon+z)^{n+\rho-1} - z^{n+\rho-1}]}{\epsilon(n+\rho-1)(\epsilon+z)^{n+\rho-1}}, \psi(n+1)+\gamma\right\}$$

and

$$T_n(z,\epsilon,\rho) = \frac{z^{n+\rho}}{(n+\rho)(\epsilon+z)^{n+\rho}} F\left(n+\rho, 1; n+\rho+1; \frac{z}{\epsilon+z}\right).$$

For large z and fixed n, the optimum value for ϵ is given by

$$\epsilon = \frac{c_n}{n(c_{n-1} + |a_{n-1}|)}. \quad (34)$$

The remainder $R_{n,1}(\rho, \sigma; z)$ in expansion (23) satisfies the bound (33) with ρ replaced by $\rho + \sigma$ and z by cz.

Proof: From $|f_{n-1}(t)| \leq c_{n-1}t^{-n} \forall\, t \in [0,\infty]$ and $f_n(t) = f_{n-1}(t) - a_{n-1}t^{-n}$ we obtain $|f_n(t)| \leq (c_{n-1} + |a_{n-1}|)t^{-n} \forall\, t \in [0,\infty]$. To obtain the bound (30) we divide the integral defining $f_{n,n}(t)$ in (5) by a fixed point $u = \epsilon \geq t$ and use the bound $|f_n(t)| \leq (c_n - 1 + |a_{n-1}|)t^{-n}$ in the integral over $[t, \epsilon]$ and the bound $|f_n(t)| \leq c_n t^{-n-1}$ in the integral over $[\epsilon, \infty)$. Using $u - t \leq u$ in the integral over $[t, \epsilon]$ we obtain

$$|f_{n,n}(t)| \leq \frac{1}{(n-1)!}\left[(c_{n-1} + |a_{n-1}|)\log\left(\frac{\epsilon}{t}\right) + \frac{c_n}{\epsilon}\right] \qquad \forall\, t \in [0,\epsilon], \quad \epsilon > 0. \quad (35)$$

On the other hand in, $\forall\, t \in [0,\infty]$ we introduce the bound $|f_n(t)| \leq c_n t^{-n-1}$ in the integral definition of $f_{n,n}(t)$ and perform the change of variable $u = tv$. We obtain

$$|f_{n,n}(t)| \leq \frac{c_n}{n!}\frac{1}{t} \qquad \forall \qquad t \in [0,\infty]. \quad (36)$$

We divide the integral in the right-hand side of (17) at the point $t = \epsilon$ and use the bound (36) in the integral over $[\epsilon, \infty]$ and the bound (35) in the integral over $[0, \epsilon]$. We obtain

$$|R_{n,1}(\rho;z)| \leq \frac{(\rho)_n}{n!}$$
$$\left[nc_n \int_0^1 \frac{dt}{|\epsilon t + z|^{n+\rho}} + n\epsilon(c_{n-1} + |a_{n-1}|)\int_0^1 \frac{\log(t^{-1})dt}{|\epsilon t + z|^{n+\rho}} + c_n \int_1^\infty \frac{dt}{t|\epsilon t + z|^{n+\rho}}\right]. \quad (37)$$

Removing a factor $|z|^{n+\rho}$ from the denominator in the integrand of the two first integrals on the right-hand side of (37) and using the bound $|\epsilon t/z+1|\geq\Theta(z)$, we easily conclude that those two integrals are bounded by $[|z|\Theta(z)]^{-n-\rho}$. On the other hand, we perform the change of variable $t\to|z|t$ in the third integral on the right-hand side of (37) and integrate by parts to obtain

$$|z|^{n+\rho}\int_1^\infty\frac{dt}{t|\epsilon t+z|^{n+\rho}}=\frac{\log|z|}{|1+\epsilon/z|^{n+\rho}}+\epsilon(n+\rho)$$

$$\times\int_{|z|^{-1}}^\infty\frac{\{\epsilon t+\cos[\mathrm{Arg}(z)]\}\log t dt}{\{(\epsilon t+\cos[\mathrm{Arg}(z)])^2+\sin^2[\mathrm{Arg}(z)]\}^{(n+\rho)/2+1}}.$$

Now, with the change of variable $t\to t/\epsilon+|z|^{-1}$ and using $-\log|z|\leq\log(t/\epsilon+|z|^{-1})\leq t/\epsilon+|z|^{-1}-1\ \forall\, t\in[0,\infty]$ and ([12], p. 309, Eq. 7) we obtain (30).

To obtain (29) we use $|f_n(t)|\leq c_n t^{-n-1}$ and $|f_n(t)|\leq(c_{n-1}+|a_{n-1}|)\,t^{-n}$. Then, we have $f_n(t)\leq c_n t^{-n-1/2}$ if $t\geq 1$ and $f_n(t)\leq(c_{n-1}+|a_{n-1}|)\,t^{-n-1/2}$ if $t\leq 1$. Therefore, $f_n(t)\leq\bar{c}_n t^{-n-1/2}\ \forall\, t\in[0,\infty]$. Then, $f_n(t)$ satisfies the bound required in Proposition 1 with $s=1$, $\alpha=1/2$ and c_n replaced by $\bar{c}_n$. Repeating now the calculations of the proof of Proposition 1 we obtain (29).

If we get rid of irrelevant terms for large z, the right-hand side of (30), as function of ϵ, has a minimum for ϵ given in (31).

Bounds (32) are obtained using the inequality $|t+az|^{-\rho}|t+bz|^{-\sigma}\leq|t+az|^{-\rho-\sigma}+|t+bz|^{-\rho-\sigma}$ in the definition of $R_{n,s}(\rho,\sigma;z)$ and formulas (28), (29), and (30).

Bounds (33–34) and the bound for $R_{n,1}(\rho,\sigma;z)$ for real positive a, b, z have been obtained in [10]. ■

The following two lemmas introduce two families of functions $f(t)$ that verify the bound $|f_n(t)|\leq c_n t^{-n/s-\alpha}\ \forall\, t\in[0,\infty]$. Moreover, for these functions $f(t)$, the constants c_n can be easily obtained from $f(t)$.

LEMMA 5. *Suppose* $f(t)$ *satisfies* (4) *with* $0<\alpha\leq 1/s$, *and consider the function* $g(u)\equiv u^{-\alpha s}f(u^{-s})-\sum_{k=K}^{-1}a_k u^k$. *If* $g(w)$ *is a bounded analytic function in the region of the complex* w*-plane consisting of all points at a distance* $<r$ *from the positive real axis (see Figure* 1), *then,*

$$|f_n(t)|\leq C\epsilon^{-n}t^{-n/s-\alpha},$$

where C *is a bound of* $|g(w)|$ *in that region and* $0<\epsilon<r$.

Proof: From the asymptotic expansion (4) and the Lagrange formula for the remainder in the Taylor expansion of $g(u)$ in $u=0$, we have

$$g(u)=\sum_{k=0}^{n-1}a_k u^k+R_n(u),$$

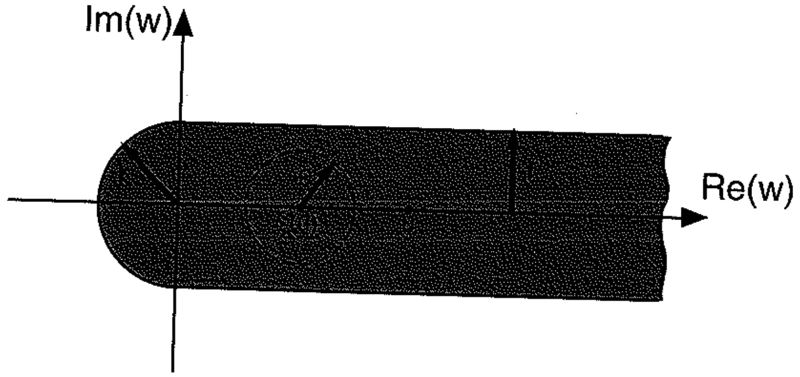

Figure 1. Analyticity region for the function $g(u)$ considered in Lemma 5. The integration variable u in (5) is real and unbounded; therefore, the analyticity region for $g(u)$ must contain the positive real axis. The circle of radius' ϵ centered at $\xi(u)$, with $0 < \xi(u) < u$, used in the proof of Lemma 5 must be contained in this region and therefore, $\epsilon < r$.

where

$$R_n(u) = \frac{1}{n!}\frac{d^n g(u)}{du^n}\bigg|_{u=\xi} u^n, \qquad \xi \in (0,u).$$

Using the Cauchy formula for the derivative of an analytic function,

$$\frac{d^n g(u)}{du^n} = \frac{n!}{2\pi i}\int_{\mathscr{C}} \frac{g(w)}{(w-\xi)^{n+1}}\,dw,$$

where $\mathscr{C}$ is a circle of radius ϵ around ξ into the region of analyticity of the function $g(w)$ (where it is also bounded). Then, for fixed ξ and ϵ, performing the change of variable $w = \xi + \epsilon e^{i\theta}$, and using $|g(\xi + \epsilon e^{i\theta})| \leq C$ for $\theta \in [0, 2\pi)$ with C independent of θ, ϵ, and ξ, we obtain the desired result. ■

LEMMA 6. *If the expansion* (4) (*with* $0 < \alpha \leq 1/s$) *satisfies the error test, then*

$$|f_n(t)| \leq |a_n| t^{-n/s-\alpha} \qquad \text{and} \qquad |f_n(t)| \leq |a_{n-1}| t^{-(n-1)/s-\alpha}.$$

Proof: A proof of the first inequality can be found in ([13], p. 68). The second inequality follows from the first one, from sign $[f_n(t)] \neq$ sign $[f_{n-1}(t)]$ and

$$f_n(t) = f_{n-1}(t) - \frac{a_{n-1}}{t^{(n-1)/s+\alpha}}.$$

■

COROLLARY 1. *If $f(t)$ satisfies the hypotheses of Lemma 5, then $R_{n,s}(\rho; z)$ and $R_{n,s}(\rho, \sigma; z)$ satisfy the bounds given in Propositions 1 and 2 with $c_n = C\epsilon^{-n}$.*

Moreover, these expansions are convergent when the parameter $|z|^{1/s}$ (*or* $|cz|^{1/s}$ *with* $c \equiv min\{|a|,|b|\}$) *is longer than the inverse of the width of the region considered in Lemma* 5 (*see Figure* 1), *more precisely, when*

$$\varepsilon^s|z| \geq 1 \quad \text{if} \quad \rho < 1 \quad \text{or} \quad \varepsilon^s|z| > 1 \quad \text{if} \quad \rho \geq 1$$

in Theorems 1 or 2,

$$\varepsilon^s|cz| \geq 1 \quad \text{if} \quad \rho+\sigma < 1 \quad \text{or} \quad \varepsilon^s|cz| > 1 \quad \text{if} \quad \rho+\sigma \geq 1$$

in Theorems 3 or 4.

For $\alpha = s = 1$, *the convergence of expansions* (19) *and* (23) *also requires that* $\lim_{n\to\infty} n^{\rho-1} a_n z^{-n} = 0$ *and* $\lim_{n\to\infty} n^{\rho+\sigma-1} a_n (cz)^{-n} = 0$, *respectively.*

COROLLARY 2. *If the expansion* (4) *of* $f(t)$ *verifies the error test, then* $R_{n,s}(\rho; z)$ *and* $R_{n,s}(\rho, \sigma; z)$ *satisfy the bounds given in Propositions* 1 *and* 2 *when replacing* c_n *by* $|a_n|$ *and* c_{n-1} *by* 0. *Moreover, the expansions given in Theorems* 1 *and* 2 *are convergent when the coefficients* a_n *in the asymptotic expansion* (4) *verify* $\lim_{n\to\infty} n^{\rho-1} a_n z^{-n/s} = 0$. *The expansions given in Theorems* 3 *and* 4 *are convergent when the coefficients* a_n *verify* $\lim_{n\to\infty} n^{\rho+\sigma-1} a_n (cz)^{-n/s} = 0$, $c \equiv min\{|a|,|b|\}$.

4. Examples

An important family of functions $f(t)$ in many applications, (Feynman diagrams in quantum field theory ([6], Ch. 6), ([7], Ch. 7) or symmetric elliptic integrals ([14], Ch. 12) for example are defined by integrals of the form (3) with $f(t)$ given in (37).) has the form

$$f(t) = \prod_{k=1}^{N} \frac{1}{(t^{1/s} + x_k)^{\mu_k}}, \tag{38}$$

where $s \in \mathbb{N}$, $x_1, \ldots, x_N \geq 0$, $\mu_1 x_1, \ldots, \mu_{N-1} x_{N-1} > 0$, $\mu_N x_N \geq 0$ and $\mu_N < s$ if $x_N = 0$. This family of functions trivially satisfy the hypotheses of Lemma 5 with $r^{-1} = \text{Max}\{x_1, \ldots, x_N\}$. Therefore, the bounds of Corollary 1 apply to the remainders in the expansions given in Theorems 1–4 when $f(t)$ has the form (38). However, we can show that they also verify the hypotheses of Lemma 6 and then Corollary 2 applies, too. Define

$$\mu \equiv \sum_{k=1}^{N} \mu_k.$$

For $\mu \notin \mathbb{N}$, the asymptotic expansion of $f(t)$ in $t = \infty$ is given, for $n = s, 2s, 3s, \ldots, n \geq [\mu]$ by (4) with $\alpha \equiv (\mu - [\mu])/s$ and $K \equiv [\mu]$,

$$f(t) = \sum_{k=[\mu]}^{n-1} \frac{a_k}{t^{(k+\mu-[\mu])/s}} + f_n(t), \tag{39}$$

where empty sums must be understood as zero. For $k = 0, 1, 2, \ldots$

$$a_{k+[\mu]} = \lim_{u \to 0} \frac{1}{k!} \frac{d^k}{du^k} [u^{-\mu} f(u^{-s})] \tag{40}$$

and $f_n(t) = \mathcal{O}(t^{-(n+\mu-[\mu])/s})$ as $t \to \infty$. Then, we have the following lemma.

LEMMA 7. *For $\mu \notin \mathbb{N}, n \geq [\mu]$, and $\forall\, t \in [0, \infty]$, the remainder term $f_n(t)$ in the expansion* (39) *of the function $f(t)$ defined in* (38) *satisfies the error test. More precisely, $sign[f_n(t)] = sign(a_n) = sign[(-1)^{n-[\mu]}]$ and then,*

$$|f_n(t)| \leq \frac{|a_n|}{t^{(n+\mu-[\mu])/s}} \qquad for \qquad n \geq [\mu],$$
$$|f_n(t)| \leq \frac{|a_{n-1}|}{t^{(n+\mu-[\mu]-1)/s}} \qquad for \qquad n \geq [\mu] + 1.$$

Proof: The Taylor expansion of $u^{-\mu} f(u^{-s})$ at $u = 0$ is given by

$$u^{-\mu} f(u^{-s}) \equiv \prod_{k=1}^{N} (1 + x_k u)^{-\mu_k} = \sum_{k=0}^{n-[\mu]-1} a_{k+[\mu]} u^k + u^{-\mu} f_n(u^{-s}).$$

Applying the binomial formula for the derivative of a product we realize that the nth u-derivative of $u^{-\mu} f(u^{-s})$ has the same sign as $(-1)^n\ \forall\, u \in [0, \infty]$. Then, $\text{sign}(a_n) = \text{sign}(-1)^{n-[\mu]}$ for $n \geq [\mu]$ and, by the Lagrange formula for the remainder $u^{-\mu} f_n(u^{-s})$ we obtain that $\text{sign}[f_n(t)] = \text{sign}(-1)^{n-[\mu]}$ for $n \geq [\mu]$ and $\forall\, t \in [0, \infty]$. Therefore, two consecutive error terms $f_n(t)$ and $f_{n+1}(t)$ in the expansion of $f(t)$ have opposite sign. After applying Lemma 6, we obtain the above inequalities.

On the other hand, for $\mu \in \mathbb{N}$, the asymptotic expansion in $t = \infty$ of the function $f(t)$ defined in (38) is given, for $n = s, 2s, 3s, \ldots$ and $n \geq \mu - 1$, by (4) with $\alpha \equiv 1/s$ and $K \equiv \mu - 1$,

$$f(t) = \sum_{k=\mu-1}^{n-1} \frac{a_k}{t^{(k+1)/s}} + f_n(t), \tag{41}$$

where empty sums must be understood as zero. For $k = 0, 1, 2, \ldots,$

$$a_{k+\mu-1} = \lim_{u \to 0} \frac{1}{k!} \frac{d^k}{du^k} [u^{-\mu} f(u^{-s})] \tag{42}$$

and $f_n(t) = \mathcal{O}(t^{-(n+1)/s})$ as $t \to \infty$. Then, we have the following lemma.

LEMMA 8. *For* $\mu \in \mathbb{N}, n \geq \mu - 1$ *and* $\forall\, t \in [0, \infty]$, *the remainder term* $f_n(t)$ *in the expansion* (41) *of the function* $f(t)$ *defined in* (38) *satisfies the error test. More precisely,* $sign[f_n(t)] = sign(a_n) = sign[(-1)^{n-\mu+1}]$ *and*

$$|f_n(t)| \leq \frac{|a_n|}{t^{(n+1)/s}} \quad \text{for} \quad n \geq \mu - 1, \qquad |f_n(t)| \leq \frac{|a_{n-1}|}{t^{n/s}} \quad \text{for} \quad n \geq \mu.$$

Proof: Similar to the proof of Lemma 7 when replacing $[\mu]$ by $\mu - 1$. ∎

COROLLARY 3. *Theorems* 1 *and* 3 *hold for the function* $f(t)$ *given in* (38) *if* $\mu \notin \mathbb{N}$ *with* $\alpha \equiv (\mu - [\mu])/s$, $K \equiv [\mu]$ *and* a_k *as given in* (40). *Moreover, for* $n = s, 2s, 3s, \ldots$ *and* $n \geq [\mu]$, *the error terms* $R_{n,s}(\rho; z)$ *and* $R_{n,s}(\rho, \sigma; z)$ *in the expansions* (16) *and* (21) *satisfy the bounds given in Proposition* 1 *with* $c_n = |a_n|$.

COROLLARY 4. *Theorems* 2 *and* 4 *hold for the function* $f(t)$ *given in* (38) *if* $\mu \in \mathbb{N}$ *with* $\alpha \equiv 1/s$, $K \equiv \mu - 1$ *and* a_k *given in* (42). *Moreover, for* $n = s, 2s, 3s, \ldots$ *and* $n \geq \mu - 1$, *the error terms* $R_{n,s}(\rho, \sigma; z)$ *in the expansions* (16) *and* (21) *satisfy the bounds given in propositions* 1 *and* 2 *when replacing* c_n *by* $|a_n|$ *and* c_{n-1} *by* 0.

Two specific examples that show the accuracy of the expansions given in Section 2, and the error bounds of Section 3 are given below.

4.1. The tadpole in the theory of the scalar field in 3 + 1 dimensions

The mass renormalization of the scalar field in 3+1 dimensions regularized by means of high derivatives ([15], Ch. 4, Sec. 4) requires the calculation of the integral

$$I_{s,\rho}(m, \Lambda) \equiv \frac{2s}{m^4} \int_0^\infty \frac{p^3\, dp}{(p^2 + m^2)(p^{2s} + \Lambda^{2s})^\rho},$$

where m is the bare mass of the scalar field, Λ is the regulator parameter, and the parameters $\rho > 0$ and $s \in \mathbb{N}$ verify $s\rho > 1$. Physical observables are defined for large values of the regulator parameter, and then an approximation of the integral for large values of Λ is required. By means of a simple change of variable, this integral reads

$$I_{s,\rho}(m, \Lambda) \equiv \frac{1}{m^4} \int_0^\infty \frac{t^{2/s-1}\, dt}{(t^{1/s} + m^2)(t + \Lambda^{2s})^\rho}.$$

Therefore, it has the form considered in Theorem 2 with $z \equiv \Lambda^{2s}$ and

$$f(t) \equiv \frac{t^{2/s-1}}{m^4(t^{1/s}+m^2)} \sim \sum_{k=s-2}^{\infty} \frac{(-m^2)^{k-s}}{t^{(k+1)/s}}, \qquad t \to \infty. \tag{43}$$

If $s = 1$ then $I_{s,\rho}(m, \Lambda)$ may be approximated by Wong's methods. Therefore, we consider the case $s \geq 2$. Then, the asymptotic expansion of $f(t)$ for large t has the form considered in Theorem 2 with $K \equiv s-2$ and

$$\begin{aligned} &a_0 = a_1 = \cdots = a_{s-3} = 0 \\ &a_k = (-m^2)^{k-s}, \qquad k = s-2,\ s-1, s, \ldots \end{aligned} \tag{44}$$

Then, applying Theorem 2, we have

$$\begin{aligned} I_{s,\rho}(m,\Lambda) = \sum_{k=0}^{n/s-1} \Bigg\{ &\frac{(-1)^{k(s+1)-1}(\rho)_k m^{2(sk-1)}}{k!\Lambda^{2s(k+\rho)}} \left[\log\left(\frac{\Lambda}{m}\right)^{2s} - \gamma - \psi(k+\rho)\right] \\ &+ \sum_{j=j_0}^{s-2} \frac{(-1)^{k(s+1)+j-s}\Gamma[(\rho+k+(j+1)/s-1)\Gamma(1-(j+1)/s]}{m^{2[s(1-k)-j]}[(j+1)/s]_k\Gamma(\rho)\Lambda^{2[s(\rho+k-1)+j+1]}} \\ &+ \frac{d_{s(k+1)}(\rho)_k}{\Lambda^{2s(\rho+k)}} \Bigg\} + R_{n,s,\rho}(m;\Lambda), \end{aligned} \tag{45}$$

where $j_0 \equiv \max[0, s(1-k)-2]$, $d_{s(k+1)}$ is given by (10), (11), or (20) and the remainder term is given by (17) with $z \equiv \Lambda^{2s}$. Using (11), ([14], p. 41, Eq. 3.2) and the duplication formula of the gamma function we obtain

$$\begin{aligned} d_{s(k+1)} = \frac{(-1)^{k+s}}{k!m^{2s}} \Bigg\{ &\sum_{j=ks}^{s(k+1)-2} \frac{(-1)^j m^{2j}}{k-(j+1)/s+1} + k! \sum_{j=0}^{s-2} \frac{(-1)^{sk+j} m^{2(sk+j)}}{((j+1)/s-1)_{k+1}} \\ &+ \sum_{j=1}^{k} \sum_{l=ks}^{s(k+1)-1} \frac{(-1)^l (k-j+2)_{j-1} m^{2l}}{[(l+1)/s-j]_j} \Bigg\}. \end{aligned}$$

The function (43) has the form (38) with $N = 2$, $\mu_1 = 1$, $\mu_2 = s-2$ and a_k given in (44). Therefore, applying Corollary 4, we obtain, for $n \geq s-2$,

$$|R_{n,s,\rho}(m;\Lambda)| \leq \frac{\pi m^{2(n-s)}\Gamma[(n+1)/s+\rho-1]}{\Gamma(\rho)\sin(\pi/s)\Gamma[(n+1)/s]\Lambda^{2(s(\rho-1)+n+1)}}. \tag{46}$$

This bound shows that expansion (45) is convergent for $m < \Lambda$ if $\rho \geq 1$ and for $m \leq \Lambda$ if $\rho < 1$.

4.2. The third symmetric elliptic integral with two parameters large

The third symmetric standard elliptic integral is defined by ([14], Ch. 12)

$$R_J(x,y,z,p) = \frac{3}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}(t+p)},$$

where the parameters x, y, z, and p are non-negative. The integral (2/3) $R_J(x, az, bz, p)$ with z large (and $|az| \le |bz|$) has the form considered in Theorem 3 with $s = 1$, $\rho = \sigma = \alpha = 1/2$, $K = 0$ and

$$f(t) = \frac{1}{\sqrt{t+x}(t+p)} = \sum_{k=0}^{n-1} \frac{a_k}{t^{k+1/2}} + f_n^J(t).$$

Therefore, the asymptotic expansion of (2/3) $R_J(x, az, bz, p)$ for large z follows from Equation (21) in Theorem 3. Coefficients $a_k \equiv (-1)^k A_k^J(x,p)$ are trivially given by $A_0^J(x, p) = 0$ and, for $k = 1, 2, 3, \ldots$,

$$A_k^J(x,p) = -\sum_{j=0}^{k-1} \frac{(1/2)_j}{j!} x^j p^{k-j-1}. \tag{47}$$

The Mellin transform $M[f; k+1]$ in the coefficients C_k in formula (21) can be obtained from ([12], p. 303, Eq. 24). Therefore, applying Equation (21) we obtain

$$R_J(x,az,bz,p) = \frac{3}{2}\sum_{k=0}^{n-1}\left[\frac{A_k^J(x,p)B_k(a,b)}{z^{k+1/2}} + \frac{2(-1)^k x^{k+1/2} C_k(a,b)\Gamma(1/2-k)}{p\sqrt{\pi ab}z^{k+1}} \times F\left(\begin{matrix} k+1, 1 \\ 3/2 \end{matrix}\middle| 1-\frac{x}{p}\right)\right] + R_n^J(x,az,bz,p), \tag{48}$$

where, for $k = 1, 2, 3, \ldots$, the coefficients $B_k(a, b)$ and $C_k(a, b)$ are given by

$$B_k(a,b) = \sum_{j=0}^{k} \frac{\Gamma(j+1/2)\Gamma(k-j+1/2)}{j!(k-j)!a^j b^{k-j+1/2}} F\left(\begin{matrix} 1/2, k-j+1/2 \\ k+1 \end{matrix}\middle| 1-\frac{a}{b}\right)$$

and

$$C_k(a,b) = \sum_{j=0}^{k} \binom{k}{j} \frac{(1/2)_j (1/2)_{k-j}}{a^j b^{k-j}}.$$

Function $f(t)$ satisfies the conditions of Corollary 3 with $\mu = 3/2$. Therefore, for $x, p, az, bz, \in, \mathbb{C} \setminus \mathbb{R}^-$ and $n = 1, 2, 3, \ldots$, the bound (27) holds for (2/3) $R_J(x, az, bz, p)$ setting $s = 1$, $\rho = \sigma = \alpha = 1/2$ and $c_n = |A_n^J(x, p)|$ given in (47). Therefore,

$$|R_n^J(x, az, bz, p)| \leq \frac{3\pi |A_n^J(x,p)|}{2|az|^{n+1/2}} F\left[\begin{matrix} 1/2, n+1/2 \\ n/2+1 \end{matrix} \middle| \frac{1}{2}\left(1 - \frac{r}{|az|}\right)\right], \tag{49}$$

where $r \equiv \mathrm{Min}[\mathrm{Re}(az), \mathrm{Re}(bz)]$.

5. Conclusions

Wong's distributional method was introduced in [2, 3] to derive asymptotic approximations of Stieltjes and generalized Stieltjes transforms for large argument. Following Wong's proposal ([3], Example 1), the distributional approach was used in [4] and [5] to derive alternative proofs for the asymptotic expansion of generalized Stieltjes transforms considered in [2]. In any case, only transforms of functions $f(t)$ with an asymptotic expansion for large t in descending integer powers of t have been considered in [4, 5, 3, 2].

In Theorems 1–4 of this article, we have generalized those results to functions $f(t)$ having an asymptotic expansion for large t in descending rational powers of t. The asymptotic character of the expansions obtained in those theorems has been proved in Theorem 5. This kind of integrals frequently appears in the perturbative calculations of physical observables in quantum field theory (a specific example has been considered in Section 4.1).

In Propositions 1 and 2, we derived error bounds for these expansions when the remainder term $f_n(t)$ in the expansion of the function $f(t)$ can be bounded in the form $|f_n(t)| \leq c_n t^{-n/s-\alpha}$ $\forall\, t \in [0, \infty]$. When the function $g(u)$ introduced in Lemma 5 [closely related to $f(t)$] is analytic and bounded around the positive real axis or when the expansion (4) of the function $f(t)$ verifies the error test, then the constant c_n may be easily obtained from $f(t)$ (Lemmas 5 and 6 and Corollaries 1 and 2).

In particular, in Corollaries 3 and 4, an error bound for the remainder term of the expansions given in Theorems 1–4 has been obtained for the family of functions $f(t)$ defined in (38). This family (which verifies Lemmas 5 and 6) is particularly important in practice and two specific examples have been considered in Section 4. The bounds given in Propositions 1 and 2 have been obtained from the error test, and, as numerical computations show (see Tables 1, 2 and 3), they exhibit remarkable accuracy.

Table 1

Numerical Experiments on the Approximation (45) for $s = \rho = 2$, $m = 1$ and Several Real Values of Λ

Λ^4	$I_{2,2}(1, \Lambda)$	First Order Approx.	Relative Error	Relative Error Bound	Second Order Approx.	Relative Error	Relative Error Bound
2^4	14465043.26381	17619517.90992	−0.218	0.318	14127238.00830	0.0234	0.0331
5^4	86177.57828258	866103.2069155	−0.00502	0.00560	86176.41412147	1.35e-5	1.49e-5
10^4	1488.239180724	1488.692923075	−0.000305	0.000317	1488.239104858	5.10e-8	5.28e-8
20^4	24.11421735583	24.11467193843	−1.89e-5	1.91e-5	24.11421735109	1.97e-10	1.99e-10
50^4	0.1001559256284	0.1001559737591	−4.81e-7	4.82e-7	.1001559256283	1.28e-13	1.28e-13

Second, third, and sixth columns represent $10^9 I_{2,2}(1,\Lambda)$, approximation (45) for $n = 2$ and approximation (45) for $n = 4$, respectively. Fourth and seventh columns represent the absolute value of the respective relative errors in (45). Fifth and last columns represent the respective error bounds given by Equation (46).

Table 2
The Same Experiment as in Table 1 for $\mathrm{Arg}(\Lambda^4) = \pi/4$

$\lvert\Lambda^4\rvert$	$I_{2,2}(1, \Lambda)$	First Order Approx.	Relative Error	Relative Error Bound	Second Order Approx.	Relative Error	Relative Error Bound
2^4	7269609.05681 −12712112.345i	6324502.95367 −15751240.555i	0.217	0.347	7572112.25554 −12555422.101i	0.0233	0.0366
5^4	36614.6667884 −78551.4624166i	36460.9154143 −78957.8566451i	5.01e-3	6.15e-3	36615.7331615 −78550.9844846i	1.35e-5	1.66e-5
10^4	593.432806120 −1368.70126320i	593.263748250 −1369.12317235i	3.05e-4	3.49e-4	593.432876009 −1368.70123338i	5.09e-8	5.89e-8
20^4	9.36195732452 −22.2459736570i	9.36178491369 −22.2463945833i	1.88e-5	2.11e-5	9.36195732889 −22.2459736552i	1.97e-10	2.22e-10
50^4	0.03845144690 −0.09250346522i	0.03845142852 −0.09250350971i	4.81e-7	5.32e-7	0.03845144690 −0.09250346522i	1.28e-13	1.44e-13

Table 3
Numerical Example of the Approximation (48)

z	$R_J(1, az, bz, 2)$	Second Order Approx.	Relative Error	Relative Error Bound	Third Order Approx.	Relative Error	Relative Error Bound
10	0.0896732127 −0.1176762413i	0.0680162162 −0.1261349134i	0.157	0.396	0.0868656595 −0.1146833997i	0.0277	0.0919
20	0.0467096544 −0.0697992616i	0.0424695054 −0.0712571145i	0.0538	0.123	0.0464520698 −0.0695021674i	0.00527	0.0143
50	0.09190154710 −0.0327173122i	0.0185433492 −0.0328572966i	0.0150	0.0278	0.0190047588 −0.0327038995i	4.54e-4	12.9e-4
100	0.0094934359 −0.0177602227i	0.0094057093 −0.0177839292i	0.00451	0.00917	0.0094924792 −0.0177589675i	7.84e-5	21.3e-5
200	0.0047141435 −0.0094192751i	0.0046980692 −0.009423307i	0.00157	0.00307	0.0047140585 −0.0094191595i	1.36e-5	3.56e-5

Second, third, and sixth columns represent $R_J(1,az,bz,2)$ for $a = e^{i\pi/4}$ and $b = e^{i\pi/2}$, approximation (48) for $n = 2$ and approximation (48) for $n = 3$, respectively. Fourth and seventh columns represent the respective relative errors in (48). Fifth and last columns represent the respective error bounds given by Equation (49).

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References

1. A. I. ZAYED, *Handbook of Function and Generalized Function Transformations*, CRC, New York, 1996.
2. R. WONG, *Asymptotic Approximations of Integrals*, Academic Press, New York, 1989.
3. R. WONG, Explicit error terms for asymptotic expansions of Mellin convolutions, *J. Math. Anal. Appl.* 72:740–756 (1979).
4. J. L. LÓPEZ, Asymptotic expansions of symmetric standard elliptic integrals, *SIAM J. Math. Anal.* 31(4):754–775 (2000).
5. J. L. LÓPEZ, Uniform asymptotic expansions of symmetric elliptic integrals, *Const. Approx.*, in press.
6. C. ITZYKSON and J. B. ZUBER, *Quantum Field Theory*, McGraw-Hill, New York, 1980.
7. M. KAKU, *Quantum Field Theory: A Modern Introduction*, Oxford University Press, New York, 1993.
8. L. ALVAREZ-GAUMÉ, J. M. F. LABASTIDA, and A. V. RAMALLO, *Nucl. Phys.* B 334:103 (1990).
9. M. ASOREY, and F. FALCETO, *Phys. Lett.* B241:31 (1990).
10. W. CHEN, G. W. SEMENOFF, and Y.-S. WU, *Phys. Rev.* D46:5521 (1992).
11. M. L. GLASSER, Some Bessel function integrals, *Kyungpook Math. J.* 13:171–174 (1973).
12. A. P. PRUDNIKOV, Yu. A. BRYCHKOV, and O. I. MARICHEV, *Integrals and Series*, vol. 1, Gordon and Breach Science, Russia, 1990.
13. F. W. J. OLVER, *Asymptotics and Special Functions*, Academic Press, New York, 1974.
14. N. M. TEMME, *Special Functions: An Introduction to the Classical Functions of Mathematical Physics*, Wiley, New York, 1996.
15. L. D. FADDEEV and A. A. SLAVNOV, *Gauge Fields: Introduction to Quantum Theory*, The Benjamin/Cummings Pub. Co., London, 1980.
16. M. ABRAMOWITZ and I. A. STEGUN, *Handbook of Mathematical Functions*, Dover, New York, 1970.

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