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Incomplete gamma functions for large values of their variables

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Abstract

We consider the incomplete gamma functions $\Gamma(a,z)$ and $\gamma(a,z)$ for large values of their variables a and z. We derive four complementary asymptotic expansions which approximate these functions for large $|a|$ and $|z|$ with $|\mathrm{Arg}(a)| \leqslant \pi$ and $|\mathrm{Arg}(z)| < \pi$. Three of these expansions are given in terms of decreasing powers of $a-z$ and are not valid near the transition point $a=z$. A fourth expansion is given in terms of decreasing powers of a and error functions and is valid for a near z. These expansions have a simpler structure than other expansions previously given in the literature.

Keywords: Incomplete gamma function; Uniform asymptotic expansions

1. Introduction

Incomplete gamma functions are defined by the integrals

$$\Gamma(a,z) \equiv \int_z^\infty e^{-t} t^{a-1}\,\mathrm{d}t, \quad z \in \mathbb{C} \setminus \mathbb{R}^-,\ a \in \mathbb{C}, \tag{1}$$

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and

$$\gamma(a,z) \equiv \int_0^z e^{-t} t^{a-1}\,\mathrm{d}t, \quad z \in \mathbb{C} \setminus \mathbb{R}^-,\ \Re a > 0. \tag{2}$$

In both definitions, the integration paths do not cross the negative real axis. These functions play an important role in applied mathematics, physics, statistics and other scientific disciplines.

In this paper we are concerned with their approximation for large values of a and z. The asymptotic expansion of $\Gamma(a,z)$ and $\gamma(a,z)$ when either a or z is large is well known and quite straightforward (see for example [11, Section 11.2.1]). When both, a and z are large, the situation is different. We can find several uniform asymptotic expansions in the literature that are more complicated than the non-uniform expansions. That complexity is more evident at the transition point $a = z$, where the asymptotic structure of $\Gamma(a,z)$ and $\gamma(a,z)$ suffers an abrupt change. Several authors have obtained different asymptotic expansions of these functions for large a and z valid in different regions of $\mathrm{Arg}(a)$ and of $\mathrm{Arg}(z)$. Some of them are not valid near the transition point [2,6,12]. Other expansions are valid at that point [7,9,10,12], [11, Section 11.2.4], and involve a complementary error function which takes account of that abrupt change in the asymptotic behaviour. Many of those expansions are based on standard uniform asymptotic methods for integrals and are complicated owing to the difficult changes of variables involved in those methods. But recently, Paris has suggested a different uniform asymptotic treatment for $\Gamma(a,z)$ and $\gamma(a,z)$ which simplifies the analysis [8]. His method does not require any change of variable, but just an appropriate factorization of the integrand and an expansion of one of the factors. Paris has obtained in this way simpler uniform asymptotic expansions of $\Gamma(a,z)$ and $\gamma(a,z)$ valid at the transition point $a = z$ which involve a complementary error function. Moreover, Paris explains how the expansions given in [2,6,12] (not valid at $a = z$) can be derived from his expansions.

On the other hand, Temme and López have proposed in specific examples (Bernoulli, Euler, Charlier, Laguerre and Jacobi polynomials) a modification of the saddle point method which simplifies the analysis [3,5]. This simplification consists of just expanding the non-exponential part of the integrand at the saddle point(s) of the exponential part avoiding any change of variable. In this way, simpler uniform asymptotic expansions than the classical ones are obtained for those polynomials.

In this paper we combine both simplifier ideas (the one of Paris and the one of Temme and López) in order to derive quite simple asymptotic expansions of $\Gamma(a,z)$ and $\gamma(a,z)$ for large a and z. Our strategy consists of: (i) a factorization of the integrand in $\Gamma(a,z)$ or $\gamma(a,z)$ in an exponential factor times another factor and (ii) an expansion of this second factor at the asymptotically relevant point of the exponential factor (saddle point or end point). The main benefit of this procedure is the derivation of easy asymptotic expansions with easily computable coefficients. We require three or four different expansions to cover the region $(a,z) \in \mathbb{C} \times (\mathbb{C} \setminus \mathbb{R}^-)$ depending on the value of $a - z$. Because of the relation $\Gamma(a,z) + \gamma(a,z) = \Gamma(a)$ valid for $a \in \mathbb{C} \setminus \mathbb{Z}^-$, we will only derive expansions for either $\Gamma(a,z)$ or $\gamma(a,z)$ at every one of the four different regions mentioned above.

In the following section we consider $\Gamma(a,z)$ and derive an expansion valid in the region $\Re a < 0$ and away from the transition point $a = z$. In Section 3 we derive an expansion valid in the region $\Re a > -1$, $\Re(a) > \Re(z)$ and away from the transition point $a = z$. In Section 4 we derive an expansion valid in the region $\Re a > -1$, $\Re(a) < \Re(z)$ and away from the transition point $a = z$. In Section 5 we obtain an asymptotic expansion valid around the transition point $a = z$. In Section 6 we present some conclusions and a few comments. In what follows $\mathrm{Arg}(z)$ denotes the principal argument of z: $\mathrm{Arg}(z) \in (-\pi, \pi]$.

2. $\Re$(a) < 0 and away from the transition point

We consider in this section the function $\Gamma(a,z)$ and an integral representation different from (1) [11, p. 280, Eq. (11.13)]:

$$\Gamma(a,z) = \frac{e^{-z}}{\Gamma(1-a)} \int_0^\infty \frac{e^{-zt} t^{-a}}{1+t} \, \mathrm{d}t, \quad \Re a < 0,\ \Re z > 0. \tag{3}$$

After the change of variable $t = u/z$, it reads

$$\Gamma(a,z) = \frac{z^a e^{-z}}{\Gamma(1-a)} \int_0^{z\infty} \frac{e^{-u} u^{-a}}{u+z} \, \mathrm{d}u, \quad \Re a < 0,\ \Re z > 0.$$

The integrand is an analytic function of u in the sector $\mathrm{Arg}(u) \in (0, \mathrm{Arg}(z))$. In that sector $\Re u > 0$. Then, using the Cauchy's Residue Theorem we have

$$\Gamma(a,z) = \frac{z^a e^{-z}}{\Gamma(1-a)} \int_0^\infty \frac{e^{-u} u^{-a}}{u+z} \, \mathrm{d}u, \quad \Re a < 0,\ \left|\mathrm{Arg}(z)\right| < \pi/2.$$

The right-hand side above is an analytic function of z in the sector $|\mathrm{Arg}(z)| < \pi$ and therefore defines the analytic continuation of $\Gamma(a,z)$ to that sector. Thus, in what follows, we assume $|\mathrm{Arg}(z)| < \pi$.

We perform the change of variable $u = -at$ in the above integral:

$$\Gamma(a,z) = \left(-\frac{a}{z}\right)^{1-a} \frac{e^{-z}}{\Gamma(1-a)} \int_0^{-\infty/a} \frac{e^{at} t^{-a}}{1-\alpha t} \, \mathrm{d}t, \quad \alpha \equiv \frac{a}{z}.$$

The integrand is an analytic function of t in the sector $\mathrm{Arg}(t) \in (0, \mathrm{Arg}(-1/a))$ and has a simple pole at $t = \alpha^{-1}$. This pole is in that sector if $\mathrm{Arg}(z)/\mathrm{Arg}(a) \geqslant 1$. In that sector $\Re(at) < 0$. Then, using the Cauchy's Residue Theorem we have

$$\Gamma(a,z) = \Gamma_0(a,z) - \frac{2\pi i R}{\Gamma(1-a)} e^{i\pi a \,\mathrm{sgn}(\mathrm{Arg}(a))} \mathrm{sgn}\big(\mathrm{Arg}(a)\big), \tag{4}$$

where

$$\Gamma_0(a,z) \equiv \left(-\frac{a}{z}\right)^{1-a} \frac{e^{-z}}{\Gamma(1-a)} \int_0^\infty \frac{e^{a(t-\log t)}}{1-\alpha t}\,\mathrm{d}t \tag{5}$$

and

$$R \equiv \begin{cases} 1 & \text{if } \mathrm{Arg}(z)/\mathrm{Arg}(a) > 1, \\ 1/2 & \text{if } \mathrm{Arg}(z)/\mathrm{Arg}(a) = 1, \\ 0 & \text{if } \mathrm{Arg}(z)/Arg(a) < 1. \end{cases}$$

The unique saddle point of the exponent $t - \log t$ is located at $t = 1$. This is the unique asymptotically relevant point and then, the asymptotic approximation of $\Gamma_0(a,z)$ requires an expansion of the non-exponential part of the integrand in (5) at the point $t = 1$ [5]:

$$\frac{1}{1-\alpha t} = \frac{1}{1-\alpha} \sum_{k=0}^{n-1} \left(\frac{1-t}{\alpha-1}\right)^k \alpha^k + \frac{1}{1-\alpha t}\left(\frac{1-t}{\alpha-1}\right)^n \alpha^n. \tag{6}$$

Substituting (6) in (5) and interchanging summation and integration we obtain

$$\Gamma_0(a,z) = -z^a e^{-z}\left[\sum_{k=0}^{n-1} \frac{\Phi_k(a)}{(a-z)^{k+1}} + R_n(a,z)\right], \tag{7}$$

where the functions $\Phi_k(a)$ are given by

$$\Phi_k(a) \equiv \frac{(-a)^{k+1-a}}{\Gamma(1-a)} \int_0^\infty e^{a(t-\log t)}(t-1)^k\,\mathrm{d}t = \sum_{j=0}^{k} \binom{k}{j} \frac{(1-a)_j}{a^{j-k}}. \tag{8}$$

The first few functions $\Phi_k(a)$ are detailed in Table 1. The real part of the exponent in the above integrand has an absolute maximum at $t = 1$. Applying the Laplace method and using [1, Eq. 6.1.37] we deduce that $\Phi_k(a) = \mathcal{O}(a^{k/2})$ as $a \to \infty$. Therefore, the sequence $(a-z)^{-k-1}\Phi_k(a)$ in (7) is an asymptotic sequence for large a and z if $z - a = \mathcal{O}(a^{1/2+\varepsilon})$ for any $\varepsilon > 0$: $(a-z)^{-k-1}\Phi_k(a) = \mathcal{O}(a^{k/2}(a-z)^{-k-1})$.

Table 1
First few functions $\Phi_k(a)$ defined in (8) and used in (7)

k	$\Phi_k(a)$	k	$\Phi_k(a)$
0	1	4	$24 - 26a + 3a^2$
1	1	5	$120 - 154a + 35a^2$
2	$2 - a$	6	$720 - 1044a + 340a^2 - 15a^3$
3	$6 - 5a$	7	$5040 - 8028a + 3304a^2 - 315a^3$

Table 2
Numerical experiments about the relative errors in the approximation of $\Gamma(a,z)$ using (7) for real a and z with a away from z and different values of n

a	z	$n=1$	$n=3$	$n=5$	$n=7$	$n=9$
−50	20	0.004	0.0004	0.00003	4.0e−6	5.4e−7
−100	50	0.002	0.00008	3.3e−6	1.5e−7	8.3e−9
−200	100	0.001	0.00002	4.1e−7	9.2e−9	2.4e−10
−500	200	0.0004	4.1e−6	3.6e−8	3.6e−10	4.0e−12
−1000	500	0.0002	8.8e−7	3.2e−9	1.4e−11	1.8e−12

Table 3
Numerical experiments about the relative errors in the approximation of $\Gamma(a,z)$ using (7) for complex a and z with a away from z and different values of n

z	a	$n=1$	$n=3$	$n=5$	$n=7$	$n=9$
$10-30i$	$-5+5i$	0.02	0.0006	0.00002	2.0e−6	2.1e−7
$50-30i$	$-20+5i$	0.009	0.0001	5.1e−6	1.9e−7	1.0e−8
$100-30i$	$-50+5i$	0.004	0.00005	9.2e−7	1.9e−8	5.5e−10
$200-30i$	$-100+5i$	0.002	0.00001	1.2e−7	1.3e−9	1.7e−11
$500-30i$	$-200+5i$	0.001	2.4e−6	7.3e−9	2.9e−11	1.3e−13

The remainder term in (7) is given by

$$R_n(a,z) \equiv \frac{(-a)^{n+1-a}}{(a-z)^n\Gamma(1-a)} \int_0^\infty \frac{e^{a(t-\log t)}}{at-z}(t-1)^n \, \mathrm{d}t. \tag{9}$$

The real part of the exponent has an absolute maximum at $t=1$. Applying Laplace's method to this integral we deduce that $R_n(a,z) = \mathcal{O}(a^{n/2}(a-z)^{-n-1})$ as $a \to \infty$. Therefore, (7) is an asymptotic expansion for large a and z provided that $a - z = \mathcal{O}(a^{1/2+\varepsilon})$, $\varepsilon > 0$.

Apart from the explicit formula (8), integrating by parts in (8) we find the recurrence:

$$\Phi_k(a) = k\Phi_{k-1}(a) + a(1-k)\Phi_{k-2}(a), \quad k = 2, 3, 4, \ldots.$$

From this recurrence and $\Phi_0(a) = \Phi_1(a) = 1$, it may be shown by induction that $\Phi_k(a) = \mathcal{O}(a^{\lfloor k/2 \rfloor})$ as $a \to \infty$. Therefore, the asymptotic properties of the sequence $(a-z)^{-k-1}\Phi_k(a)$ are better than those deduced by using the Laplace method:

$$(a-z)^{-k-1}\Phi_k(a) = \mathcal{O}\big(a^{\lfloor k/2 \rfloor}(a-z)^{-k-1}\big).$$

3. $\Re(a) > -1$, $\Re(a) > \Re(z)$ and away from the transition point

We consider in this section the function $\gamma(a+1,z)$. From (2) we have

$$\gamma(a+1,z) = \int_0^z u^a\, e^{-u}\, \mathrm{d}u, \quad \Re a > -1. \tag{10}$$

After the change of variable $u = z(1-t)$ and defining again $\alpha \equiv a/z$ we have,

$$\gamma(a+1,z) = z^{a+1} e^{-z} \int_0^1 e^{zf(t)}\, \mathrm{d}t, \quad f(t) \equiv t + \alpha \log(1-t). \tag{11}$$

The real part of the exponent in the integrand of (11) reads

$$\Re\big[zf(t)\big] = t\Re z + \Re a \log(1-t). \tag{12}$$

It has a relative maximum at the point $t = 1 - \Re a/\Re z$. Therefore, for $\Re a > \Re z > 0$, the maximum value of the exponent in the integrand of (11) is attained at $t = 0$. This indicates that the main contribution of the integrand to the integral (11) comes from the point $t = 0$ and the integral may be approximated replacing $f(t)$ in (11) by its Taylor polynomial at $t = 0$ [5]. We split the function $f(t)$ in two terms:

$$f(t) = f_1(t) + f_2(t) \tag{13}$$

where $f_1(t) \equiv (1-\alpha)\, t$ is the Taylor polynomial of degree 1 of $f(t)$ at $t = 0$ and

$$f_2(t) \equiv f(t) - (1-\alpha)t = \alpha\big[t + \log(1-t)\big]. \tag{14}$$

With this splitting, the integral in (11) reads

$$\gamma(a+1,z) = z^{a+1} e^{-z} \int_0^1 e^{(z-a)t} e^{zf_2(t)}\, \mathrm{d}t. \tag{15}$$

We expand

$$e^{zf_2(t)} = e^{at}(1-t)^a = \sum_{k=0}^{\infty} c_k(a) t^k, \quad |t| < 1, \tag{16}$$

with

$$c_k(a) \equiv \sum_{j=0}^{k} \frac{(-a)_j}{j!} \frac{a^{k-j}}{(k-j)!}. \tag{17}$$

Table 4
First few coefficients $c_k(a)$ defined in (17) and used in (20)

k	$c_k(a)$	k	$c_k(a)$
0	1	4	$(-2a+a^2)/8$
1	0	5	$(-6a+5a^2)/30$
2	$-a/2$	6	$(-24a+26a^2-3a^3)/144$
3	$-a/3$	7	$(-120a+154a^2-35a^3)/840$

The first few coefficients $c_k(a)$ are detailed in Table 4. From the differential equation $(1-t)f' + atf = 0$, satisfied by $e^{zf_2(t)}$, we derive:

$$c_{k+1}(a) = \frac{1}{k+1}\bigl[kc_k(a) - ac_{k-1}(a)\bigr]. \tag{18}$$

From $f_2(0) = f_2'(0) = 0$ we see that [4, Lemma 2.1]

$$c_0 = 1, \qquad c_1 = 0, \qquad c_k(a) = \mathcal{O}\bigl(a^{\lfloor k/2 \rfloor}\bigr), \quad |a| \to \infty. \tag{19}$$

The expansion in (16) converges uniformly with respect to t on $[0,1]$ (when $\Re a > 0$). Hence we can substitute (16) in (15) to obtain, after interchanging summation and integration,

$$\gamma(a+1, z) = e^{-z} z^{a+1} \sum_{k=0}^{\infty} c_k(a) \Phi_k(z-a), \tag{20}$$

where the coefficients $c_k(a)$ are given in (17) or (18)–(19) and

$$\Phi_k(z-a) \equiv \int_0^1 e^{(z-a)t} t^k \,\mathrm{d}t = \frac{\gamma(k+1, a-z)}{(a-z)^{k+1}}. \tag{21}$$

Taking into account that

$$\begin{aligned} \Phi_k(z-a) &= \frac{\mathrm{d}^k}{\mathrm{d}(z-a)^k} \int_0^1 e^{(z-a)t}\,\mathrm{d}t = \frac{\mathrm{d}^k}{\mathrm{d}(z-a)^k}\left[\frac{e^{(z-a)}-1}{z-a}\right] \\ &= \frac{k!}{(a-z)^{k+1}} - e^{z-a} \sum_{j=0}^{k} \binom{k}{j} \frac{j!}{(a-z)^{j+1}} \end{aligned} \tag{22}$$

and that e^{z-a} is exponentially small for $\Re z < \Re a$, we see that

$$\Phi_k(z-a) = \mathcal{O}\bigl((z-a)^{-k-1}\bigr), \quad |z-a| \to \infty. \tag{23}$$

Table 5
First few functions $\Phi_k(z-a)$ defined in (22) and used in (20)

k	$\Phi_k(z-a)$	k	$\Phi_k(z-a)$
0	$\frac{1}{a-z}-\frac{e^{z-a}}{a-z}$	2	$\frac{2}{(a-z)^3}-\frac{e^{z-a}}{a-z}\left[1+\frac{2}{a-z}+\frac{2}{(a-z)^2}\right]$
1	$\frac{1}{(a-z)^2}-\frac{e^{z-a}}{a-z}\left[1+\frac{1}{a-z}\right]$	3	$\frac{6}{(a-z)^4}-\frac{e^{z-a}}{a-z}\left[1+\frac{3}{a-z}+\frac{6}{(a-z)^2}+\frac{6}{(a-z)^3}\right]$

Table 6
Numerical experiments about the relative errors in the approximation of $\gamma(a,z)$ using (20) for real a and z with a away from z and different values of n

a	z	$n=1$	$n=3$	$n=6$	$n=9$	$n=12$
50	10	0.03	0.001	0.0004	0.00003	3.2e−6
100	40	0.02	0.001	0.0003	9.4e−6	6.3e−8
200	80	0.01	0.0003	0.00003	2.9e−7	1.1e−6
500	180	0.004	0.00004	1.7e−6	1.8e−9	1.3e−10
1000	500	0.003	0.00003	9.5e−7	3.0e−10	4.2e−11

Table 7
Numerical experiments about the relative errors in the approximation of $\gamma(a,z)$ using (20) for complex a and z with a away from z and different values of n

a	z	$n=1$	$n=3$	$n=6$	$n=9$	$n=12$
$50+20i$	$20-5i$	0.03	0.002	0.0006	0.00005	8.5e−6
$100+20i$	$50-5i$	0.03	0.002	0.0005	0.00001	9.3e−6
$200+20i$	$80-5i$	0.01	0.0003	0.00003	2.5e−7	5.0e−8
$500+20i$	$200-5i$	0.005	0.00005	2.5e−6	2.9e−9	2.7e−10
$1000+20i$	$500-5i$	0.003	0.00003	9.4e−7	3.3e−10	4.0e−11

On the other hand, integrating by parts in (21) we find the recurrence:

$$\Phi_k(z-a)=\frac{1}{z-a}\left[e^{z-a}-k\Phi_{k-1}(z-a)\right],\quad k=1,2,3,\ldots.$$

The first few functions $\Phi_k(a)$ are detailed in Table 5.

The expansion (20) is convergent. But from (19) and (23) we see that it is also asymptotic for large $a-z$ provided that $a-z=\mathcal{O}(a^{1/2+\varepsilon})$, $\varepsilon>0$ and $\Re z<\Re a$.

3.1. An alternative expansion

We consider in this subsection the formula [11, p. 279, Eq. (11.9)]

$$\gamma(a+1,z)=z^{a+1}e^{-z}\sum_{k=0}^{\infty}\frac{z^k}{(a+1)_{k+1}} \tag{24}$$

as an alternative expansion to expansion (20). As well as (20), this series is convergent. It is also asymptotic for large a and z provided $z = \mathcal{O}(a^{1-\varepsilon})$, $\varepsilon > 0$. We compare below convergence and asymptotic properties of (20) and (24).

We consider large k (in particular $k \gg \Re(a)$). From (17) we write

$$c_k(a) = \sum_{j=0}^{j_0} \frac{(-a)_j}{j!}\frac{a^{k-j}}{(k-j)!} + \sum_{j=j_0+1}^{k/2} \frac{(-a)_j}{j!}\frac{a^{k-j}}{(k-j)!} + \sum_{j=k/2+1}^{k} \frac{(-a)_j}{j!}\frac{a^{k-j}}{(k-j)!},$$

with $j_0 \equiv \lfloor \Re(a) \rfloor + 2$. For large k, the first sum is clearly, at most, of the order $\mathcal{O}((1+a^k)/(k-j_0)!)$. Using $(-a)_j = \Gamma(j-a)/\Gamma(-a)$, $|\Gamma(j-a)| \leqslant |\Gamma(j-\Re(a))|$ and $\Gamma(j-\Re(a)) < j!$ for $j > j_0$, we see that the second sum above is, at most, of the order $\mathcal{O}((1+a^k)/(k/2-1)!)$. Finally, for the last sum we write:

$$\sum_{j=\frac{k}{2}+1}^{k} \frac{|(-a)_j|}{j!}\frac{|a|^{k-j}}{(k-j)!} \leqslant \frac{\Gamma(k/2+1-\Re(a))}{|\Gamma(-a)|\Gamma(k/2+2)} \sum_{j=\frac{k}{2}+1}^{k} \frac{|a|^{k-j}}{(k-j)!} \leqslant e^{|a|}\frac{\Gamma(k/2+1-\Re(a))}{|\Gamma(-a)|\Gamma(k/2+2)}.$$

Using the Stirling approximation of the gamma function, we see that this last sum is of the order $\mathcal{O}(k^{-1-\Re(a)})$ for large k. Therefore, using these estimations, we have $c_k(a) = \mathcal{O}(k^{-1-\Re(a)})$ as $k \to \infty$. From (21) we see that $|\Phi_k(z-a)| \leqslant (k+1)^{-1}$. Therefore, expansion (20) has the following rate of convergence: $c_k(a)\Phi_k(a) \sim k^{-2-\Re(a)}$ as $k \to \infty$, whereas expansion (24) has a stronger factorial rate of convergence.

On the other hand, the asymptotic properties of (20) are better than those of (24): the condition $z = \mathcal{O}(a^{1-\varepsilon})$, $\varepsilon > 0$ is more stringent than the condition $a - z = \mathcal{O}(a^{1/2+\varepsilon})$, $\varepsilon > 0$. In particular, the asymptotic character of (24) for large a and z requires $|z| < |a|$, whereas expansion (20) is asymptotic for large a and z when $|z| > |a|$ provided $\Re(a) > \Re(z)$ and a away from z.

Tables 8 and 9 illustrate the different rate of convergence and asymptotic accuracy of expansions (20) and (24).

Table 8
Numerical experiments comparing the rate of convergence of expansions (20) and (24) when approximating $e^z z^{-a-1}\gamma(a+1,z)$ for fixed $a = 20$ and $z = 10$

n	approximation (20)	approximation (24)
5	0.0935918	0.0847679
10	0.0909609	0.0851108
20	0.0866497	0.0851121
30	0.0845521	0.0851121
40	0.0850625	0.0851121
∞	0.0851121	0.0851121

Second and third columns represent the approximations supplied by (20) and (24), respectively, for several number of summands n. The last row ($n = \infty$) represents the exact value of $e^z z^{-a-1}\gamma(a+1,z)$ up to 7 digits.

Table 9
Numerical experiments comparing the asymptotic accuracy of expansions (20) and (24) when approximating $e^z z^{-a-1}\gamma(a+1,z)$ for several values of a and z and fixed number of summands $n=3$

z	a	exact value	approximation (20)	approximation (24)
10	100	0.01097590	0.01097390	0.01108720
10	1000	0.00100907	0.00100907	0.00101009
100	200	0.00980752	0.00980000	0.00871903
100	1000	0.00110974	0.00110974	0.00110987
1000	2000	0.00099801	0.00099800	0.00087469

The third column represent the exact value of $e^z z^{-a-1}\gamma(a+1,z)$ up to 8 digits. Fourth and fifth columns represent the approximations supplied by (20) and (24), respectively.

4. $\Re(a) > -1$, $\Re(a) < \Re(z)$ and away from the transition point

We consider in this section the function $\Gamma(a+1,z)$:

$$\Gamma(a+1,z)=\int_z^{\infty} u^a e^{-u}\,\mathrm{d}u, \qquad \left|\mathrm{Arg}(z)\right| < \pi. \tag{25}$$

After the change of variable $u=z(1+t)$ and using the Cauchy residue theorem we have

$$\Gamma(a+1,z)=z^{a+1}e^{-z}\int_0^{\infty e^{i\theta}} e^{-zf(t)}\,\mathrm{d}t, \qquad f(t)\equiv t-\alpha\log(1+t), \tag{26}$$

for any $|\theta|<\pi/2$ with $|\theta+\mathrm{Arg}(z)|<\pi/2$.

The asymptotic features of (26) for $\Re a < \Re z$ are those of the integral (15) for $\Re a > \Re z$. Then, repeating the same arguments as in the previous section we obtain the formal expansion

$$\Gamma(a+1,z)\sim e^{-z}z^{a+1}\sum_{k=0}^{\infty}\frac{c_k(a)}{(z-a)^{k+1}}, \tag{27}$$

with

$$c_k(a)\equiv(-1)^k\sum_{j=0}^{k}\binom{k}{j}\frac{(-a)_j}{j!}\frac{a^{k-j}}{(k-j)!} \tag{28}$$

and

$$c_{k+1}(a)=-k\left[c_k(a)+a\,c_{k-1}(a)\right], \qquad k=1,2,3,\ldots. \tag{29}$$

Again,

$$c_0=1, \qquad c_1=0, \qquad c_k(a)=\mathcal{O}\left(a^{\lfloor k/2\rfloor}\right), \quad |a|\to\infty. \tag{30}$$

Table 10
First few coefficients $c_k(a)$ defined in (28) and used in (27)

k	$c_k(a)$	k	$c_k(a)$
0	1	4	$-3(2a-a^2)$
1	0	5	$4(6a-5a^2)$
2	$-a$	6	$-5(24a-26a^2+3a^3)$
3	$2a$	7	$6(120a-154a^2+35a^3)$

The first few coefficients $c_k(a)$ are detailed in Table 10.

Expansion (27) is not convergent because we cannot apply the arguments of uniform convergence used in Section 3. Nevertheless it is asymptotic for large $a-z$ provided that $a-z=\mathcal{O}(a^{1/2+\varepsilon})$ for any $\varepsilon>0$. We do not offer any proof of this here because expansion (27) is just the expansion previously derived by Tricomi [12].

5. The transition point

The expansions given in the three preceding sections are not valid near the transition point $a=z$. We consider here another variation of the integral representation (1) of $\Gamma(a+1,z)$ in order to derive an asymptotic expansion valid at the transition point:

$$\Gamma(a+1,z)=\int_{\mathcal{C}_1} t^a e^{-t}\,dt. \tag{31}$$

In this formula, $\mathcal{C}_1$ is a simple path joining the point $t=z$ with the point $t=\infty e^{i\theta}$ with $-\pi/2<\theta<\pi/2$ and which does not cross the negative real axis (see Fig. 1(left)).

After the change of variable $t=a(1+u)$, (31) is transformed in

$$\Gamma(a+1,z)=e^{-a}a^{a+1}\int_{\mathcal{C}_2} e^{-a(u-\log(1+u))}\,du, \tag{32}$$

where $\mathcal{C}_2$ is a simple path joining the point $u=z/a-1$ with the point $t=\infty e^{i(\theta-\mathrm{Arg}(a))}$ and which does not cross the line $[-1,-\infty/a)$ (see Fig. 1(right)).

The function $f(u)\equiv u-\log(1+u)$ in the exponent in (32) has a simple saddle point at $u=0$, where $f(0)=0$. The main contribution of the integrand to the integral (32) comes from the point $u=0$ and the integral may be approximated replacing $f(u)$ in (32) by its Taylor polynomial at $u=0$ [5]. We split the function $f(u)$ in two terms: $f(u)=\frac{1}{2}u^2+f_3(u)$, where $\frac{1}{2}u^2$ is the Taylor polynomial of degree 2 of $f(u)$ at $u=0$ and $f_3(u)\equiv u-\log(1+u)-\frac{1}{2}u^2$. With this splitting, the integral in (32) reads

$$\Gamma(a+1,z)=e^{-a}a^{a+1}\int_{\mathcal{C}_2} e^{-\frac{a}{2}u^2-af_3(u)}\,du. \tag{33}$$

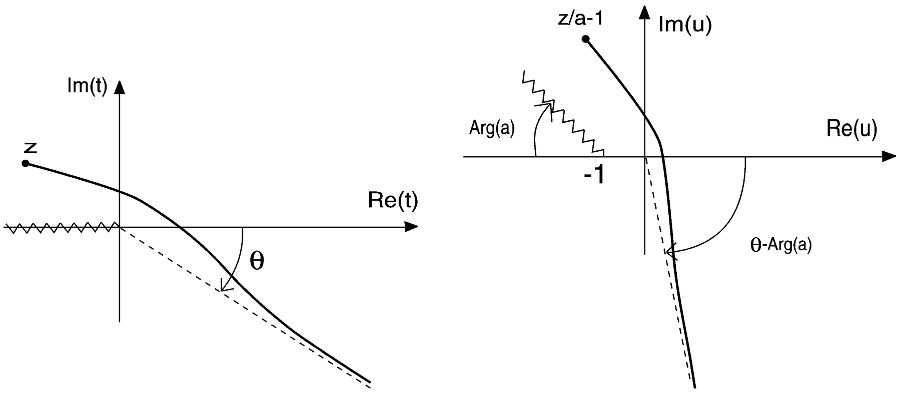

Fig. 1. Integration path $\mathcal{C}_1$ in formula (31) (left). Integration path $\mathcal{C}_2$ in formula (32) (right).

Table 11
First few coefficients $c_k(a)$ defined in (35) and used in (36)

k	$c_k(a)$	k	$c_k(a)$
0	1	4	$-a/4$
1	0	5	$a/5$
2	0	6	$(-3a+a^2)/18$
3	$a/3$	7	$(12a-7a^2)/84$

We expand

$$e^{-af_3(u)} = e^{au^2/2-au}(1+u)^a = \sum_{k=0}^{\infty} c_k(a)u^k, \quad |u|<1, \tag{34}$$

with

$$c_k(a) = (-1)^k \sum_{j=0}^{k} \sum_{l=0}^{\lfloor j/2 \rfloor} \frac{(-a)_{j-2l}}{(j-2l)!l!} \frac{a^{k-j}}{(k-j)!} \left(\frac{a}{2}\right)^l. \tag{35}$$

The first few coefficients $c_k(a)$ are detailed in Table 11. From the differential equation $(1+u)f' - au^2 f = 0$, satisfied by $e^{-af_3(u)}$, we derive:

$$c_{k+1}(a) = \frac{1}{k+1}\left[ac_{k-2}(a) - kc_k(a)\right].$$

From $f_3(0)=f_3'(0)=f_3''(0)=0$ we see that [4, Lemma 2.1] $c_0(a)=1$ and $c_1(a)=c_2(a)=0$. Using also the above recurrence we see that $c_k(a)=\mathcal{O}(a^{\lfloor k/3\rfloor})$ as $a\to\infty$. Replacing the expansion (34) in (33) we obtain

$$\Gamma(a+1,z)=e^{-a}a^{a+1}\left[\sum_{k=0}^{n-1}c_k(a)\Phi_k(a,z)+R_n(a,z)\right], \tag{36}$$

with

$$\Phi_k(a,z)\equiv\int_{\mathcal{C}_2}u^k e^{-\frac{a}{2}u^2}\,\mathrm{d}u \tag{37}$$

and

$$R_n(a,z)\equiv\int_{\mathcal{C}_2}e^{-\frac{a}{2}u^2}\left[e^{-af_3(u)}-\sum_{k=0}^{n-1}c_k(a)u^k\right]\mathrm{d}u. \tag{38}$$

The convergence of integrals (37) and (38) requires the additional hypothesis $2\theta-\pi/2<\operatorname{Arg}(a)<2\theta+\pi/2$. After a straightforward computation we find

$$\Phi_0(a,z)=\sqrt{\frac{\pi}{2a}}\operatorname{erfc}\left(\frac{z-a}{\sqrt{2a}}\right),\qquad \Phi_1(a,z)=\frac{e^{-(z-a)^2/(2a)}}{a} \tag{39}$$

and, for $k=1,2,3,\ldots,$

$$\Phi_{2k}(a,z)=(-2)^k\sqrt{\frac{\pi}{2}}\frac{\mathrm{d}^k}{\mathrm{d}a^k}\left[\frac{1}{\sqrt{a}}\operatorname{erfc}\left(b\sqrt{\frac{a}{2}}\right)\right]\bigg|_{b=(z-a)/a} \tag{40}$$

and

$$\Phi_{2k+1}(a,z)=(-2)^k\frac{\mathrm{d}^k}{\mathrm{d}a^k}\left[\frac{e^{-ab^2/2}}{a}\right]\bigg|_{b=(z-a)/a}. \tag{41}$$

On the other hand, integrating by parts in (37) we find that, for $k=2,3,4,\ldots,$

$$\Phi_k(a,z)=\frac{1}{a}\left[(k-1)\Phi_{k-2}(a,z)+\left(\frac{z-a}{a}\right)^{k-1}e^{-(z-a)^2/(2a)}\right]. \tag{42}$$

From (40) and (41) or from (39) and (42) we see that

$$\Phi_k(a,z)=e^{-(z-a)^2/(2a)}\begin{cases}\mathcal{O}(a^{-(k+1)/2}) & \text{if } |z-a|^2\leqslant|a|,\\ \mathcal{O}(a^{-k}(z-a)^{k-1}) & \text{if } |z-a|^2\geqslant|a|.\end{cases} \tag{43}$$

Table 12
Asymptotic order of the first 9 terms of the sequence $c_k(a)\Phi_k(a,z)$ in (36) (the factor $e^{-(z-a)^2/(2a)}$ is omitted)

k	0	1	2	3	4	5	6	7	8
$\lvert z-a\rvert^2 \leqslant \lvert a\rvert$	$a^{-1/2}$	0	0	a^{-1}	$a^{-3/2}$	a^{-2}	$a^{-3/2}$	a^{-2}	$a^{-5/2}$
$\lvert z-a\rvert^2 \geqslant \lvert a\rvert$	$a^{-1/2}$	0	0	$(z-a)^{-2}$	$(z-a)^{-3}$	$(z-a)^{-4}$	$(z-a)^{-3}$	$(z-a)^{-4}$	$(z-a)^{-5}$

Therefore, using that $c_k(a) = \mathcal{O}(a^{\lfloor k/3\rfloor})$ as $a \to \infty$, we have

$$c_k(a)\Phi_k(a,z) = e^{-(z-a)^2/(2a)} \begin{cases} \mathcal{O}(a^{\lfloor k/3\rfloor-(k+1)/2}) & \text{if } |z-a|^2 \leqslant |a|, \\ \mathcal{O}(a^{\lfloor k/3\rfloor-k}(z-a)^{k-1}) & \text{if } |z-a|^2 \geqslant |a|. \end{cases} \tag{44}$$

Then, the sequence $c_k(a)\Phi_k(a,z)$ is an asymptotic sequence if $z-a = o(a^{2/3})$ as $a \to \infty$.

Finally, we show that the remainder $R_n(a,z)$ in (36) (defined in (38)) is of the same order than the first neglected term in the expansion (36). For this purpose we write

$$e^{-af_3(u)} - \sum_{k=0}^{n-1} c_k(a)u^k = \sum_{k=n}^{\infty} c_k(a)u^k, \quad |u| < 1. \tag{45}$$

We deform and split the integration path in the right-hand side of (38): $\mathcal{C}_2 \to \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5$, where $\mathcal{C}_3$ is a straight line joining the points $u = z/a - 1$ and $u = 0$; $\mathcal{C}_4$ is a straight line joining the points $u = 0$ and $u = s$, $s := e^{i(\theta - \text{Arg}(a))}$; and $\mathcal{C}_5$ is a straight line joining the points $u = s$ and $u = s\cdot\infty$ (see Fig. 2):

$$R_n(a,z) = R_n^{(1)}(a,z) + R_n^{(2)}(a,z), \tag{46}$$

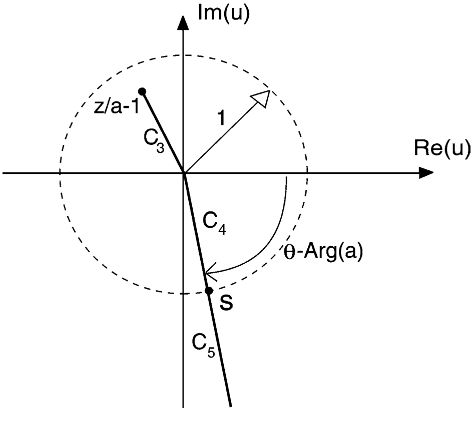

Fig. 2. Integration paths $\mathcal{C}_3$, $\mathcal{C}_4$ and $\mathcal{C}_5$ in formulas (47)–(48).

with

$$R_n^{(1)}(a,z) \equiv \int_{\mathcal{C}_3 \cup \mathcal{C}_4} e^{-\frac{a}{2}u^2}\left[e^{-af_3(u)} - \sum_{k=0}^{n-1} c_k(a)u^k\right]\mathrm{d}u \tag{47}$$

and

$$R_n^{(2)}(a,z) \equiv \int_{\mathcal{C}_5} e^{-\frac{a}{2}u^2}\left[e^{-af_3(u)} - \sum_{k=0}^{n-1} c_k(a)u^k\right]\mathrm{d}u. \tag{48}$$

If $|z-a| < |a|$, then the series in the right-hand side of (45) converges uniformly with respect to u on $\mathcal{C}_3 \cup \mathcal{C}_4$. Introducing (45) in (47) and interchanging the orders of summation and integration we obtain

$$\begin{aligned} R_n^{(1)}(a,z) &= \sum_{k=n}^{\infty} c_k(a) \int_{z/a-1}^{s} u^k e^{-au^2/2}\,\mathrm{d}u \\ &= \sum_{k=\lfloor (n+1)/2 \rfloor}^{\infty} c_{2k}(a)(-2)^k \sqrt{\frac{\pi}{2}} \frac{\mathrm{d}^k}{\mathrm{d}a^k}\left[\frac{1}{a}\left(\operatorname{erfc}\left(b\sqrt{\frac{a}{2}}\right) - \operatorname{erfc}\left(s\sqrt{\frac{a}{2}}\right)\right)\right]_{b=(z-a)/a} \\ &\quad + \sum_{k=\lfloor n/2 \rfloor}^{\infty} c_{2k+1}(a)(-2)^k \frac{\mathrm{d}^k}{\mathrm{d}a^k}\left[\frac{e^{-ab^2/2}}{a} - \frac{e^{-as^2/2}}{a}\right]_{b=(z-a)/a}. \end{aligned} \tag{49}$$

For large a, this expression is of the order of the first term of the series, $\mathcal{O}(c_n(a)\Phi_n(a,z))$, provided that $\Re(as^2) \geqslant \Re((z-a)^2/a)$.

On the other hand, after the change of variable $u \to r$ in the integral (48) defined by $u = sr$, $r \in [1,\infty)$, we have

$$\left|R_n^{(2)}(a,z)\right| \leqslant \int_1^{\infty}\left[e^{-|a|g_1(r,\operatorname{Arg}(a),\theta)} + e^{-|a|g_2(r,\operatorname{Arg}(a),\theta)}\left|\sum_{k=0}^{n-1} c_k(a) r^k s^k\right|\right]\mathrm{d}r, \tag{50}$$

with

$$\begin{aligned} g_1\big(r,\operatorname{Arg}(a),\theta\big) &\equiv r\cos\theta - \cos\big(\operatorname{Arg}(a)\big)\log\sqrt{1+r^2+2r\cos(\theta-\operatorname{Arg}(a))} \\ &\qquad + \sin\big(\operatorname{Arg}(a)\big)\arctan\left(\frac{r\sin(\theta-\operatorname{Arg}(a))}{1+r\cos(\theta-\operatorname{Arg}(a))}\right), \end{aligned} \tag{51}$$

$$g_2\big(r,\operatorname{Arg}(a),\theta\big) \equiv \frac{1}{2}r^2\cos\big(2\theta-\operatorname{Arg}(a)\big). \tag{52}$$

The derivative of the function $g_1(r, \mathrm{Arg}(a), \theta)$ with respect to r is

$$\frac{\mathrm{d}g_1(r, \mathrm{Arg}(a), \theta)}{\mathrm{d}r} = \frac{\cos(2\theta - \mathrm{Arg}(a)) + r\cos\theta}{1 + r^2 + 2r\cos(\theta - \mathrm{Arg}(a))} r. \tag{53}$$

This is non-negative thanks to the hypotheses $-\pi/2 < \theta < \pi/2$ and $2\theta - \pi/2 < \mathrm{Arg}(a) < 2\theta + \pi/2$ and therefore, the function $g_1(r, \mathrm{Arg}(a), \theta)$ is an increasing function of r. Moreover, within this hypotheses, the function $g_2(r, \mathrm{Arg}(a), \theta)$ is also an increasing function of r. Therefore, both exponentials in the right-hand side of (50) attain their maximum inside the integration interval at the point $r = 1$. Then,

$$R_n^{(2)}(a, z) = \mathcal{O}\big(e^{-|a|g_1(1, \mathrm{Arg}(a), \theta)} + e^{-|a|g_2(1, \mathrm{Arg}(a), \theta)}\big) = \mathcal{O}\big(e^{-as} + e^{-as^2}\big). \tag{54}$$

For large a, this expression is of the order $\mathcal{O}(c_n(a)\Phi_n(a, z))$, provided that $\Re(as) \geqslant \Re((z-a)^2/a)$. Then, from (46), (47), (48), (49) and (54) we have that $R_n(a, z) = \mathcal{O}(c_n(a)\Phi_n(a, z))$ as $a, z \to \infty$ with $a - z = o(a^{2/3})$ provided that

$$|z - a|^2 \cos\big(2\mathrm{Arg}(z - a) - \mathrm{Arg}(a)\big) \leqslant |a|^2 \mathrm{Min}\big\{\cos\theta, \cos\big(2\theta - \mathrm{Arg}(a)\big)\big\}.$$

This inequality holds for large enough a thanks to the hypotheses $|\theta| < \pi/2$, $|2\theta - \mathrm{Arg}(a)| < \pi/2$ and $z - a = o(a^{2/3})$. The optimum value for θ is attained at $\theta = \mathrm{Arg}(a)/3$.

Tables 13 and 14 show some numerical experiments about the approximation supplied by (36).

Table 13
Relative errors in the approximation of $\gamma(a, z)$ supplied by (36) for $a = 100$, z on a semicircle of radius 2 around a and different values of n

a	z	$n = 1$	$n = 4$	$n = 10$	$n = 16$
100	102	0.06	0.001	0.0006	0.00003
100	$100 + 2e^{i\pi/4}$	0.06	0.0009	0.0005	0.00003
100	$100 + 2e^{i\pi/2}$	0.05	0.0008	0.0005	0.00002
100	$100 + 2e^{i3\pi/4}$	0.04	0.0007	0.0004	0.00002
100	98	0.04	0.0007	0.0004	0.00002

Table 14
Relative errors in the approximation of $\gamma(a, z)$ supplied by (36) for different values of a, z near a and different values of n

a	z	$n = 1$	$n = 4$	n= 10	$n = 16$
$200 - 100i$	$180 - 105i$	0.01	0.0003	0.00006	4.4e−6
$500 + 5i$	$450 + 6i$	0.003	0.00007	0.00001	5.0e−7
$1000 - 30i$	$980 - 25i$	0.01	0.00005	4.6e−6	9.8e−8
$4 + 5000i$	$10 + 4900i$	0.005	0.00003	1.4e−7	7.0e−10
10000	$9900 + i$	0.002	3.5e−6	4.4e−8	1.0e−10

6. Concluding remarks

The main results of the paper are the following:

(A) For $\Re a < 0$, $|\mathrm{Arg}(z)| < \pi$ and large a and z with $a - z = \mathcal{O}(a^{1/2+\varepsilon})$, $\varepsilon > 0$,

$$\Gamma(a, z) \sim -\frac{2\pi i R}{\Gamma(1-a)} e^{i\pi a \,\mathrm{sgn}(\mathrm{Arg}(a))} \mathrm{sgn}\big(\mathrm{Arg}(a)\big) - z^a e^{-z} \sum_{k=0}^{\infty} \frac{\Phi_k(a)}{(a-z)^{k+1}},$$

with

$$\Phi_0(a) = 1, \qquad \Phi_1(a) = 1,$$
$$\Phi_k(a) = k\,\Phi_{k-1}(a) + a(1-k)\,\Phi_{k-2}(a), \quad k = 2, 3, 4, \ldots$$

and

$$R \equiv \begin{cases} 1 & \text{if } \mathrm{Arg}(z)/\mathrm{Arg}(a) > 1, \\ 1/2 & \text{if } \mathrm{Arg}(z)/\mathrm{Arg}(a) = 1, \\ 0 & \text{if } \mathrm{Arg}(z)/\mathrm{Arg}(a) < 1. \end{cases}$$

(B) For $\Re z > \Re a > -1$, $|\mathrm{Arg}(z)| < \pi$ and large a and z with $a - z = \mathcal{O}(a^{1/2+\varepsilon})$, $\varepsilon > 0$,

$$\Gamma(a+1, z) \sim e^{-z} z^{a+1} \sum_{k=0}^{\infty} \frac{c_k(a)}{(z-a)^{k+1}},$$

with

$$c_0 = 1, \qquad c_1 = 0, \qquad c_{k+1}(a) = -k\big[c_k(a) + a c_{k-1}(a)\big], \quad k = 1, 2, 3, \ldots.$$

(C) For $\Re a > -1$, $|\mathrm{Arg}(z)| < \pi$ and $\Re z < \Re a$,

$$\gamma(a+1, z) = e^{-z} z^{a+1} \sum_{k=0}^{\infty} c_k(a) \Phi_k(z-a),$$

with

$$c_0 = 1, \qquad c_1 = 0, \qquad c_{k+1}(a) = \frac{1}{k+1}\big[k c_k(a) - a c_{k-1}(a)\big], \quad k = 1, 2, 3, \ldots$$

and

$$\Phi_0(z-a) = \frac{1 - e^{z-a}}{a - z},$$
$$\Phi_k(z-a) = \frac{1}{z-a}\big[e^{z-a} - k\Phi_{k-1}(z-a)\big], \quad k = 1, 2, 3, \ldots.$$

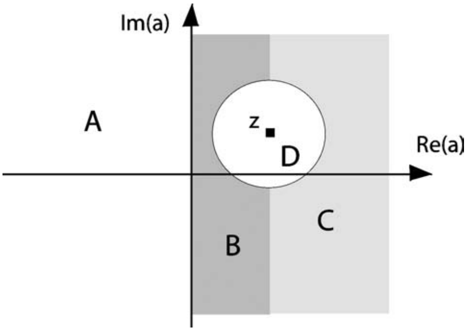

Fig. 3. Regions of validity of the above expansions depending on the value of a and the relative value of a and z. Asymptotically, the radius r of the region D must be of an order between $\mathcal{O}(a^{1/2})$ and $\mathcal{O}(a^{2/3})$. An intermediate value is $\mathcal{O}(a^{5/12})$.

The above expansion is an asymptotic expansion for large a and z provided that $a-z=\mathcal{O}(a^{1/2+\varepsilon})$, $\varepsilon>0$.

(D) For $|\mathrm{Arg}(z)|<\pi$ and large a and z with $a-z=o(a^{2/3})$,

$$\Gamma(a+1,z)\sim e^{-a}a^{a+1}\sum_{k=0}^{\infty}c_k(a)\Phi_k(a,z),$$

with

$$c_0(a)=1,\qquad c_1(a)=c_2(a)=0,$$

$$\Phi_0(a,z)=\sqrt{\frac{\pi}{2a}}\operatorname{erfc}\left(\frac{z-a}{\sqrt{2a}}\right),\qquad \Phi_1(a,z)=\frac{e^{-(z-a)^2/(2a)}}{a}$$

and, for $k=2,3,4,\ldots,$

$$c_{k+1}(a)=\frac{1}{k+1}\left[ac_{k-2}(a)-kc_k(a)\right],\quad k=2,3,4,\ldots,$$

$$\Phi_k(a,z)=\frac{1}{a}\left[(k-1)\Phi_{k-2}(a,z)+\left(\frac{z-a}{a}\right)^{k-1}e^{-(z-a)^2/(2a)}\right].$$

As it was stated in the introduction, the main benefit of the above expansions is their simplicity. As far as we know, expansions (A), (C) and (D) are new, whereas (B) was previously obtained by Tricomi [12]. The asymptotic sequences in the expansions (A) and (C) are different from those in [2,6,12]. The coefficients are simpler and the expansions are valid in different regions for (a,z). Moreover, expansion (C) is convergent. The expansion (D) (valid for a near z), as well as Paris' expansion, is given in terms of complementary error functions, but the coefficients are simpler than in Paris' expansion.

Moreover, it is uniformly valid in $\Re(a-z)$, whereas Paris' expansion distinguishes between $\Re(a-z) \geqslant 0$ and $\Re(a-z) \leqslant 0$ [8, Eqs. (2.15), (3.4)]. On the other hand, for a given precision, Paris' expansion requires less terms than expansion (D).

If the asymptotic sequence $\Phi_k(z-a)$ in (C) is replaced by its asymptotic approximation for large $a-z$ (with $\Re(a-z) > 0$) given by the first term on the right-hand side of (22), our expansion reduces to Tricomi's expansion for $\gamma(a,z)$ [12], which is not convergent. In fact, the exponentially small terms in $\Phi_k(z-a)$ are necessary for the convergence of (C).

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