Available at WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE @DIRECT•

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 151 (2003) 235–256

www.elsevier.com/locate/cam

Asymptotic expansions of the Lauricella hypergeometric function F_D

Chelo Ferreira[a], José L. López[b,*]

[a]Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Zaragoza, 50013 Zaragoza, Spain
[b]Departamento de Matemática e Informática, Campus de Arrosadia S/N, Universidad Pública de Navarra, 31006 Pamplona, Spain

Received 3 January 2002; received in revised form 18 September 2002

Abstract

The Lauricella hypergeometric function $F_D^r(a, b_1, \ldots, b_r; c; x_1, \ldots, x_r)$ with $r \in \mathbb{N}$, is considered for large values of one variable: x_1, or two variables: x_1 and x_2. An integral representation of this function is obtained in the form of a generalized Stieltjes transform. Distributional approach is applied to this integral to derive four asymptotic expansions of this function in increasing powers of the asymptotic variable(s) $1 - x_1$ or $1 - x_1$ and $1 - x_2$. For certain values of the parameters a, b_i and c, two of these expansions also involve logarithmic terms in the asymptotic variable(s). For large x_1, coefficients of these expansions are given in terms of the Lauricella hypergeometric function $F_D^{r-1}(a, b_2, \ldots, b_r; c; x_2, \ldots, x_r)$ and its derivative with respect to the parameter a, whereas for large x_1 and x_2 those coefficients are given in terms of $F_D^{r-2}(a, b_3, \ldots, b_r; c; x_3, \ldots, x_r)$ and its derivative. All the expansions are accompanied by error bounds for the remainder at any order of the approximation. Numerical experiments show that these bounds are considerably accurate.

MSC: 41A60; 33C65

Keywords: Lauricella hypergeometric function; Asymptotic expansions; Distributional approach; Generalized Stieltjes transforms

1. Introduction

The Lauricella hypergeometric function $F_D^r(a, b_1, \ldots, b_r; c; x_1, \ldots, x_r)$ is defined by means of the multiple power series [7]

* Corresponding author. Tel.: +34-948-169-534; fax: +34-948-169-521.
E-mail address: jl.lopez@unavarra.es (J.L. López).

0377-0427/03/$ - see front matter
PII: S0377-0427(02)00814-2

$$F_D^r(a,b_1,\dots,b_r;c;x_1,\dots,x_r)$$
$$\equiv \sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m_1+\cdots+m_r}(b_1)_{m_1}\cdots(b_r)_{m_r}}{(c)_{m_1+\cdots+m_r}m_1!\cdots m_r!}x_1^{m_1}\cdots x_r^{m_r},\quad |x_i|<1,\ i=1,2,\dots,r.$$

Two important particular cases of this function are the case $r=1$: the Gauss hypergeometric function, and the case $r=2$: the Appell function.

The Lauricella function has several applications in mathematical physics due to its role in the theory of Lie group representations [13] and its astrophysical and quantum chemical applications [19]. There is an extensive mathematical literature devoted to the study of these functions: Srivastava derived polynomial expansions of the Lauricella functions using expansions involving Bessel functions [20]. Integral formulas involving the Lauricella functions are established in [18]. The behavior of F_D^3 near certain of its logarithmic singularities is analyzed in [17]. Several formulas involving series and integrals of the Lauricella functions and their application to derive bilateral and bilinear generating functions of Jacobi, Laguerre and Sylvester polynomials are derived in [12]. In [1], a family of generalized elliptic-type integrals is introduced, which admits an explicit representation in terms of the Lauricella function. Carlson explains how the Lauricella hypergeometric function is connected with symmetric elliptic integrals and their advantages for numerical and symbolic integration [3]. The Lauricella functions have also probabilistic interpretations, which arise from the evaluation of certain product moments of some multivariate distributions. It is shown that these product moments may be expressed as transformations on the Lauricella functions [15]. Other specific probabilistic and statistic problems are studied in [11]. Generalized hypergeometric functions of several variables are needed to study Goursat–Darboux problems for third-order differential equations [4]. A comprehensive reference to the Lauricella function is [22].

Approximations of the Lauricella functions have been investigated by many authors: Complete power series expansions at $x_i=0$, $i=1,2,3,\dots,r-1$ of F_D^r in terms of products of the Gauss hypergeometric function and F_D^{r-1} have been obtained by Srivastava and Goyal [21], and the later is extended by Joshi [6]. Approximations by using branched continued fractions are studied in [14]. Al-Zamel et al. have obtained asymptotic expansions of F_D when one of its variables approaches 1 in terms of multiple series involving elementary functions [1]. On the other hand, asymptotic expansions of these functions for large values of the variables have not been exhaustively investigated. Complete convergent expansions of F_D^r have been obtained by Carlson using Mellin transforms techniques [2]. Although these expansions have an attractively simple structure, explicit computation of the terms of the expansions is not straightforward and the upper bound on the truncation error is not quite satisfactory [2, Section 5]. Asymptotic approximations of a class of generalized hypergeometric functions, which includes F_D^r may be found in [4]. In this paper we consider the problem of finding complete asymptotic expansions of $F_D^r(a,b_1,\dots,b_r;c;x_1,\dots,x_r)$ for large values of one or two of its variables x_i. We attempt to obtain easy algorithms to compute the coefficients of these expansions as well as error bounds at any order of the approximation.

The starting point is the integral representation [7, Eq. (25)]

$$F_D^r(a,b_1,\dots,b_r;c;x_1,\dots,x_r)=\frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)}\int_0^1\frac{s^{a-1}(1-s)^{c-a-1}\,\mathrm{d}s}{(1-sx_1)^{b_1}\cdots(1-sx_r)^{b_r}},\qquad (1)$$

where $\Re(a) > 0$, $\Re(c-a) > 0$, $x_i \notin [1,\infty)$ if $b_i \geqslant 1$ for $i = 1,2,3,\ldots,r$. This integral defines the analytical continuation of $F_D^r(a,b_1,\ldots,b_r;c;x_1,\ldots,x_r)$ to the cut complex x_i–planes $\mathbb{C}\backslash[1,\infty)$ [23, Theorem 2.3, p. 30].

The first step in the analysis is to write the aforementioned integral as a generalized Stieltjes transform. For that purpose, we perform the change of variable $s = (1+t)^{-1}$ in (1) obtaining

$$F_D^r(a,b_1,\ldots,b_r;c;x_1,\ldots,x_r) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)}\int_0^\infty \frac{f_r(t)}{(t+1-x_1)^{b_1}}\,\mathrm{d}t \tag{2}$$

or

$$F_D^r(a,b_1,\ldots,b_r;c;x_1,\ldots,x_r) \equiv \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)}\int_0^\infty \frac{f_{r-1}(t)}{(t+1-x_1)^{b_1}(t+1-x_2)^{b_2}}\,\mathrm{d}t, \tag{3}$$

where

$$f_{r-1}(t) \equiv \frac{t^{c-a-1}(1+t)^{b_1+b_2+\cdots+b_r-c}}{\prod_{k=3}^r (t+1-x_k)^{b_k}}, \quad f_r(t) \equiv \frac{f_{r-1}(t)}{(t+1-x_2)^{b_2}}. \tag{4}$$

Then, up to a factor, the Lauricella function F_D^r is a generalized Stieltjes transform of $f_{r-1}(t)$ or $f_r(t)$. For $\Re(c-a) > 0$, the function $f_{r-1}(t)$ is a locally integrable function on $[0,\infty)$ and satisfies

$$f_{r-1}(t) = \sum_{k=0}^{n-1} \frac{A_k}{t^{k-b_1-b_2+a+1}} + f_n^{r-1}(t), \tag{5}$$

where

$$A_k \equiv \sum_{m_1=0}^{k}\sum_{m_2=0}^{m_1}\cdots\sum_{m_{r-2}=0}^{m_{r-3}} \binom{\sum_{j=1}^r b_j - c}{k-m_1}\binom{-b_3}{m_1-m_2}\cdots\binom{-b_{r-1}}{m_{r-1}-m_{r-2}} \times \binom{-b_r}{m_{r-2}}(1-x_r)^{m_{r-2}}\prod_{i=1}^{r-3}(1-x_{i+2})^{m_i-m_{i+1}} \tag{6}$$

and $f_n^{r-1}(t) = \mathcal{O}(t^{-n+b_1+b_2-a-1})$ when $t \to \infty$.

On the other hand, for $\Re(c-a) > 0$ and $x_2 - 1 \notin \mathbb{R}^+ \cup \{0\}$ if $\Re b_2 \geqslant 1$, $f_r(t)$ is a locally integrable function on $[0,\infty)$ and satisfies

$$f_r(t) = \sum_{k=0}^{n-1} \frac{B_k}{t^{k-b_1+a+1}} + f_n^r(t), \tag{7}$$

where

$$B_k \equiv \sum_{j=0}^{k} A_{k-j}\binom{-b_2}{j}(1-x_2)^j \tag{8}$$

and $f_n^r(t) = \mathcal{O}(t^{-n+b_1-a-1})$ when $t \to \infty$.

Classical asymptotic methods do not apply to integrals (2) and (3). But the asymptotic methods based on the distributional approach [24,25], Chapter [6,8,9] may be generalized in order to be applied to these integrals. This generalization is performed in [5,10], and also in this paper. In Section 2, we summarize the main theorems obtained in [5,10] and derive new theorems that we will need in this paper. In Section 3 we apply these theorems to obtain asymptotic expansions of integrals (2) and (3) with error bounds, for both: large x_1 and fixed x_j, $j \neq 1$, and large x_1 and x_2 and fixed x_j, $j \neq 1,2$. Several numerical examples are shown as illustrations.

2. Asymptotic expansions of generalized Stieltjes transforms

Let $f(t)$ be a locally integrable function on $[0,\infty)$ which satisfies

$$f(t) = \sum_{k=K}^{n-1} \frac{a_k}{t^{k+s}} + f_n(t), \tag{9}$$

where $K \in \mathbb{Z}$, $0 < \Re s \leqslant 1$, $\{a_k,\ k = K, K+1, K+2, \ldots\}$ is a sequence of complex numbers and $f_n(t) = \mathcal{O}(t^{-n-s})$ when $t \to \infty$.

Then, asymptotic expansions (including error bounds) of the generalized Stieltjes transforms of $f(t)$

$$S_f(w;z) \equiv \int_0^\infty \frac{f(t)}{(t+z)^w}\,dt, \qquad S_f(w_1,w_2;z) \equiv \int_0^\infty \frac{f(t)}{(t+xz)^{w_1}(t+yz)^{w_2}}\,dt \tag{10}$$

for large z and fixed x and y are given in [25, Chapter 6,8,9] for real w, w_1, w_2 and s and complex x, y, z. In [5,10], these theorems are generalized to the case of complex w, w_1, w_2 and s. Therefore, in the following pages, we consider that the parameters w, w_1, w_2, x, y and z are complex and that $f(t)$ is a locally integrable function on $[0,\infty)$ which satisfies (9). In the following pages, we use the notation introduced in [25].

2.1. *Asymptotic expansion of $S_f(w;z)$ and $S_f(w_1,w_2;z)$ for large z*

Asymptotic expansions of $S_f(w;z)$ for large z are given in the following two theorems proved in [10, Theorems 1–3].

Theorem 1. *Let $f(t)$ be a locally integrable function on $[0,\infty)$ which satisfies* (9) *with $0 < \Re s \leqslant 1$, $s \neq 1$. Then, for $z \in \mathbb{C}\setminus\mathbb{R}^- \cup \{0\}$, $\Re(s+w)+K > 1$ and $n = 1,2,3,\ldots,$*

$$\int_0^\infty \frac{f(t)}{(t+z)^w}\,dt = \sum_{k=K}^{n-1} \frac{(-1)^k \pi a_k \Gamma(w+s+k-1)}{\Gamma(s+k)\Gamma(w)\sin(\pi s) z^{w+s+k-1}} + \sum_{k=0}^{n-1} \frac{(-1)^k (w)_k M[f;k+1]}{k!\, z^{k+w}} + R_n(w;z), \tag{11}$$

where empty sums must be understood as zero and $(w)_k$ denotes the Pochhammer symbol. In this formula, $M[f;z]$ denotes the Mellin transform of f: $\int_0^\infty t^{z-1} f(t)\,dt$ or its analytic continuation.

The remainder term is defined by

$$R_n(w;z) \equiv (w)_n \int_0^\infty \frac{f_{n,n}(t)\,\mathrm{d}t}{(t+z)^{n+w}}, \tag{12}$$

where $f_{n,n}(t)$ is defined recursively by

$$f_{n,0}(t) \equiv f_n(t), \quad f_{n,k+1}(t) \equiv -\int_t^\infty f_{n,k}(u)\,\mathrm{d}u = \frac{(-1)^{k+1}}{k!}\int_t^\infty (u-t)^k f_n(u)\,\mathrm{d}u. \tag{13}$$

The remainder term verifies $R_n(w;z) = \mathcal{O}(z^{-n-w-s+1})$ when $z \to \infty$.

Theorem 2. *Let $f(t)$ be a locally integrable function on $[0,\infty)$ which satisfies (9) with $s=1$. Then, for $z \in \mathbb{C}\backslash\mathbb{R}^- \cup \{0\}$, $\Re w + K > 0$ and $n = 1,2,3,\ldots,$*

$$\int_0^\infty \frac{f(t)}{(t+z)^w}\,\mathrm{d}t = \sum_{k=K}^{-1} a_k \frac{\Gamma(w+k)\Gamma(-k)}{\Gamma(w)z^{w+k}} + \sum_{k=0}^{n-1} \frac{(-1)^k (w)_k}{k! z^{k+w}} \times [a_k(\log(z) - \gamma - \psi(k+w)) + b_{k+1}] + R_n(w;z), \tag{14}$$

where empty sums must be understood as zero and for $k=0,1,2,\ldots,$ the coefficients b_{k+1} are given by

$$b_{k+1} \equiv \lim_{z\to k}\left[M[f;z+1] + \frac{a_k}{z-k}\right] + a_k(\gamma + \psi(k+1)). \tag{15}$$

The remainder term $R_n(w;z)$ is given in (12) and verifies $R_n(w;z) = \mathcal{O}(z^{-n-w}\log z)$ when $z \to \infty$.

To obtain now asymptotic expansions of $S_f(w_1,w_2;z)$ for large z we use the same techniques that we used in [10]. We denote by $\mathscr{S}$ the space of rapidly decreasing functions and by $\langle \Lambda, \varphi \rangle$ the image of a tempered distribution Λ acting over a function $\varphi \in \mathscr{S}$. Since $f(t)$ in (9) and (10) is a locally integrable function on $[0,\infty)$, it defines a distribution $\mathbf{f}$:

$$\langle \mathbf{f}, \varphi \rangle \equiv \int_0^\infty f(t)\varphi(t)\,\mathrm{d}t.$$

The distributions associated with t^{-k-s}, $k = 0,1,2,\ldots,n-1$ are given by [25, Chapter 5, 10]

$$\langle \mathbf{t}^{-k-s}, \varphi \rangle \equiv \frac{1}{(s)_k}\int_0^\infty t^{-s}\varphi^{(k)}(t)\,\mathrm{d}t \quad \text{if } 0 < \Re s < 1,$$

$$\langle \mathbf{t}^{-k-s}, \varphi \rangle \equiv \frac{1}{(\mathrm{i}\Im s)_{k+1}}\int_0^\infty t^{-\mathrm{i}\Im s}\varphi^{(k+1)}(t)\,\mathrm{d}t \quad \text{if } 1 \neq s = 1 + \mathrm{i}\Im s,$$

where $(s)_k$ denotes the Pochhammer symbol of s, and

$$\langle \mathbf{t}^{-k-1}, \varphi \rangle \equiv -\frac{1}{k!}\int_0^\infty \log(t)\varphi^{(k+1)}(t)\,\mathrm{d}t.$$

To assign a distribution to the function $f_n(t)$ introduced in (9), we use the recursive definition of $f_{n,n}(t)$ given in (13) [25, Chapter 5, 10]:

$$\langle \mathbf{f}_n, \varphi\rangle \equiv (-1)^n \langle \mathbf{f}_{n,n}, \varphi^{(n)}\rangle \equiv (-1)^n \int_0^\infty f_{n,n}(t)\varphi^{(n)}(t)\,\mathrm{d}t.$$

Following the same steps in [10], we define a particular function of $\mathcal{S}$:

$$\varphi_\eta(t) \equiv \frac{\mathrm{e}^{-\eta t}}{(t+xz)^{w_1}(t+yz)^{w_2}} \in \mathcal{S},$$

where $xz \notin \mathbb{R}^- \cup \{0\}$ if $\Re w_1 \geqslant 1$, $yz \notin \mathbb{R}^- \cup \{0\}$ if $\Re w_2 \geqslant 1$ and $\eta > 0$. In the following lemma we calculate the image of the tempered distributions defined above and of the delta distribution acting over $\varphi_\eta(t)$, and take the limit $\eta \to 0$.

Lemma 1. *Let $f(t)$ verify (9). Then, for $0 < \Re s \leqslant 1$, $k=0,1,2,\ldots$ and $n=1,2,3,\ldots$, the following identities hold*:

$$\lim_{\eta\to 0} \langle \mathbf{f}, \varphi_\eta\rangle = \int_0^\infty \frac{f(t)}{(t+xz)^{w_1}(t+yz)^{w_2}}\,\mathrm{d}t \quad \textit{for } \Re(s+w_1+w_2)+K > 1.$$

$$\lim_{\eta\to 0} \langle \delta, \varphi_\eta^{(k)}\rangle = \frac{(-1)^k}{z^{k+w_1+w_2}} \sum_{j=0}^{k} \binom{k}{j} \frac{(w_1)_j (w_2)_{k-j}}{x^{w_1+j} y^{w_2+k-j}},$$

$$\lim_{\eta\to 0} \langle \mathbf{t}^{-s}, \varphi_\eta^{(k)}\rangle = \frac{\Gamma(1-s)\Gamma(k+w_1+w_2+s-1)}{\Gamma(w_1+w_2) z^{k+w_1+w_2+s-1}} \frac{(-1)^k}{x^{w_1+k+s-1} y^{w_2}}$$
$$\times F\left(\left.\begin{matrix} 1-s-k, w_2 \\ w_1+w_2 \end{matrix} \right| 1-\frac{x}{y}\right) \quad \textit{for } \Re(s+w_1+w_2)+k>1,\ s\neq 1,$$

where

$$F\left(\left.\begin{matrix} \alpha, \beta \\ \delta \end{matrix} \right| z\right) \equiv {}_2F_1(\alpha,\beta,\delta;z)$$

denotes the Gauss hypergeometric function,

$$\lim_{\eta\to 0} \langle \log(\mathbf{t}), \varphi_\eta^{(k+1)}\rangle = \frac{(-1)^{k+1}}{(k+w_1+w_2) z^{k+w_1+w_2}} \sum_{j=0}^{k+1} \binom{k+1}{j} \frac{(w_1)_j (w_2)_{k+1-j}}{x^{w_1+j-1} y^{w_2+k+1-j}}$$
$$\times \left[(\log(xz) - \gamma - \psi(k+w_1+w_2)) F\left(\left.\begin{matrix} 1, k+1+w_2-j \\ k+1+w_1+w_2 \end{matrix} \right| 1-\frac{x}{y}\right) \right.$$
$$\left. + F'\left(\left.\begin{matrix} 1, k+1+w_2-j \\ k+1+w_1+w_2 \end{matrix} \right| 1-\frac{x}{y}\right) \right], \quad \textit{for } \Re(s+w_1+w_2) > 0,$$

where

$$F'\left(\begin{matrix}\alpha,\beta\\ \delta\end{matrix}\middle| z\right) \equiv \frac{\mathrm{d}}{\mathrm{d}\alpha}F\left(\begin{matrix}\alpha,\beta\\ \delta\end{matrix}\middle| z\right)$$

and

$$\lim_{\eta\to 0}\langle \mathbf{f}_{n,n},\varphi_\eta^{(n)}\rangle = (-1)^n\sum_{j=0}^{n}\binom{n}{j}\int_0^\infty \frac{(w_1)_j(w_2)_{n-j}f_{n,n}(t)}{(t+xz)^{j+w_1}(t+yz)^{n-j+w_2}}\,\mathrm{d}t$$

for $\Re(s+w_1+w_2)+n>1$.

Proof. It is a straightforward generalization of the proof of in [5, Lemma 4] from real to complex values of s, w_1 and w_2. □

Using the aforementioned Lemma 1 and Lemmas 1 and 2 of [10], we prove the following theorems.

Theorem 3. *Let $f(t)$ be a locally integrable function on $[0,\infty)$ which satisfies (9) with $0<\Re s\leqslant 1$, $s\neq 1$. Then, for $xz, yz\in\mathbb{C}\setminus\mathbb{R}^-\cup\{0\}$, $\Re w_1, \Re w_2>0$, $\Re(s+w_1+w_2)+K>1$ and $n=1,2,3,\ldots$,*

$$\int_0^\infty \frac{f(t)}{(t+xz)^{w_1}(t+yz)^{w_2}}\,\mathrm{d}t = \sum_{k=K}^{n-1}\frac{A_k}{z^{k+w_1+w_2+s-1}} + \sum_{k=0}^{n-1}\frac{B_k}{z^{k+w_1+w_2}} + R_n(w_1,w_2;z), \tag{16}$$

where empty sums must be understood as zero. The coefficients A_k, B_k and C_k are defined by

$$A_k \equiv a_k\frac{\Gamma(1-s-k)\Gamma(w_1+w_2+s+k-1)}{\Gamma(w_1+w_2)x^{w_1+s+k-1}y^{w_2}}F\left(\begin{matrix}1-s-k,w_2\\ w_1+w_2\end{matrix}\middle| 1-\frac{x}{y}\right)$$

and

$$B_k \equiv \frac{(-1)^k M[f;k+1]}{k!}\sum_{j=0}^{k}\binom{k}{j}\frac{(w_1)_j(w_2)_{k-j}}{x^{w_1+j}y^{k+w_2-j}}.$$

The remainder term is defined by

$$R_n(w_1,w_2;z) \equiv \sum_{j=0}^{n}\binom{n}{j}(w_1)_j(w_2)_{n-j}\int_0^\infty \frac{f_{n,n}(t)\,\mathrm{d}t}{(t+xz)^{j+w_1}(t+yz)^{n+w_2-j}}, \tag{17}$$

where $f_{n,n}(t)$ is defined in (13) and verifies $R_n(w_1,w_2;z)=\mathcal{O}(z^{-n-w_1-w_2-s+1})$ when $z\to\infty$.

Proof. The proof of formulas (16) and (17) is a straightforward generalization of the proof of Theorem 3 in [5] from real to complex parameters. It uses previous Lemma 1 instead of Lemma 4 as used there, which is valid only for real parameters. The proof of the asymptotic character of expansion (16) is a straightforward generalization of the proof of [Theorem 5 in [5]] to the case of complex parameters (Theorem 5 in [5] is proved only for real parameters). □

Theorem 4. *Let $f(t)$ be a locally integrable function on $[0,\infty)$ which satisfies (9) with $s=1$. Then, for $xz, yz \in \mathbb{C}\setminus\mathbb{R}^- \cup \{0\}$, $\Re w_1, \Re w_2 > 0$, $\Re(w_1+w_2)+K>0$ and $n=1,2,3,\ldots,$*

$$\int_0^\infty \frac{f(t)}{(t+xz)^{w_1}(t+yz)^{w_2}}\,dt = \sum_{k=K}^{-1}\frac{A_k}{z^{w_1+w_2+k}} + \sum_{k=0}^{n-1}\frac{(-1)^k}{k!z^{k+w_1+w_2}}$$
$$\times[B_k(\log(xz)-\gamma-\psi(k+w_1+w_2))+B_k'+C_k]$$
$$+R_{n,s}(w_1,w_2;z), \tag{18}$$

where empty sums must be understood as zero,

$$A_k \equiv a_k\frac{\Gamma(-k)\Gamma(w_1+w_2+k)}{\Gamma(w_1+w_2)x^{w_1+k}y^{w_2}}F\left(\begin{matrix}-k,w_2\\ w_1+w_2\end{matrix}\middle| 1-\frac{x}{y}\right),$$

$$B_k \equiv \frac{a_k}{(k+w_1+w_2)}\sum_{j=0}^{k+1}\binom{k+1}{j}\frac{(w_1)_j(w_2)_{k+1-j}}{x^{w_1+j-1}y^{w_2+k+1-j}}F\left(\begin{matrix}1,k+1+w_2-j\\ k+1+w_1+w_2\end{matrix}\middle| 1-\frac{x}{y}\right),$$

$$B_k' \equiv \frac{a_k}{(k+w_1+w_2)}\sum_{j=0}^{k+1}\binom{k+1}{j}\frac{(w_1)_j(w_2)_{k+1-j}}{x^{w_1+j-1}y^{w_2+k+1-j}}F'\left(\begin{matrix}1,k+1+w_2-j\\ k+1+w_1+w_2\end{matrix}\middle| 1-\frac{x}{y}\right)$$

and

$$C_k \equiv b_{k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{(w_1)_j(w_2)_{k-j}}{x^{w_1+j}y^{k+w_2-j}},$$

where b_{k+1} is given in (15). The remainder term $R_{n,s}(w_1,w_2;z)$ is given in (17) and verifies $R_n(w_1,w_2;z)=\mathcal{O}(z^{-n-w_1-w_2}\log z)$ when $z\to\infty$.

Proof. The proof of the previous formulas is a straightforward generalization of the proof of Theorem 4 in [5] from real to complex parameters. Again, it uses previous Lemma 1 instead of Lemma 4 as used there. And again, the proof of the asymptotic character of expansion (18) is a straightforward generalization of the proof of [Theorem 5 in [5]] to the case of complex parameters. □

2.2. Error bounds

The following two propositions show that, if $\exists c_n > 0$ such that $|f_n(t)| \leqslant c_n t^{-n-\Re s}$ $\forall t \in [0,\infty)$, then bounds for the remainder terms in the expansions above can be calculated in terms of the constant c_n.

Proposition 1. *If, for $0<\Re s<1$, the remainder $f_n(t)$ in expansion (9) of the function $f(t)$ satisfies the bound $|f_n(t)| \leqslant c_n t^{-n-\Re s}$ $\forall t\in[0,\infty)$ for some positive constant c_n, then the remainder*

$R_n(w;z)$ in expansion (11) satisfies

$$|R_n(w;z)| \leqslant \frac{c_n\pi(|w|)_n\Gamma(n+\Re w+\Re s-1)h(z,w)}{\Gamma(n+\Re s)\Gamma(n+\Re w)|\sin(\pi\Re s)||z|^{n+\Re w+\Re s-1}} \times F\left(\begin{matrix}1-\Re s, n+\Re s+\Re w-1\\ (n+\Re w+1)/2\end{matrix}\middle|\sin^2\left(\frac{\mathrm{Arg}(z)}{2}\right)\right) \tag{19}$$

and the remainder $R_{n,s}(w_1,w_2;z)$ in expansion (16) satisfies

$$|R_n(w_1,w_2;z)| \leqslant \frac{c_n\pi(|w_1|+|w_2|)_n\Gamma(n+\Re(w_1+w_2+s)-1)h(xz,w_1)h(yz,w_2)}{\Gamma(n+\Re s)\Gamma(n+\Re w_1+\Re w_2)|\sin(\pi\Re s)|\,|vz|^{n+\Re(w_1+w_2+s)-1}} \times F\left(\begin{matrix}1-\Re s, n+\Re(s+w_1+w_2)-1\\ (n+\Re w_1+\Re w_2+1)/2\end{matrix}\middle|\frac{1}{2}\left(1-\frac{u}{|vz|}\right)\right),$$

where

$$v \equiv \mathrm{Min}\{|x|,|y|\}, \quad u \equiv \mathrm{Min}\{\Re(xz),\Re(yz)\} \tag{20}$$

and

$$h(z,w) \equiv \begin{cases} 1 & \text{if } \mathrm{Arg}(z)\Im w \geqslant 0,\\ e^{|\mathrm{Arg}(z)\Im w|} & \text{if } \mathrm{Arg}(z)\Im w < 0. \end{cases} \tag{21}$$

Proof. Eq. (19) is proved in [10, Proposition 1]. The second bound is obtained using (19), the inequalities $|t+xz|^2$, $|t+yz|^2 \geqslant t^2+2ut+|vz|^2$ in definition (17) of $R_{n,s}(w_1,w_2;z)$, formula [16, Eq. 7, p. 309] and the equality

$$\sum_{k=0}^{n}\binom{n}{k}(|w_1|)_k(|w_2|)_{n-k} = (|w_1|+|w_2|)_n. \qquad \square \tag{22}$$

Proposition 2. *If, for $\Re s = 1$, each remainder $f_n(t)$ in expansion (9) of the function $f(t)$ satisfies the bound $|f_n(t)| \leqslant c_n t^{-n-1} \forall t \in [0,\infty)$ for some positive constant c_n, then the remainder $R_n(w;z)$ in expansions (11) and (14) satisfies*

$$|R_n(w;z)| \leqslant \frac{\bar{c}_n\pi(|w|)_n\Gamma(n+\Re w-1/2)h(z,w)}{\Gamma(n+1/2)\Gamma(n+\Re w)|z|^{n+\Re w-1/2}} \times F\left(\begin{matrix}1/2, n+\Re w-1/2\\ (n+\Re w+1)/2\end{matrix}\middle|\sin^2\left(\frac{\mathrm{Arg}(z)}{2}\right)\right) \equiv \mathbf{R}_n^{(1)}(w;z), \tag{23}$$

where $\bar{c}_n \equiv \mathrm{Max}\{c_n, c_{n-1}+|a_{n-1}|\}$ and

$$|R_n(w;z)| \leqslant \frac{(|w|)_n}{|z|^{n+\Re w}}\left\{\frac{\varepsilon(c_{n-1}+|a_{n-1}|)+c_n}{(n-1)!\Theta(z,\varepsilon)^{n+\Re w}} + \frac{c_n}{n!}\left|1+\frac{\varepsilon}{z}\right|^{-n-\Re w}\right.$$

$$\times\left[\log|z| + \frac{(n+\Re w)[(2\varepsilon+\Re z+|\Re z|)(|z|^{-1}-1)+(|\Re z|-\Re z)\log|z|]}{2(n+\Re w+1)|z+\varepsilon|}H_1\right.$$

$$\left.\left. + \frac{4\varepsilon+\Re z+|\Re z|-2\varepsilon|z|}{2\varepsilon(n+\Re w+1)|z|}H_0 + \frac{2|\varepsilon+z|H_{-1}}{\varepsilon((n+\Re w)^2-1)|z|}\right]\right\} h(z,w) \equiv \mathbf{R}_n^{(2)}(w;z), \quad (24)$$

where ε is an arbitrary positive number,

$$H_m \equiv F\left(\begin{array}{c|} 2-m, n+\Re w+m \\ (n+\Re w+3)/2 \end{array} \sin^2\left(\frac{\mathrm{Arg}(z+\varepsilon)}{2}\right)\right) \quad (25)$$

and

$$\Theta(z,\varepsilon) \equiv \begin{cases} 1 & \text{if } \Re z \geqslant 0, \\ |\sin(\mathrm{Arg}(z)| & \text{if } \varepsilon \geqslant -\Re z > 0, \\ |1+\varepsilon/z| & \text{if } -\Re z > \varepsilon > 0. \end{cases} \quad (26)$$

For large z and fixed n, the optimum value for ε is given approximately by

$$\varepsilon^2 = \frac{c_n}{n(c_{n-1}+|a_{n-1}|)}\left[\frac{2H_{-1}}{(n+\Re w)^2-1} + \frac{(\Re z+|\Re z|)H_0}{2(n+\Re w+1)|z|}\right]. \quad (27)$$

The remainder $R_n(w_1,w_2;z)$ in expansions (16) *and* (18) *satisfies*

$$|R_n(w_1,w_2;z)| \leqslant \mathbf{R}_n^{(i)}(w_1+w_2;xz) + \mathbf{R}_n^{(i)}(w_1+w_2;yz) \quad (28)$$

with either $i=1$ or 2. If x, y, z, w, w_1 and w_2 are positive real numbers, then

$$|R_n(w;z)| \leqslant [n\varepsilon(c_{n-1}+|a_{n-1}|) + c_n(S_n(z,\varepsilon,w)+T_n(z,\varepsilon,w))]\frac{(w)_n}{n!z^{n+w}}, \quad (29)$$

where ε is again an arbitrary positive number,

$$S_n(z,\varepsilon,w) \equiv \mathrm{Min}\left\{\frac{nz\left[(\varepsilon+z)^{n+w-1}-z^{n+w-1}\right]}{\varepsilon(n+w-1)(\varepsilon+z)^{n+w-1}}, \psi(n+1)+\gamma\right\}$$

and

$$T_n(z,\varepsilon,w) \equiv \frac{z^{n+w}}{(n+w)(\varepsilon+z)^{n+w}}F\left(n+w,1;n+w+1;\frac{z}{\varepsilon+z}\right).$$

For large z and fixed n, the optimum value for ε is given by

$$\varepsilon = \frac{c_n}{n(c_{n-1}+|a_{n-1}|)}. \quad (30)$$

The remainder $R_n(w_1,w_2;z)$ in expansions (16) *and* (18) *satisfies bound* (29) *replacing w by w_1+w_2 and z by vz, with $v \equiv \mathrm{Min}\{|x|,|y|\}$.*

Proof. Bounds (23)–(27) for $R_n(w;z)$ are obtained in [10, Proposition 2].

On the other hand, bound (28) is obtained using the inequality $|t+xz|^{-\Re w_1}|t+yz|^{-\Re w_2} \leqslant |t+xz|^{-\Re w_1-\Re w_2} + |t+yz|^{-\Re w_1-\Re w_2}$ in definition (17) of $R_n(w_1,w_2;z)$ and formulas (22)–(24).

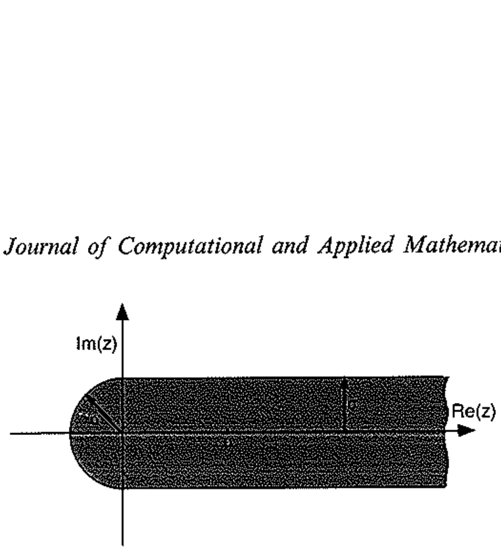

Fig. 1. Analyticity region W for the function $g(z)$ considered in Proposition 3.

The bounds for $R_n(w;z)$ and $R_n(w_1,w_2;z)$ for real positive x, y, w, w_1, w_2 and z have been obtained in [9, Propositions 2 and 4]. □

In the following two propositions, two families of functions $f(t)$ are given, which verify the bound $|f_n(t)| \leqslant c_n t^{-n-\Re s}$ required in the aforementioned Propositions 1 and 2. Moreover, the constant c_n is easily obtained from $f(t)$. These propositions are proved in Lemmas 5 and 6 and Corollaries 1 and 2 in [5].

Proposition 3. *If $f(t)$ verifies (9) with $\Re s > 0$ and the function $g(u) \equiv u^{-s-K} f(u^{-1})$ is a bounded analytic function in the region W of the complex z-plane comprised by the points situated at a distance $< \sigma$ from the positive real axis (see Fig. 1), then $R_{n,s}(w;z)$ and $R_{n,s}(w_1,w_2;z)$ satisfy the bounds given in Propositions 1 and 2 with $c_n = Cr^{-n}$, where C is a bound of $|g(z)|$ in W and $0 < r < \sigma$. Moreover, the expansions given in Theorems 1 and 2 are convergent when the parameter $|z|$ is longer than the inverse of the width of the region W: when $\sigma|z| \geqslant 1$ if $\Re w < 1$ or $\sigma|z| > 1$ if $\Re w \geqslant 1$. The expansions given in Theorems 3 and 4 are convergent when the parameter $|vz|$, with $v \equiv \mathrm{Min}\{|x|,|y|\}$, is longer than the inverse of the width of that region: when $\sigma|vz| \geqslant 1$ if $\Re w_1 + \Re w_2 < 1$ or $\sigma|vz| > 1$ if $\Re w_1 + \Re w_2 \geqslant 1$.*

For $\Re s = 1$, the convergence of these expansions requires also that $\lim_{n\to\infty} n^{w-1} a_n z^{-n} = 0$ or $\lim_{n\to\infty} n^{w_1+w_2-1} a_n (vz)^{-n} = 0$, respectively.

Proposition 4. *If expansion (9) of $f(t)$ verifies the error test, then $R_n(w;z)$ and $R_{n,s}(w_1,w_2;z)$ satisfy the bounds given in Propositions 1 and 2 replacing c_n by $|a_n|$ and c_{n-1} by 0. Moreover, the expansions given in Theorems 1 and 2 are convergent when the coefficients a_n in the asymptotic expansion (9) verify $\lim_{n\to\infty} n^{w-1} a_n z^{-n} = 0$. The expansions given in Theorems 3 and 4 are convergent when $\lim_{n\to\infty} n^{w_1+w_2-1} a_n (vz)^{-n} = 0$, $v \equiv \mathrm{Min}\{|x|,|y|\}$.*

3. Asymptotic expansions of the Lauricella function F_D^r

In order to obtain asymptotic expansions of $F_D^r(a,b_1,\ldots,b_r;c;x_1,\ldots,x_r)$ for $|x_1| \to \infty$ and/or $|x_2| \to \infty$ we just apply Theorems 1–4 to integral (2) or (3). Error bounds for the remainders are obtained from Propositions 3 and 4, because the functions $f_{r-1}(t)$ and $f_r(t)$, which define F_D^r in (2) and (3), belong to the class of functions considered in those propositions.

Corollary 1. *For* $\Re a > 0$, $\Re(c-a) > 0$, $1+a-b_1 \notin \mathbb{Z}$, $|\mathrm{Arg}(z)| < \pi$ *and* $x_i - 1 \notin \mathbb{R}^+ \cup \{0\}$ *if* $\Re b_i \geqslant 1$ *for* $2 \leqslant i \leqslant r$,

$$F_D^r(a,b_1,\ldots,b_r;c;1-z,x_2,\ldots,x_r)$$
$$=\frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)}\left\{\sum_{k=0}^{n-1}\frac{\Gamma(b_1+k)}{\Gamma(b_1)}\frac{(-1)^k C_k}{k!z^{k+b_1}}\right.$$
$$\left.+\frac{\pi}{\Gamma(b_1)\sin(\pi s)}\sum_{k=0}^{n-K-1}\frac{(-1)^k\Gamma(k+a)}{\Gamma(k-b_1+a+1)}\frac{B_k}{z^{k+a}}+R_n(a,b_1,\ldots,b_r;c;z,x_2,\ldots,x_r)\right\}, \quad (31)$$

where $K \equiv \mathrm{Int}(\Re(1-b_1+a))$ *and the coefficients* B_k *are defined in* (8). *Coefficients* C_k *are given by*

$$C_k \equiv \frac{\Gamma(k+c-a)\Gamma(a-k-b_1)}{\Gamma(c-b_1)\prod_{j=2}^r(1-x_j)^{b_j}}$$
$$\times F_D^{r-1}\left(k-a+c;b_2,\ldots,b_r;c-b_1,\frac{x_2}{x_2-1},\ldots,\frac{x_r}{x_r-1}\right), \quad (32)$$

where $s \equiv Fr(\Re(1+a-b_1)) + i\Im(1+a-b_1)$.

If $\Re(1+a-b_1) \notin \mathbb{Z}$ *and* $n \in \mathbb{N}$, *a bound for the remainder is given by*

$$R_n(a,b_1,\ldots,b_r;c;z,x_2,\ldots,x_r)|$$
$$\leqslant \frac{c_n\pi(|b_1|)_n\Gamma(n+\Re(b_1+s)-1)}{\Gamma(n+\Re s)\Gamma(n+\Re b_1)|\sin(\pi\Re s)|}$$
$$\times\frac{h(z,b_1)}{|z|^{n+\Re(b_1+s)-1}}F\left(\left.\begin{matrix}1-\Re s, n+\Re(s+b_1)-1\\(n+\Re b_1+1)/2\end{matrix}\right|\sin^2\left(\frac{\mathrm{Arg}(z)}{2}\right)\right), \quad (33)$$

where $h(z,b_1)$ *was defined in* (21) *and we can take* $c_n = |B_{n-K}|$ *if the following conditions over the parameters hold:*

$$a,c,b_1,\ldots,b_r \in \mathbb{R},\ \sum_{j=1}^r b_j \leqslant c,\ b_j \geqslant 0,\ \Re(1-x_j) > 0,\ j=2,3,4,\ldots,r. \quad (34)$$

In any case, we can take $c_n = Cr^{-n}$, *where*

$$C \geqslant \mathrm{Sup}_{u\in W}\left|(1+u)^{b_1+b_2+\cdots+b_r-c}\left(\prod_{j=2}^r(1+(1-x_j)u)^{-b_j}\right)\right|, \quad (35)$$

W *is the region considered in Proposition* 3 *for* $g(u) = u^{b_1-a-1}f_{r-1}(u^{-1})$ *with*

$$0 < r < \mathrm{Min}\{1, |1-x_2|^{-1}\xi(b_2),\ldots,|1-x_r|^{-1}\xi(b_r)\}, \quad (36)$$

$$\xi(b) \equiv \begin{cases}1 & \text{if } b \notin \mathbb{Z}^- \cup \{0\},\\ +\infty & \text{if } b \in \mathbb{Z}^- \cup \{0\}.\end{cases}$$

On the other hand, if $\Re(1+a-b_1)\in\mathbb{Z}$ *and* $n\in\mathbb{N}$, *two bounds for the remainder are given by*

$$|R_n(a,b_1,\dots,b_r;c;z,x_2,\dots,x_r)| \leqslant \frac{\bar{c}_n\pi(|b_1|)_n\Gamma(n+\Re b_1-1/2)h(z,b_1)}{\Gamma(n+1/2)\Gamma(n+\Re b_1)|z|^{n+\Re b_1-1/2}} \times F\left(\left.\begin{matrix}1/2, n+\Re b_1-1/2\\ (n+\Re b_1+1)/2\end{matrix}\right|\sin^2\left(\frac{\mathrm{Arg}(z)}{2}\right)\right) \equiv \mathbf{R}_n^{(1)}(B_n,b_1;z) \qquad (37)$$

and

$$|R_n(a,b_1,\dots,b_r;c;z,x_2,\dots,x_r)| \leqslant \frac{(|b_1|)_n}{|z|^{n+\Re b_1}}\left\{\frac{c_n}{n!}\left|1+\frac{\varepsilon}{z}\right|^{-n-\Re b_1}\left[\log|z| + \frac{(n+\Re b_1)[(2\varepsilon+\Re z+|\Re z|)(|z|^{-1}-1)+(|\Re z|-\Re z)\log|z|]}{2(n+\Re b_1+1)|z+\varepsilon|}H_1 + \frac{4\varepsilon+\Re z+|\Re z|-2\varepsilon|z|}{2\varepsilon(n+\Re b_1+1)|z|}H_0 + \frac{2|\varepsilon+z|}{\varepsilon((n+\Re b_1)^2-1)|z|}H_{-1}\right] + \frac{\varepsilon(c_{n-1}+|B_{n-K-1}|)+c_n}{(n-1)!\Theta(z,\varepsilon)^{n+\Re b_1}}\right\}h(z,b_1) \equiv \mathbf{R}_n^{(2)}(Cr^{-n},B_n,b_1;z). \qquad (38)$$

In these formulas, $\bar{c}_n=\mathrm{Max}\{|B_{n-K}|,|B_{n-K-1}|\}$ *with* $c_n=|B_{n-K}|$ *and* $c_{n-1}=0$ *if conditions* (34) *hold. In any case, we can take* $\bar{c}_n=\mathrm{Max}\{c_n,c_{n-1}+|B_{n-K-1}|\}$, *with* $c_n=Cr^{-n}$ *given above. In* (38), ε *is an arbitrary positive number,* $\Theta(z,\varepsilon)$ *is given in* (26) *and* H_k *is given in* (25) *setting* $w=b_1$. *For large* z *and fixed* n, *the optimum value for* ε *is given approximately by* (27) *setting* $w=b_1$. *Moreover, expansion* (31) *is convergent when* $\mathrm{Max}\{|1-x_2|\xi(b_2)^{-1},\dots,|1-x_r|\xi(b_r)^{-1},1\}<|z|$.

Proof. To obtain expansion (31), we just apply Theorem 1 to integral (2) with $f(t)=f_r(t)$ given in (4), $a_k=B_{k-K}$ given in (8), $w=b_1$ and s and K given above. After the change of variable $t=u(1-u)^{-1}$, the mellin transform of $f_r(t)$ reads

$$M[f_r;k+1]=\left(\prod_{i=2}^{r}(1-x_i)^{-b_i}\right)\int_0^1\frac{u^{k+c-a-1}}{(1-u)^{k+b_1-a+1}}\prod_{i=2}^{r}\left(1+\frac{x_i}{1-x_i}u\right)^{-b_i}\,du.$$

Then, the first term in (32) follows from (1).

If (34) holds, then, by [8, Lemmas 3 and 4], the function $f_r(t)$ verifies the error test. Therefore, by Proposition 4, the remainder in expansion (31) verifies the bounds given in Propositions 1 and 2 with $c_n=|B_{n-K}|$, $c_{n-1}=0$. In any case, by Proposition 3, the remainder in expansion (31) verifies the bounds given in Propositions 1 and 2 with $c_n=Cr^{-n}$, C and r verifying (35) and (36), respectively. Therefore, bounds (33), (37) and (38) hold.

Now introduce the bound

$$|B_n| \leqslant C \begin{pmatrix} \Re(b_1 - c) \\ n \end{pmatrix} [\mathrm{Max}\{1, |1 - x_2|\xi(b_2)^{-1}, \ldots, |1 - x_r|\xi(b_r)^{-1}\}]^n,$$

where C is independent of n, in (33) and (37). Then, using (36), we obtain that $\lim_{n\to\infty} R_n(a, b_1, \ldots, b_r; c; z, x_2, \ldots, x_r) = 0$ if $\mathrm{Max}\{|1 - x_2|\xi(b_2)^{-1}, \ldots, |1 - x_r|\xi(b_r)^{-1}, 1\} < |z|$. □

Corollary 2. *For* $\Re a$, $\Re(c - a) > 0$, $x_i - 1 \notin \mathbb{R}^+ \cup \{0\}$ *if* $\Re b_i \geqslant 1$ *for* $2 \leqslant i \leqslant r$, $1 + a - b_1 \in \mathbb{Z}$ *and* $|\mathrm{Arg}(z)| < \pi$,

$$\begin{aligned} &F_D^r(a, b_1, \ldots, b_r; c; 1 - z, x_2, \ldots, x_r) \\ &= \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \left\{ \sum_{k=0}^{n-1} \frac{(-1)^k (b_1)_k}{k! z^{k+b_1}} [B_{k+b_1-a}(\log(z) - \gamma - \psi(k + b_1)) + C_k] \right. \\ &\left. + \sum_{k=0}^{b_1-a-1} B_k \frac{\Gamma(k+a)\Gamma(b_1 - k - a)}{\Gamma(b_1) z^{k+a}} + R_n(a, b_1, \ldots, b_r; c; z, x_2, \ldots, x_r) \right\}, \end{aligned} \tag{39}$$

where the coefficients B_k *are given in* (8) *and the coefficients* C_k *are given by*

$$\begin{aligned} C_k \equiv &\frac{(-1)^{k+b_1-a-1}\Gamma(k+c-a)}{\Gamma(c-b_1)(k+b_1-a)! \prod_{i=2}^r (1-x_i)^{b_i}} \left\{ [\psi(k+c-a) \right. \\ &- \psi(k+b_1-a+1)] F_D^{r-1}\left(k - a + c, b_2, \ldots, b_r; c - b_1; \frac{x_2}{x_2 - 1}, \ldots, \frac{x_r}{x_r - 1}\right) \\ &\left. + F_D^{r-1\prime}\left(k - a + c, b_2, \ldots, b_r; c - b_1; \frac{x_2}{x_2 - 1}, \ldots, \frac{x_r}{x_r - 1}\right)\right\} \\ &+ B_{k+b_1-a}(\gamma + \psi(k+1)), \end{aligned} \tag{40}$$

where $F_D^{r-1\prime}$ *represents the derivative of the Lauricella function* F_D^{r-1} *with respect to the first parameter.*

For $n \in \mathbb{N}$, *two bounds for the remainder are given by* (37) *and* (38) *in Corollary* 1 *replacing* B_{n-K} *by* B_{n-a+b_1}. *Expansion* (39) *is convergent if* $\mathrm{Max}\{|1 - x_2|\xi(b_2)^{-1}, \ldots, |1 - x_r|\xi(b_r)^{-1}, 1\} < |z|$, *where* $\xi(b_i)$ *is defined in* (36).

Proof. To obtain expansion (39), just apply Theorem 2 to integral (2) with $f(t) = f_r(t)$ given in (4), $s = 1$, $K = a - b_1$, $a_k = B_{k+b_1-a}$ and $w = b_1$.

On the other hand, the coefficients B_k in expansion (7) of $f_r(t)$ may be written as

$$B_k = \frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\,[t^{b_1-a-1}f_r(t^{-1})]_{t=0}.$$

Using the Cauchy formula for the derivative of an analytic function, we obtain

$$B_{k+b_1-a} = \frac{\mathrm{d}^{k+b_1-a}}{\mathrm{d}t^{k+b_1-a}}\left[\frac{t^k(1-t)^{b_1-a-1}}{(k+b_1-a)!}f_r\left(\frac{t}{1-t}\right)\right]_{t=1}. \tag{41}$$

The coefficient C_n in (39) is just b_{n+1} given by (15) with $a_n = B_{n+b_1-a}$. The Mellin transform $M[f_r; z+1]$ in formula (15) is given by

$$\frac{\Gamma(z+c-a)\Gamma(a-z-b_1)}{\Gamma(c-b_1)\prod_{j=2}^r(1-x_j)^{b_j}}F_D^{r-1}\left(z-a+c, b_2,\ldots,b_r; c-b_1; \frac{x_2}{x_2-1},\ldots,\frac{x_r}{x_r-1}\right).$$

Then, when $z\to n$, there are two singular terms in the limit in (15): $B_{n+b_1-a}/(z-n)$ and $\Gamma(a-z-b_1)$. Setting $z=n+\eta$, expanding these terms at $\eta=0$ and using (41) we obtain (40).

Bounds (37) and (38) are obtained as in Corollary 1 (but using only Proposition 2). □

Corollary 3. *For* $\Re a>0$, $\Re(c-a)>0$, $1+a-b_1-b_2\notin\mathbb{Z}$, $|\mathrm{Arg}(xz)|<\pi$ *and* $|\mathrm{Arg}(yz)|<\pi$,

$$\begin{aligned} &F_D^r(a,b_1,\ldots,b_r;c;1-xz,1-yz,x_3,\ldots,x_r)\\ &=\frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)}\left\{\sum_{k=0}^{n-1}\frac{(-1)^kB_k}{k!z^{k+b_1+b_2}}+\sum_{k=0}^{n-K-1}\frac{\Gamma(b_1+b_2-a-k)\Gamma(a+k)}{\Gamma(b_1+b_2)x^{a+k-b_2}y^{b_2}z^{k+a}}A_k\right.\\ &\left.F\left(\begin{matrix}b_1+b_2-a-k, b_2\\ b_1+b_2\end{matrix}\middle|1-\frac{x}{y}\right)+R_n(a,b_1,\ldots,b_r;c;xz,yz,x_3,\ldots,x_r)\right\}, \end{aligned} \tag{42}$$

where $K\equiv \mathrm{Int}(\Re(1+a-b_1-b_2))$ *and the coefficients* A_k *are defined in* (6). *Coefficients* D_k *are given by*

$$\begin{aligned} B_k \equiv &\sum_{j=0}^{k}\binom{k}{j}\frac{(b_1)_j(b_2)_{k-j}}{x^{b+j}y^{k+c-j}}\frac{\Gamma(k+c-a)\Gamma(-k+a-b_1-b_2)}{\Gamma(c-b_1-b_2)\prod_{i=3}^r(1-x_i)^{b_i}}\\ &\times F_D^{r-2}\left(k+c-a, b_3,\ldots,b_r; c-b_1-b_2; \frac{x_3}{x_3-1},\ldots,\frac{x_r}{x_r-1}\right), \end{aligned} \tag{43}$$

where $s\equiv Fr(\Re(1-b_1-b_2+a))+\mathrm{i}\Im(1-b_1-b_2+a)$.

If $\Re(1+a-b_1-b_2)\notin\mathbb{Z}$ and $n\geqslant 0$, a bound for the remainder is given by

$$
\begin{aligned}
&|R_n(a,b_1,\ldots,b_r;c;xz,yz,x_3,\ldots,x_r)|\\
&\quad\leqslant \frac{c_n\pi(|b_1|+|b_2|)_n\Gamma(n+\Re(b_1+b_2+s)-1)h(xz,b_1)h(yz,b_2)}{\Gamma(n+\Re s)\Gamma(n+\Re(b_1+b_2))|\sin(\pi\Re s)||vz|^{n+\Re(b_1+b_2+s)-1}}\\
&\quad\times F\left(\begin{matrix}1-\Re s, n+\Re(s+b_1+b_2)-1\\ (n+\Re(b_1+b_2)+1)/2\end{matrix}\,\middle|\,\frac{1}{2}\left(1-\frac{u}{|vz|}\right)\right),
\end{aligned}\tag{44}
$$

where $h(z,w)$ was defined in (21) and u and v were defined in (20).

In formula (44) we can take $c_n=|A_{n-K}|$ if

$$a,c,b_1,\ldots,b_r\in\mathbb{R},\quad \sum_{j=1}^{r}b_j\leqslant c,\quad b_j\geqslant 0,\ \Re(1-x_j)>0,\ j=3,4,5,\ldots,r. \tag{45}$$

In any case, we can take $c_n=C$, where

$$C>\mathrm{Sup}_{u\in W}\left|(1+u)^{\sum_{j=1}^{r}b_j-c}\left(\prod_{j=3}^{r}(1+(1-x_j)u)^{-b_j}\right)\right| \tag{46}$$

and W is the region considered in Proposition 3 for $g(u)=u^{b_1+b_2-a-1}f_{r-1}(u^{-1})$.

On the other hand, if $\Re(1+a-b_1-b_2)\in\mathbb{Z}$ and $n\geqslant 0$, the remainder in expansion (42) satisfies

$$
\begin{aligned}
|R_n(a,b_1,\ldots,b_r;c;xz,yz,x_3,\ldots,x_r)|&\leqslant \mathbf{R}_n^{(1)}(A_n,b_1+b_2;1-xz)\\
&\quad+\mathbf{R}_n^{(1)}(A_n,b_1+b_2;1-yz)
\end{aligned}\tag{47}
$$

or

$$
\begin{aligned}
|R_n(a,b_1,\ldots,b_r;c;xz,yz,x_3,\ldots,x_r)|&\leqslant \mathbf{R}_n^{(2)}(C,A_n,b_1+b_2;1-xz)\\
&\quad+\mathbf{R}_n^{(2)}(C,A_n,b_1+b_2;1-yz),
\end{aligned}\tag{48}
$$

where $\mathbf{R}_n^{(1)}$ and $\mathbf{R}_n^{(2)}$ were defined in (37) and (38). Moreover, expansion (42) is convergent if $v|z|>\mathrm{Max}\{|1-x_3|\xi(b_3)^{-1},\ldots,|1-x_r|\xi(b_r)^{-1},1\}$.

Proof. To obtain expansion (42), just apply Theorem 3 to integral (3) with $f(t)=f_{r-1}(t)$ given in (4), $a_k=A_{k-K}$ given in (6), $w_1=b_1$, $w_2=b_2$ and s and K given above. The calculation of coefficient C_k in formula (16) of Theorem 3 requires the calculation of the Mellin transform $M[f_{r-1};k+1]$. After trivial manipulations, and using (1) we obtain

$$
\begin{aligned}
M[f_{r-1};k+1]&=\frac{\Gamma(k+c-a)\Gamma(-k+a-b_1-b_2)}{\Gamma(c-b_1-b_2)\prod_{i=3}^{r}(1-x_i)^{b_i}}\\
&\quad\times F_D^{r-2}\left(k+c-a,b_3,\ldots,b_r;c-b_1-b_2;\frac{x_3}{x_3-1},\ldots,\frac{x_r}{x_r-1}\right)
\end{aligned}
$$

and (43) follows.

If (45) holds, then, by [8, Lemmas 3 and 4], the function $f_{r-1}(t)$ verifies the error test. Therefore, by Proposition 4, the remainder in expansion (42) verifies the bounds given in Propositions 1 and 2 with $c_n = |A_{n-K}|$, $c_{n-1} = 0$. In any case, by Proposition 3, the remainder in expansion (42) verifies the bounds given in formula (28) of Proposition 2 with $w_1 = b_1$, $w_2 = b_2$, $r = 1$ and $c_n = C$, C verifying (46). Therefore, bound (47) holds.

Finally, using the same argument that we used at the end of the proof of the Corollary 1, we obtain that, if $\mathrm{Min}\{|xz|,|yz|\} > \mathrm{Max}\{|1-x_3|\xi(b_3)^{-1},\dots,|1-x_r|\xi(b_r)^{-1},1\}$, then $\lim_{n\to\infty} R_n^{(i)} = 0$ and therefore, $\lim_{n\to\infty} R_n(a,b_1,\dots,b_r;c;xz,yz,x_3,\dots,x_r) = 0$. □

Corollary 4. *For* $\Re a > 0$, $\Re(c-a) > 0$, $1+a-b_1-b_2 \in \mathbb{Z}$, $|\mathrm{Arg}(xz)| < \pi$ *and* $|\mathrm{Arg}(yz)| < \pi$,

$$
\begin{aligned}
&F_D^r(a,b_1,\dots,b_r;c;1-xz,1-yz,x_3,\dots,x_r)\\
&=\frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)}\left\{\sum_{k=0}^{n-1}\frac{(-1)^k C_{k+1}}{k!z^{k+b_1+b_2}} + \sum_{k=0}^{b_1+b_2-a-1}\frac{\Gamma(-k+b_1+b_2-a)\Gamma(k+a)A_k}{\Gamma(b_1+b_2)x^{k+a-b_2}y^{b_2}z^{a+k}}\right.\\
&\quad\times F\left(\begin{matrix} b_1+b_2-a-k, b_2\\ b_1+b_2\end{matrix}\middle| 1-\frac{x}{y}\right) + \sum_{k=0}^{n-1}\frac{(-1)^k A_{k+b_1+b_2-a}}{k!(k+b_1+b_2)z^{k+b_1+b_2}}\\
&\quad\left.\times[D_k(\log(xz)-\gamma-\psi(k+b_1+b_2)) + D_k'] + R_n(a,b_1,\dots,b_r;c;xz,yz,x_3,\dots,x_r)\right\},
\end{aligned}\tag{49}
$$

where coefficients A_k *are given in* (6) *and coefficients* D_k, D_k' *and* C_{k+1} *are given by*

$$
D_k \equiv \sum_{j=0}^{k+1}\binom{k+1}{j}\frac{(b_1)_j(b_2)_{k+1-j}}{x^{b_1+j-1}y^{b_2+k+1-j}}F\left(\begin{matrix}1,k+1+b_2-j\\ k+1+b_1+b_2\end{matrix}\middle| 1-\frac{x}{y}\right),
$$

$$
D_k' \equiv \sum_{j=0}^{k+1}\binom{k+1}{j}\frac{(b_1)_j(b_2)_{k+1-j}}{x^{b_1+j-1}y^{b_2+k+1-j}}F'\left(\begin{matrix}1,k+1+b_2-j\\ k+1+b_1+b_2\end{matrix}\middle| 1-\frac{x}{y}\right)
$$

and

$$
\begin{aligned}
C_{k+1} \equiv &\left\{\frac{\Gamma(k+c-a)(-1)^{k+b_1+b_2-a-1}}{(k+b_1+b_2-a)!\Gamma(c-b_1-b_2)\prod_{i=3}^r(1-x_i)^{b_i}}\right.\\
&\times\left[(\psi(k+c-a)-\psi(k+b_1+b_2-a+1))\right.\\
&\times F_D^{r-2}\left(k+c-a,b_3,\dots,b_r;c-b_1-b_2;\frac{x_3}{x_3-1},\dots,\frac{x_r}{x_r-1}\right)\\
&\left.+F_D^{r-2\prime}\left(k+c-a,b_3,\dots,b_r;c-b_1-b_2;\frac{x_3}{x_3-1},\dots,\frac{x_r}{x_r-1}\right)\right]\\
&\left.+A_{k+b_1+b_2-a}(\gamma+\psi(k+1))\right\}\sum_{j=0}^{k}\binom{k}{j}\frac{(b_1)_j(b_2)_{k-j}}{x^{b_1+j}y^{k+b_2-j}}.
\end{aligned}\tag{50}
$$

respectively.

The remainder in expansion (49) *satisfies bounds* (47) *and* (48). *Moreover, the expansion* (3) *is convergent if* $\mathrm{Min}\{|xz|,|yz|\} > \mathrm{Max}\{|1-x_3|\xi(b_3)^{-1},\ldots,|1-x_r|\xi(b_r)^{-1},1\}$.

Proof. To obtain expansion (49), just apply Theorem 4 to integral (3) with $f(t)=f_{r-1}(t)$ given in (4), $s=1$, $K=a-b_1-b_2$, $a_k=A_{k+b_1+b_2-a}$, $w_1=b_1$ and $w_2=b_2$.

On the other hand, the coefficients A_k in expansion (5) of $f_{r-1}(t)$ may be written as

$$A_k=\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}t^k}[t^{b_1+b_2-a-1}f_{r-1}(t^{-1})]_{t=0}.$$

Using the Cauchy formula for the derivative of an analytic function, we obtain

$$A_{k+b+c-a}=\frac{\mathrm{d}^{k+b_1+b_2-a}}{\mathrm{d}t^{k+b_1+b_2-a}}\left[\frac{t^k(1-t)^{b_1+b_2-a-1}}{(k+b_1+b_2-a)!}f_{r-1}\left(\frac{t}{1-t}\right)\right]_{t=1}. \tag{51}$$

Coefficients C_{n+1} in (49) are given by b_{n+1} in (15) with $a_n=A_{n+b_1+b_2-a}$. The Mellin transform $M[f_{r-1};z+1]$ in formula (15) is given by

$$\frac{\Gamma(z+c-a)\Gamma(a-z-b_1-b_2)}{\Gamma(c-b_1-b_2)\prod_{i=3}^{r}(1-x_i)^{b_i}}F_D^{r-2}\left(z+c-a,b_3,\ldots,b_r;c-b_1-b_2;\frac{x_3}{x_3-1},\ldots,\frac{x_r}{x_r-1}\right).$$

Then, when $z\to n$, there are two singular terms in the limit in (15): $A_{n+b_1+b_2-a}/(z-n)$ and $\Gamma(a-z-b_1-b_2)$. Setting $z=n+\eta$, expanding these terms at $\eta=0$ and using (51) we obtain (50).

The bounds (47) and (48) are obtained as in Corollary 3. □

Table 1
Approximation supplied by (31) and error bounds given by (33) (Parameter values: $a=2$, $b_1=2.25$, $b_2=0.05$, $b_3=0.25$, $c=3.5$, $x_2=0.05$, $x_3=0.1$, $\mathrm{Arg}(x_1)=-3\pi/4$)

$\|x_1\|$	F_D^3	First or. approx.	Relative error	Relative er. bound	Second or. approx.	Relative error	Relative er. bound
20	0.0084304–0.0027687i	0.0011291–0.0022707i	0.198	0.388	0.00089615–0.0027532i	0.019	0.036
50	0.00012195–0.0005966i	0.00014874–0.00055883i	0.076	0.126	0.000123723–0.0005963i	0.0029	0.0048
100	2.7056e-5–0.000175i	3.1065e-5–0.0001696i	0.037	0.055	2.718e-5–0.0001747i	7.15e-4	1.05e-3
200	5.8769e-6–4.9249e-5i	6.4469e-6–4.856e-5i	0.018	0.025	5.8855e-6–4.9248e-5i	1.74e-4	2.39e-4
500	7.66297e-7–8.86727e-6i	8.0748e-7–8.8201e-6i	0.007	0.009	7.66537e-7–8.86726e-6i	2.7e-5	3.4e-5
1000	1.62758e-7–2.37149e-6i	1.682654e-7–2.36535e-6i	0.0035	0.0042	1.62774e-7–2.37149e-6i	6.6e-6	8.1e-6

Table 2
Approximation supplied by (31) and error bounds given by (33) (Parameter values: $a=1.5$, $b_1=2.25$, $b_2=0.05$, $b_3=0.25$, $b_4=0.1$ $c=3.5$, $x_2=0.25$, $x_3=0.25$, $x_4=0.1$, $\mathrm{Arg}(x_1)=\pi$)

$\|x_1\|$	F_D^4	First or. approx.	Relative error	Relative er. bound	Second or. approx.	Relative error	Relative er. bound
20	0.00761345	0.00460439	0.39	0.5	0.00730585	0.04	0.06
50	0.00224492	0.00190431	0.15	0.18	0.00223068	0.0063	0.0096
100	0.0008509	0.00078796	0.074	0.088	0.00084959	0.0015	0.0023
200	0.0003145	0.00030312	0.036	0.042	0.00031442	0.00038	0.00056
500	8.23946e-5	8.1217e-5	0.014	0.016	8.239e-5	6.e-5	8.8e-5
1000	2.9578e-5	2.9383e-5	0.007	0.008	2.9577e-5	1.49e-5	2.2e-5

Table 3
Approximation supplied by (39) and error bounds given by Min{(37),(38)} (Parameter values: $a=2$, $b_1=3$, $b_2=0.5$, $b_3=0.3$, $c=4$, $x_2=0.2$, $x_3=0.3$, $\mathrm{Arg}(x_1)=\pi$)

$\|x_1\|$	F_D^3	First or. approx.	Relative error	Relative er. bound	Second or. approx.	Relative error	Relative er. bound
20	0.00095376	0.0009806	0.028	0.118	0.000956	0.0024	0.01
50	0.00017512	0.00017612	0.0057	0.029	0.00017515	0.0002	0.001
100	4.6326e-5	4.6403e-5	0.00167	0.0099	4.6327e-5	2.87e-5	1.88e-4
200	1.19702e-5	1.19759e-5	4.78e-4	3.46e-3	1.19703e-5	4.12e-6	3.28e-5
500	1.960135e-6	1.960311e-6	9.e-5	8.65e-4	1.960136e-6	3.09e-7	3.29e-6
1000	4.944542e-7	4.944666e-7	2.5e-5	3.04e-4	4.9445423e-7	4.3e-8	5.8e-7

Table 4
Approximation supplied by (39) and error bounds given by Min{(37),(38)} (Parameter values: $a=1$, $b_1=3$, $b_2=0.5$, $b_3=0.7$, $b_4=0.25$, $c=5.25$, $x_2=0.3$, $x_3=0.1$, $x_4=0.3$, $\mathrm{Arg}(x_1)=-\pi/2$)

$\|x_1\|$	F_D^4	First or. approx.	Relative error	Relative er. bound	Second or. approx.	Relative error	Relative er. bound
20	0.00320879– 0.02424707i	0.003088756– 0.0243536i	0.006	0.037	0.0031949– 0.0242319i	0.00084	0.0054
50	0.00055794– 0.0099322i	0.00055321– 0.009935i	0.0005	0.004	0.0005578– 0.009932i	2.76e-5	0.0002
100	0.00014352– 0.00498964i	0.000143146– 0.0049898i	8.15e-5	6.88e-4	0.000143512- 0.00498963i	2.03e-6	1.97e-5
200	3.6401e-5– 0.00249847i	3.6373e-5– 0.00249848i	1.18e-5	1.22e-4	3.64e-5– 0.00249847i	1.46e-7	1.75e-6
500	5.875349e-6– 9.9988e-4i	5.874485e-6– 9.99988e-4i	9.e-7	1.2e-5	5.875348e-6– 9.9988e-4i	4.4e-9	7.1e-8
1000	1.473154e-6– 4.99983e-5i	1.47309e-6– 4.99983e-5i	1.27e-7	2.19e-6	1.4731537e-6– 4.99983e-5i	3.05e-10	6.28e-9

Table 5
Approximation supplied by (42) and error bounds given by (44) (Parameter values: $a=2$, $b_1=1.25$, $b_2=1.05$, $b_3=0.25$, $c=3.5$, $x_3=0.1$, $\mathrm{Arg}(x_1)=\pi$, $\mathrm{Arg}(x_2)=\pi$)

$\lvert x_1\rvert$, $\lvert x_2\rvert$	F_D^3	First or. approx.	Relative error	Relative er. bound	Second or. approx.	Relative error	Relative er. bound
20, 40	0.00147675	0.00125527	0.15	0.63	0.00146013	0.011	0.057
50, 80	0.000364323	0.000341343	0.063	0.178	0.000363878	0.002	0.0066
200, 100	7.664583e-5	7.460133e-5	0.0267	0.109	7.661635e-5	3.84e-4	0.002
400, 300	1.506288e-5	1.4906e-5	0.01	0.02	1.506206e-5	5.82e-5	1.32e-4
800, 600	4.02922e-6	4.008577e-6	0.005	0.009	4.029166e-6	1.43e-5	3.11e-5
1500, 1000	1.331099e-6	1.327269e-6	2.877e-3	6.45e-3	1.331093e-6	4.6e-6	1.22e-5

Table 6
Approximation supplied by (42) and error bounds given by Min{(47),(48)} (Parameter values: $a=1$, $b_1=2+0.5i$, $b_2=2-1.15i$, $b_3=0.25$, $b_4=0.25$, $c=3$, $x_3=0.25$, $x_4=0.1$, $\mathrm{Arg}(x_1)=\pi$, $\mathrm{Arg}(x_2)=\pi$)

$\lvert x_1\rvert$, $\lvert x_2\rvert$	F_D^4	First or. approx.	Relative error	Relative er. bound	Second or. approx.	Relative error	Relative er. bound
20, 50	0.00973631–0.000433265i	0.00973647–0.00043335i	1.9e-5	4.5e-5	0.00973632–0.00043327i	1.3e-6	4.8e-6
30, 70	0.006805412–0.00033956i	0.00680545–0.000339585i	6.5e-6	1.2e-5	0.006805414–0.00033956i	3.02e-7	8.1e-7
50, 90	0.00478282–0.000331843i	0.00478283–0.000331849i	2.06e-6	4.66e-6	0.004782825–0.000331843i	6.29e-8	1.99e-7
100, 200	0.002256123–1.397424e-4i	0.002256124–1.397427e-4i	2.27e-7	2.75e-7	0.002256123–1.397424e-4i	3.38e-9	4.3e-9
500, 700	5.5346533e-4–4.956593e-5i	5.5346533e-4–4.9565931e-5i	3.04e-9	3.87e-9	5.5346533e-4–4.956593e-5i	1.1e-11	2.05e-11
1000, 1100	3.1368459e-4–3.4142407e-5i	3.1368459e-4–3.4142407e-5i	5.4e-10	8.9e-10	3.1368459e-4–3.4142407e-5i	2.27e-12	3.e-12

3.1. Numerical experiments

Tables 1–8 show numerical experiments on the approximation and the accuracy of the error bounds supplied by Corollaries 1–4. In these tables, the second column represents the integral $F_D^r(a,b_1,\ldots,b_r;c;x_1,\ldots,x_r)$, for $r=3$ and 4. The third and sixth columns represent the approximation given by Corollaries 1, 2, 3 or 4 for $n=1$ and 2, respectively. The fourth and seventh columns represent the respective relative errors, and the fifth and last columns are the respective relative error bounds given in those corollaries.

Table 7
Approximation supplied by (49) and error bounds given by Min{(47), (48)} (Parameter values: $a=1$, $b_1=2$, $b_2=1$, $b_3=0.1$, $c=2.5$, $x_3=0.25$, $\mathrm{Arg}(x_1)=-\pi/2$, $\mathrm{Arg}(x_2)=\pi$)

$\lvert x_i \rvert$	F_D^3	First or. approx.	Relative error	Relative er. bound	Second or. approx.	Relative error	Relative er. bound
30, 40	0.00953984–0.0143915i	0.0095395–0.0143911i	2.83e-5	0.0015	0.00953984–0.0143915i	8.4e-7	0.0001
50, 70	0.0056695–0.0084865i	0.0056695–0.00848646i	6.69e-6	0.0004	0.0056695–0.0084865i	1.22e-7	1.6e-5
100, 150	0.00280578–0.00411922i	0.00280578–0.0041192i	9.24e-7	5.45e-5	0.00280578–0.00411922i	9.34e-9	1.1e-6
400, 300	7.073525e-4–0.00140188i	7.073525e-4–0.00140188i	3.27e-8	1.5e-5	7.073525e-4–0.00140188i	1.9e-10	1.5e-7
700, 600	4.068369e-4–7.60952e-4i	4.068369e-4–7.60952e-4i	5.86e-9	2.45e-6	4.068369e-4–7.60952e-4i	5.38e-11	1.21e-8
1200, 1000	2.369669e-4–4.4902823e-5i	2.369667e-4–4.4902823e-5i	1.28e-9	6.99e-7	2.369667e-4–4.4902823e-5i	1.88e-11	2.08e-9

Table 8
Approximation supplied by (49) and error bounds given by Min{(47), (48)} (Parameter values: $a=1$, $b_1=2$, $b_2=1$, $b_3=0.25$, $b_4=0.25$, $c=3.5$, $x_3=0.25$, $x_4=0.1$, $\mathrm{Arg}(x_1)=\pi$, $\mathrm{Arg}(x_2)=\pi$)

$\lvert x_i \rvert$	F_D^4	First or. approx.	Relative error	Relative er. bound	Second or. approx.	Relative error	Relative er.bound
20, 50	0.01672667	0.016724	1.6e-4	0.001	0.0167265	7.44e-6	8.26e-5
30, 70	0.01166452	0.01166385	5.75e-5	4.13e-4	0.0116645	1.65e-6	2.1e-5
50, 90	0.00788302	0.00788289	1.76e-5	2.24e-4	0.00788302	2.6e-7	7.5e-6
100, 200	0.00382053	0.00382052	2.45e-6	2.8e-5	0.00382053	7.86e-9	4.3e-7
500, 700	8.8603367e-4	8.8603364e-4	3.6e-8	1.5e-6	8.8603367e-4	1.6e-10	6.2e-9
1000, 1100	4.834595e-5	4.834596e-5	6.26e-9	5.6e-7	4.834595e-5	3.23e-11	1.5e-9

Acknowledgements

This work has been financially supported by the *DGCYT (BFM2000-0803)*, and the *Gobierno de Navarra (Res. 134/2002)*. Special thanks should also be given to the department of mathematics of the University Carlos III of Madrid for their help and hospitality.

References

[1] A. Al-Zamel, V.K. Tuan, S.L. Kalla, Generalized elliptic-type integrals and asymptotic formulas, Appl. Math. Comput. 114 (1) (2000) 13–25.

[2] B.C. Carlson, The hypergeometric function and the R-function near their branch points, Rend. Sem. Mat. Univ. Politec. Torino, Fascicolo speciale (1985) 63–89.
[3] B.C. Carlson, Elliptic integrals: symmetric and symbolic integration, Atti Convegni Lincei 147 (1998) 161–181.
[4] J. Carvalho e Silva, H.M. Srivastava, Asymptotic representations for a class of multiple hypergeometric functions with a dominant variable, Integral Transform. Spec. Funct. 11 (2001) 137–150.
[5] C. Ferreira, J.L. López, Asymptotic expansions of generalized Stieltjes transforms of algebraically decaying functions, Stud. Appl. Math. 108 (2002) 187–215.
[6] C.M. Joshi, N.L. Joshi, Expansions of Appell and Lauricella functions, Tamkang J. Math. 27 (4) (1996) 271–279.
[7] G. Lauricella, Sulle funzioni ipergeometriche a piú variabili, Rend. Circ. mat. Palermo 7 (1893) 111–158.
[8] J.L. López, Asymptotic expansions of symmetric standard elliptic integrals, SIAM J. Math. Anal. 31 (4) (2000) 754–775.
[9] J.L. López, Uniform asymptotic expansions of symmetric elliptic integrals, Constr. Approx. 17 (4) (2001) 535–559.
[10] J.L. López, C. Ferreira, Asymptotic expansions of the generalized Epstein–Hubbell integral, IMA J. Appl. Math. 67 (2002) 301–319.
[11] A.M. Mathai, R.K. Saxena, Various practical problems in probability and statistics where Lauricella's confluent hypergeometric function appears naturally, Gadnita Sandesh 1 (1–2) (1987) 41–48.
[12] B.L. Mathur, On some results involving Lauricella functions, Bull. Calcutta Math. Soc. 70 (4) (1978) 221–227.
[13] W. Miller Jr., Lie theory and Lauricella functions F_D, J. Math. Phys. 13 (1972) 1393–1399.
[14] N.P. Molnar, Approximation of the Lauricella hypergeometric functions F_D^N by branched continued fractions, J. Math. Sci. 90 (5) (1998) 2381–2384.
[15] S.H. Ong, P.A. Lee, Probabilistic interpretations of a transformation of Lauricella's hypergeometric function of n variables and an integral of product of Laguerre polynomials, Internat. J. Math. Statist. Sci. 9 (1) (2000) 5–13.
[16] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series, Vol. 1, Gordon and Breach Science Pub., London, 1990.
[17] M. Saigo, On properties of the Lauricella hypergeometric function F_D, Bull. Central Res. Inst. Fukuoka Univ. 104 (1988) 13–31.
[18] C.K. Sharma, I.J. Singh, Some integrals involving the Lauricella functions and the multivariable H-function, Indian. J. Pure Appl. Math 21 (6) (1990) 597–604.
[19] H.M. Srivastava, Some further reduction formulas for certain classes of generalized multiple hypergeometric series arising in physical, astrophysical, and quantum chemical applications, Astrophys. Space Sci. 181 (2) (1991) 195–202.
[20] H.M. Srivastava, Some Lauricella Multiple Hypergeometric Series Associated with the Product of Several Bessel Functions, Constantin Carathéodory: An International Tribute, Vol. I and II, World Sci. Pub., Teaneck, NJ, 1991, pp. 1304–1341.
[21] H.M. Srivastava, S.P. Goyal, Fractional Derivatives of the H-function of several variables, J. Math. Anal. Appl. 112 (1985) 641–651.
[22] H.M. Srivastava, P.W. Karlsson, Multiple Gaussian Hypergeometric Series, Wiley, New York, 1985.
[23] N.M. Temme, Special functions: an introduction to the classical functions of mathematical physics, Wiley, New York, 1996.
[24] R. Wong, Distributional derivation of an asymptotic expansion, Proc. Amer. Math. Soc. 80 (1980) 266–270.
[25] R. Wong, Asymptotic Approximations of Integrals, Academic Press, New York, 1989.