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## Asymptotic expansions of the Hurwitz–Lerch zeta function

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## Asymptotic expansions of the Hurwitz–Lerch zeta function

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### Abstract

The Hurwitz–Lerch zeta function  $\Phi(z, s, a)$  is considered for large and small values of  $a \in \mathbb{C}$ , and for large values of  $z \in \mathbb{C}$ , with  $|\operatorname{Arg}(a)| < \pi$ ,  $z \notin [1, \infty)$  and  $s \in \mathbb{C}$ . This function is originally defined as a power series in  $z$ , convergent for  $|z| < 1$ ,  $s \in \mathbb{C}$  and  $1 - a \notin \mathbb{N}$ . An integral representation is obtained for  $\Phi(z, s, a)$  which define the analytical continuation of the Hurwitz–Lerch zeta function to the cut complex  $z$ -plane  $\mathbb{C} \setminus [1, \infty)$ . From this integral we derive three complete asymptotic expansions for either large or small  $a$  and large  $z$ . These expansions are accompanied by error bounds at any order of the approximation. Numerical experiments show that these bounds are very accurate for real values of the asymptotic variables.

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**Keywords:** Hurwitz–Lerch zeta function; Analytic continuation; Asymptotic expansions

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## 1. Introduction

The Lipschitz–Lerch zeta function is defined by means of the infinite series:

$$\mathcal{R}(a, x, s) \equiv \sum_{k=0}^{\infty} \frac{e^{2k\pi ix}}{(a+k)^s}, \quad s, x, a \in \mathbb{C},$$

with  $1-a \notin \mathbb{N}$  and  $\Re x \geq 0$ . If  $\Re x > 0$  the series converges  $\forall s \in \mathbb{C}$  and represents an entire function of  $s$ . If  $\Re x = 0$  the series converges absolutely for  $\Re s > 1$ . This function was introduced and investigated by Lerch [8] and Lipschitz [9] in connection with Dirichlet's famous theorem on primes in arithmetic progression. If  $x \in \mathbb{Z}$  the Lipschitz–Lerch zeta function reduces to the meromorphic Hurwitz zeta function  $\zeta(s, a)$  [11, Section 2.3, Eq. (2)] with one single pole at  $s = 1$ . Moreover,  $\zeta(s, 1)$  is nothing but the Riemann zeta function  $\zeta(s)$ .

Erdélyi used a different notation [3, Section 1.11, Eq. (1)] for the Lipschitz–Lerch zeta function:

$$\Phi(z, s, a) \equiv \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s}, \quad 1-a \notin \mathbb{N}, s \in \mathbb{C}, |z| < 1. \quad (1)$$

This function is called Hurwitz–Lerch zeta function and is related to the Lipschitz–Lerch zeta function by the equality  $\Phi(e^{2\pi ix}, s, a) = \mathcal{R}(a, x, s)$ .

Properties of the Lipschitz and Hurwitz–Lerch zeta functions have been studied by many authors. Among other results, we remark the following. Apostol obtains functional relations for the function  $\mathcal{R}(a, x, s)$  and gives an algorithm to compute  $\mathcal{R}(a, x, -n)$  for  $n \in \mathbb{N}$  in terms of a certain kind of generalized Bernoulli polynomials [2]. The function  $\mathcal{R}(a, x, s)$  is used in [5] to generalize a certain asymptotic formula considered by Ramanujan. Asymptotic equalities for some weighted mean squares of  $\mathcal{R}(a, x, s)$  are given in [6]. Integral representations as well as functional relations and expansions for  $\Phi(z, s, a)$  may be found in [11, Section 2.5].

Asymptotic expansions of  $\mathcal{R}(a, x, s)$  have been investigated by Katsurada [4] and Klusch [7]. Klusch studies some properties of this function resulting from the Taylor expansion of  $\mathcal{R}(a + \xi, x, s)$  in the neighborhood of  $\xi = 0$ . Moreover, the author establishes the common source of various classes of summation formulas involving infinite series of zeta and Hurwitz zeta functions which are collected and unified in [10]. On the other hand, Katsurada derives power series and asymptotic series for  $\mathcal{R}(a, x, s)$  in the parameter  $a$  using Mellin transform techniques. Error bounds are not given there.

However, complete asymptotic expansions including error bounds of  $\Phi(z, s, a)$  for large and small  $a$  and large  $z$  are not fully investigated. The purpose of this paper is to generalize the above mentioned expansions for  $\mathcal{R}(a, x, s)$  to expansions for  $\Phi(z, s, a)$  including error bounds. Moreover, we investigate also asymptotic expansions of  $\Phi(z, s, a)$  for large  $z$  with error bounds.

In Section 2, we derive the integral representation of  $\Phi(z, s, a)$  from which, in Section 3, we derive complete asymptotic expansions of this function in the limits mentioned above. We use the error test and Cauchy's integral formula to obtain error bounds at any order of the approximations. Numerical examples are shown as an illustration in Section 4.

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## 2. Analytic continuation of the Hurwitz–Lerch zeta function

The starting point to derive asymptotic expansions of  $\Phi(z, s, a)$  is a suitable integral representation [11, Section 2.5, Eq. (4)]:

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} e^{-ax}}{1 - ze^{-x}} dx, \quad \Re a > 0, \Re s > 0, z \notin [1, \infty). \quad (2)$$

This integral defines the analytic continuation of  $\Phi(z, s, a)$  defined in (1) to  $z \in \mathbb{C} \setminus [1, \infty)$ . But the parameters  $a$  and  $s$  are restricted to the half-planes  $\Re a > 0$  and  $\Re s > 0$ .

We can continue analytically  $\Phi(z, s, a)$  in both,  $a$  or  $s$ , to larger regions in the complex plane. In order to continue  $\Phi(z, s, a)$  to  $\Re a \leq 0$ , we consider an angle  $\varphi$  in the set (see Fig. 1)

$$\Lambda_a \equiv \{\varphi \in (-\pi/2, \pi/2), |\operatorname{Arg}(a) + \varphi| < \pi/2\}.$$

In what follows  $\operatorname{Arg}(x)$  means the principal argument of  $x$ :  $\operatorname{Arg}(x) \in (-\pi, \pi)$ .

We define the  $z$ -set:

$$\Omega_a \equiv \begin{cases} \mathbb{C} \setminus [1, \infty) & \text{if } \Re a > 0, \\ \{z \in \mathbb{C}, |z| < 1\} & \text{if } \Re a \leq 0. \end{cases}$$

If  $|z| < 1$  then the sector  $\{w \in \mathbb{C}; 0 \leq \operatorname{Arg}(w) \leq \varphi\}$  does not contain any of the poles of the integrand in (2):  $x_n \equiv \log z + 2in\pi$ ,  $n \in \mathbb{Z}$ . Then, using the Cauchy residue theorem we obtain from (2) that, for fixed  $s$  with  $\Re s > 0$  and fixed  $z$  with  $|z| < 1$ , the analytic continuation of  $\Phi(z, s, a)$  in the  $a$  variable to  $\Re a \leq 0$  is given by

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty e^{i\varphi}} \frac{x^{s-1} e^{-ax}}{1 - ze^{-x}} dx, \quad \varphi \in \Lambda_a.$$

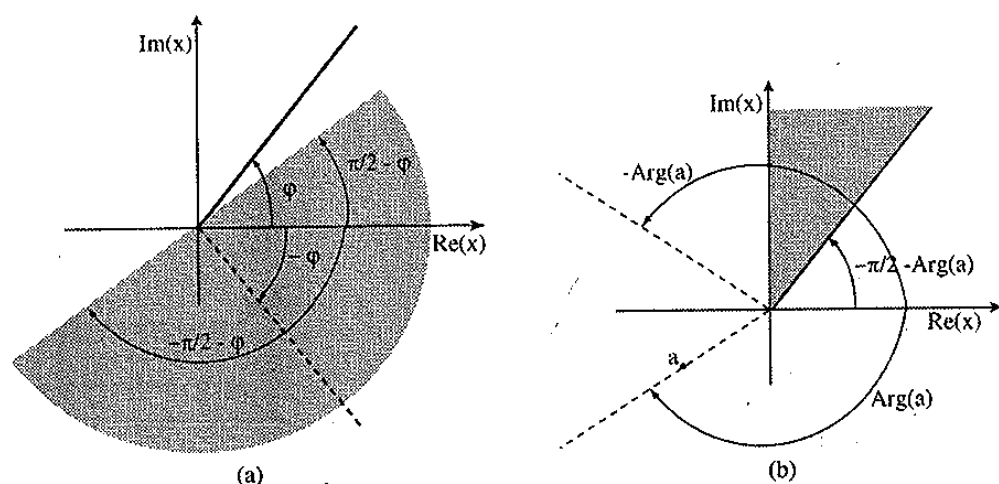


Fig. 1. For a fixed angle  $\varphi \in (-\pi/2, \pi/2)$ , the parameter  $a$  defined in  $\Lambda_a$  must be contained in the shaded region of panel (a):  $\operatorname{Arg}(a) \in (-\pi/2 - \varphi, \pi/2 - \varphi)$ . For a fixed parameter  $a$  in  $\Lambda_a$ , the angle  $\varphi$  must be contained in the shaded region of panel (b):  $\varphi \in (-\pi/2, \pi/2) \cap (-\pi/2 - \operatorname{Arg}(a), \pi/2 - \operatorname{Arg}(a))$ . Observe that  $\varphi = -\operatorname{Arg}(a)/2$  is always in  $\Lambda_a$ .

A straightforward computation obtain from (2) that, for fixed  $a \in \mathbb{C} \setminus \mathbb{R}^-$ , the analytical continuation of  $\Phi(z, s, a)$  in the complex  $s$  plane to the region  $s \in \mathbb{C} \setminus \mathbb{N}$  is given, for  $z \in \Omega_a$  and  $|\operatorname{Arg}(a)| < \pi$  by

$$\Phi(z, s, a) = \frac{i\Gamma(1-s)}{2\pi} \int_{\mathcal{L}_\varphi} \frac{(-x)^{s-1} e^{-ax}}{1 - ze^{-x}} dx, \quad (3)$$

where  $\mathcal{L}_\varphi$  is the Hankel's contour shown in Fig. 2 with  $\varphi = 0$  if  $\Re a > 0$  and  $\varphi \in \Lambda_a$  if  $\Re a \leq 0$ . This contour must not enclose any pole  $x_n$  of the integrand, and the branch of  $\operatorname{Arg}(x)$  through the contour is chosen as  $\varphi \leq \operatorname{Arg}(x) \leq \varphi + 2\pi$  with  $\operatorname{Arg}(-x) = \operatorname{Arg}(x) - \pi$ . This last formula may be proved by deforming the contour  $\mathcal{L}_\varphi$  into two half-lines which proceed from  $\infty e^{i\varphi}$  to  $0e^{i\varphi}$  and from  $0e^{i(\varphi+2\pi)}$  to  $\infty e^{i(\varphi+2\pi)}$ . Therefore, we have the following proposition.

**Proposition 1.** The analytical continuation of the Hurwitz–Lerch zeta function  $\Phi(z, s, a)$  defined in (1) to  $z \in \Omega_a$ ,  $a \in \mathbb{C} \setminus \mathbb{R}^-$  and  $s \in \mathbb{C}$ , is given by

$$\Phi(z, s, a) = L(s) \int_{\mathcal{C}_\varphi} \frac{x^{s-1} e^{-ax}}{1 - ze^{-x}} dx, \quad (4)$$

where

- (i)  $L(s) \equiv \begin{cases} \Gamma(s)^{-1} & \text{if } \Re(s) > 0, \\ i(2\pi)^{-1} e^{i\pi(1-s)} \Gamma(1-s) & \text{if } s \notin \mathbb{N}, \end{cases}$
- (ii)  $\mathcal{C}_\varphi \equiv \begin{cases} [0, \infty e^{i\varphi}) & \text{if } \Re(s) > 0, \\ \mathcal{L}_\varphi \text{ (Fig. 2)} & \text{if } s \notin \mathbb{N}, \end{cases}$
- (iii)  $\begin{cases} \varphi = 0 & \text{if } \Re(a) > 0, \\ \varphi \in \Lambda_a & \text{if } \Re(a) \leq 0. \end{cases}$

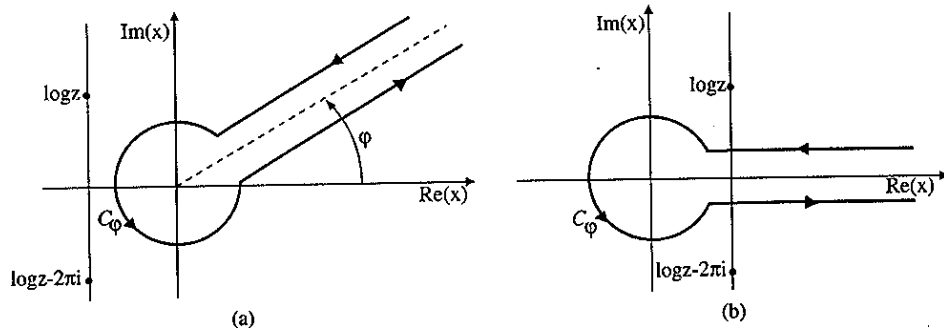


Fig. 2. Hankel's contour involved in the integral representation (3) of  $\Phi(z, s, a)$  with  $\varphi = 0$  if  $\Re a > 0$  (panel (b)) and  $\varphi \in \Lambda_a$  if  $\Re a \leq 0$  (panel (a)). It surrounds the half-line  $[0, \infty e^{i\varphi})$  in the counterclockwise direction and, for  $z \in \Omega_a$ , it does not enclose any pole  $x_n \equiv \log z + 2n\pi i$ ,  $n \in \mathbb{Z}$ , of the integrand in (3).

### 3. Asymptotic expansi

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We recall here the c  
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For  $-n \in \mathbb{N}$  or  $n = 0$   
plane as a meromorph

$$\operatorname{Li}_0(z) = \frac{1}{1-z}$$

**Theorem 1.** For  $|\operatorname{Ar}$   
for large  $a$  and fixed

$$\Phi(z, s, a) = \frac{1}{1-z}$$

$$N = 1, 2, 3$$

where  $\operatorname{Li}_n(z)$  is giv  
 $\mathcal{O}(a^{-N-s})$  as  $|a| \rightarrow \infty$   
the remainder is gi

$$|R_N(z, s, a)|$$

where

$$r(z) \equiv \begin{cases} |A| & \text{if } |z| < 1, \\ - & \text{if } |z| > 1. \end{cases}$$

$$C(z) \equiv |1 - z|$$

**Proof.** We introd

$$\frac{1}{1 - ze^{-x}}$$

into the integran  
and the fact that

$$\left( \frac{\partial}{\partial x} \right)^n \frac{1}{1 - ze^{-x}}$$

### 3. Asymptotic expansions of the Hurwitz–Lerch zeta function

The integral representation (4) given above is the starting point to derive asymptotic expansions of  $\Phi(z, s, a)$  for either large or small  $a$  or for large  $z$ . These expansions are given in Theorems 1–3. Error bounds are obtained for  $\Re s > 0$ . Empty sums must be understood as zero in the remaining of the paper.

We recall here the definition of the polylogarithm function  $\text{Li}_n(z)$  [11, p. 114] that we use in the following theorem:

$$\text{Li}_n(z) \equiv \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \quad |z| < 1, \quad n \in \mathbb{Z}. \quad (5)$$

For  $-n \in \mathbb{N}$  or  $n = 0$ , this function may be continued analytically to the whole complex plane as a meromorphic function of  $z$  with a single pole at  $z = 1$ :

$$\text{Li}_0(z) = \frac{1}{1-z}, \quad \text{Li}_{-n}(z) = z \frac{d}{dz} \text{Li}_{1-n}(z), \quad n \in \mathbb{N}. \quad (6)$$

**Theorem 1.** For  $|\text{Arg}(a)| < \pi$ ,  $s \in \mathbb{C}$  and  $z \in \Omega_a$ , an asymptotic expansion of  $\Phi(z, s, a)$  for large  $a$  and fixed  $s$  and  $z$  is given by

$$\Phi(z, s, a) = \frac{1}{1-z} \frac{1}{a^s} + \sum_{n=1}^{N-1} \frac{(-1)^n \text{Li}_{-n}(z)}{n!} \frac{(s)_n}{a^{n+s}} + R_N(z, s, a), \quad (7)$$

$$N = 1, 2, 3, \dots,$$

where  $\text{Li}_n(z)$  is given in (6) or in (5) for  $|z| < 1$ . The error term verifies  $R_N(z, s, a) = \mathcal{O}(a^{-N-s})$  as  $|a| \rightarrow \infty$  with  $\text{Arg}(a)$  fixed. More precisely, for  $\Re s > 0$ , an error bound for the remainder is given by

$$|R_N(z, s, a)| \leq \frac{2^{N+1} C(z) \Gamma(N + \Re s)}{(r(z))^N |\Gamma(s)| (\Re a)^{N + \Re s}}, \quad (8)$$

where

$$r(z) \equiv \begin{cases} |\text{Arg}(z)| & \text{if } |z| \geq 1, \\ -\log |z| & \text{if } |z| < 1, \end{cases}$$

$$C(z) \equiv |1 - e^{-B(z)}|^{-1}, \quad B(z) \equiv \begin{cases} i \text{Arg}(z)/2 & \text{if } |z| \geq 1, \\ \log |z|/2 + i \text{Arg}(z) & \text{if } |z| < 1. \end{cases} \quad (9)$$

**Proof.** We introduce the Taylor expansion

$$\frac{1}{1 - ze^{-x}} = \sum_{n=0}^{N-1} \frac{(-1)^n \text{Li}_{-n}(z)}{n!} x^n + r_N(x), \quad N = 1, 2, 3, \dots, \quad (10)$$

into the integrand in (4), where  $r_N(x) = \mathcal{O}(x^N)$  as  $x \rightarrow 0$ . This is obtained by noting (6) and the fact that (upon  $ze^{-x} = \xi$ )

$$\left( \frac{\partial}{\partial x} \right)^n \frac{1}{1 - ze^{-x}} \Big|_{x=0} = \left( -\xi \frac{\partial}{\partial \xi} \right)^n \frac{1}{1 - \xi} \Big|_{\xi=z}.$$

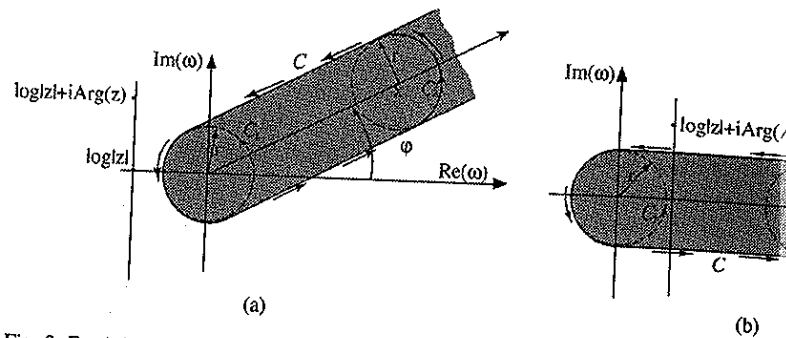


Fig. 3. For  $|x| \geq 2r$ , the contour  $C$  in the definition (12) of  $r_N(x)$  may be deformed to the disk  $r$ ,  $C_1$  and  $C_2$  centered at 0 and  $x$ , respectively. Both are contained in the shaded region  $\{\omega \in \mathbb{C}, |\omega - x| < r, 0 \leq |x| < \infty, \text{Arg}(x) = \varphi\}$ . In panel (a)  $|z| < 1$  and  $0 \leq \varphi < \pi/2$ . In panel (b)  $\varphi = 0$ .

Interchanging sum and integral we obtain (7) with

$$R_N(z, s, a) = L(s) \int_{C_\varphi} x^{s-1} e^{-ax} r_N(x) dx.$$

In the remaining part of the proof we restrict ourselves to  $\Re s > 0$ . Then,

$$R_N(z, s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty e^{i\varphi}} x^{s-1} e^{-ax} r_N(x) dx,$$

where  $\varphi = 0$  if  $\Re a > 0$  and  $\varphi = -\text{Arg}(a)/2$  if  $\Re a \leq 0$ . For any  $x \in \mathbb{C}$  an explicit formula for  $r_N(x)$  is given, using the Cauchy's integral formula, by

$$r_N(x) = \frac{x^N}{2\pi i} \int_C \frac{d\omega}{(\omega - x)\omega^N(1 - ze^{-\omega})}.$$

In this formula,  $C$  is a simple closed loop which encircles the points 0 and  $x$  in the counterclockwise direction and does not enclose any singularity of  $f(\omega) \equiv (1 - ze^{-\omega})^{-1}$  (see Figs. 3 and 4). In order to find bounds of  $R_N(z, s, a)$ , we require bounds of  $r_N(x)$  for fixed  $\text{Arg}(x) = \varphi$  with  $|\varphi| < \pi/2$  and  $0 \leq |x| < \infty$ . We define (see Figs. 3 and 4)

$$r \equiv \begin{cases} \text{Arg}(z)/2, & \text{if } |z| \geq 1, \\ -\log|z|/2 & \text{if } |z| < 1. \end{cases}$$

This number  $r$  is smaller than the distance of the  $x$ -axis to any singularity of  $f(\omega) \equiv (1 - ze^{-\omega})^{-1}$ ,  $\log|z| + i\text{Arg}(z) + 2i\pi n$ ,  $n \in \mathbb{Z}$ . We use the Cauchy residue theorem to deform  $C$  if  $|x| \geq 2r$ : we deform  $C$  to two circles  $C_1$  and  $C_2$  with radius  $r$  and centers at 0 and  $x$ , respectively (see Fig. 3). We use that  $|w| = r$  and that  $|w - x| \geq r$  for  $w \in C_1$  and that  $|w - x| = r$  and that  $|w| \geq r$  for  $w \in C_2$ . On the other hand, for  $|x| < 2r$ ,  $|w| \geq r$  and  $|w - x| \geq r$  for  $w \in C$  (see Fig. 4). Then, we obtain

$$|r_N(x)| \leq 2C(z) \frac{|x|^N}{r^N}, \quad N = 1, 2, 3, \dots,$$

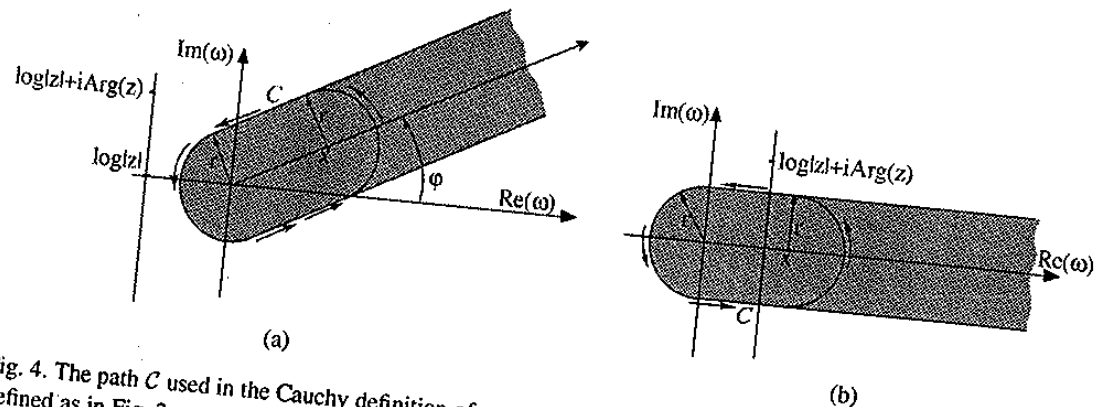


Fig. 4. The path  $C$  used in the Cauchy definition of  $r_N(x)$  in (12) for  $|x| < 2r$ , are contained in the shaded region defined as in Fig. 3.

where  $C(z)$  is a bound of  $|1 - ze^{-\omega}|^{-1}$  in the shaded region depicted in Figs. 3 and 4. The maximum of the function  $|1 - ze^{-\omega}|^{-1}$  in that region is located on the contour of the region. A simple bound  $C(z)$  is given in (9).

Introducing the bound (13) in (11) we obtain (8).

For any  $s \in \mathbb{C}$  we can obtain for  $r_N(x)$  a bound similar to (13), but with a constant  $C(z)$  much more involved. Nevertheless (7) is an asymptotic expansion for large  $a$  and any fixed  $s \in \mathbb{C}$ ,  $z \in \Omega_a$ .  $\square$

**Theorem 2.** For  $s \in \mathbb{C}$  and  $z \in \mathbb{C} \setminus [1, \infty)$ , the Taylor expansion of  $\Phi(z, s, a) - a^{-s}$  at  $a = 0$  is given by

$$\Phi(z, s, a) = \frac{1}{a^s} + \sum_{n=0}^{N-1} (-1)^n \frac{(s)_n}{n!} \text{Li}_{n+s}(z) a^n + R_N(z, s, a), \quad (14)$$

$$N = 1, 2, 3, \dots$$

The expansion is convergent for  $|a| < 1$ . The error term verifies  $R_N(z, s, a) = \mathcal{O}(a^N)$  as  $a \rightarrow 0$ . More precisely, for  $a \in [0, \infty)$  and  $\Re s > 0$ , an error bound for the remainder is given by

$$|R_N(z, s, a)| \leq \frac{\Gamma(N + \Re s)}{N! |\Gamma(s)|} |z| M(z) a^N, \quad M(z) \equiv \begin{cases} 1, & \text{if } \Re z \leq 0, \\ |z|/|\Im z|, & \text{if } \Re z > 0. \end{cases} \quad (15)$$

For  $a \in \mathbb{C}$  and  $\Re s > 0$ , an error bound for the remainder is given by

$$|R_N(z, s, a)| \leq \frac{\Gamma(N + \Re s) |z| M(z) e^r}{|\Gamma(s)| r^N} |a|^N. \quad (16)$$

**Proof.** We introduce the decomposition

$$\frac{1}{1 - ze^{-x}} = 1 + \frac{ze^{-x}}{1 - ze^{-x}}$$

into the integral representation of  $\Phi(z, s, a)$  given in (4) with  $\varphi = 0$ . Then

$$\Phi(z, s, a) = \frac{1}{a^s} + zL(s) \int_{C_0} \frac{x^{s-1} e^{-x} e^{-ax}}{1 - ze^{-x}} dx.$$

We expand  $e^{-ax}$  in power series of  $ax$ ,

$$e^{-ax} = \sum_{n=0}^{N-1} \frac{(-ax)^n}{n!} + r_N(ax).$$

Interchanging sum and integral and using [11, Eq. (71), p. 114] we obtain (14) with

$$R_N(z, s, a) = zL(s) \int_{C_0} \frac{x^{s-1} e^{-x} r_N(ax)}{1 - ze^{-x}} dx.$$

In the remaining part of the proof we restrict ourselves to  $\Re s > 0$ . Then,

$$R_N(z, s, a) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} e^{-x} r_N(ax)}{1 - ze^{-x}} dx. \quad (17)$$

For  $a \in [0, \infty)$ , the remainder  $r_N(ax)$  in the expansion of  $e^{-ax}$  verifies the error test, and therefore

$$|r_N(ax)| \leq \frac{(ax)^N}{N!}, \quad N = 1, 2, 3, \dots$$

On the other hand, using that  $0 \leq e^{-x} \leq 1$  for  $x \in [0, \infty)$ , we find that  $|1 - ze^{-x}| > M(z)^{-1}$  for  $x \in [0, \infty)$  with  $M(z)$  given in (15). Introducing these bounds in (17) we obtain the bound (15) for  $R_N(z, s, a)$ .

For any  $a \in \mathbb{C}$ , an explicit expression of  $r_N(ax)$  is given, using the Cauchy integral formula, by

$$r_N(ax) = \frac{(ax)^N}{2\pi i} \int_C \frac{e^{-\omega} d\omega}{(\omega - ax)\omega^N}, \quad (18)$$

where  $C$  is the contour of a region containing the points 0 and  $ax$  in its interior. With the same argument as in Theorem 1, we have

$$|r_N(ax)| \leq 2e^r \frac{|ax|^N}{r^N}, \quad N = 1, 2, 3, \dots, \quad (19)$$

where  $r > 0$  is an arbitrary number. Introducing (19) and  $|1 - ze^{-x}| > M(z)^{-1}$  in (17) and after trivial manipulations we obtain the bound for  $R_N(z, s, a)$  given in (16).  $|\text{Li}_{n+s}(z)|$  is bounded by some positive function  $C(z, s)$  independent of  $n$  and  $\lim_{N \rightarrow \infty} R_N(z, s, a) = 0$  for  $0 < a < 1$  by (15), then (14) gives a convergent expansion for  $|a| < 1$  when  $N \rightarrow \infty$ .  $\square$

**Remark 1.** We can use the expansion (14) to obtain the Taylor expansion of the Hurwitz–Lerch zeta function at any  $a \in \mathbb{Z}^+$ . Just introduce the expansion (14) into the right-hand side of the equality [11, Eq. (2), p. 121]:

$$\Phi(z, s, a+m) = \frac{1}{z^m} \left\{ \Phi(z, s, a) - \frac{1}{a^s} - \sum_{k=1}^{m-1} \frac{z^k}{(a+k)^s} \right\}, \quad m = 1, 2, 3, \dots$$

**Theorem 3.** For  $z \in \mathbb{C} \setminus [0, \infty)$ ,  $|z| > 1$ ,  $\Re a > 0$  and  $\Re s > 0$ , an asymptotic expansion of  $\Phi(z, s, a)$  for large  $z$  and fixed  $a$  and  $s$  is given by

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \left\{ \left[ \sum_{n=0}^{N-1} \frac{A_n(z, s, a)}{z^{n+1}} + R_N^1(z, s, a) \right] + \frac{(\log(-z))^s}{(-z)^a} \left[ \sum_{n=0}^{M-1} \frac{B_n(s, a)}{(\log(-z))^{n+1}} + R_M^2(z, s, a) \right] \right\}, \quad (20)$$

where  $N, M = 1, 2, 3, \dots$ ,

$$A_n(z, s, a) \equiv \frac{\Gamma(s, (a - n - 1) \log(-z)) - \Gamma(s)}{(a - n - 1)^s}, \quad (21)$$

and

$$B_n(s, a) \equiv \frac{1}{2} \binom{s-1}{n} \begin{cases} \psi((a+1)/2) - \psi(a/2) & \text{if } n=0, \\ n! 2^{-n} [\zeta(n+1, a/2) - \zeta(n+1, (a+1)/2)] & \text{if } n > 0. \end{cases} \quad (22)$$

In these formulas  $\Gamma(s, w)$  is the incomplete gamma function [1, Eq. (6.5.3)],  $\psi(a)$  is the digamma function [1, Eq. (6.3.1)] and  $\zeta(s, a)$  Hurwitz zeta function [11, p. 88, Eq. (1)]. The error terms  $R_N^1(z, s, a)$  and  $R_M^2(z, s, a)$  verify  $R_N^1(z, s, a) = \mathcal{O}(z^{-N-1})$  for  $N \leq a-1$  and  $R_M^2(z, s, a) = \mathcal{O}((\log z)^{-M-1})$  for  $M = 1, 2, 3, \dots$ , when  $z \rightarrow \infty$ . More precisely, for  $0 < s < 1$  and  $z \in (-\infty, -1)$ , error bounds for the remainders are given by

$$\begin{aligned} |R_N^1(z, s, a)| &\leq \left| \frac{\Gamma(s) - \Gamma(s, (\Re a - N - 1) \log(-z))}{(1 + N - \Re a)^s z^{N+1}} \right|, \\ |R_M^2(z, s, a)| &\leq \frac{M!}{2^{1+M}} \left| \binom{s-1}{M} \right| \frac{|\zeta(M+1, \Re a/2) - \zeta(M+1, (\Re a + 1)/2)|}{|\log(-z)|^{M+1}}. \end{aligned} \quad (23)$$

Nevertheless, for complex values of the parameters, error bounds for the remainders are given by

$$\begin{aligned} |R_N^1(z, s, a)| &\leq \frac{|\Gamma(\Re s) - \Gamma(\Re s, \Re a - N - 1) \log(-z)|}{(1 + N - \Re a)^{\Re s} |z|^{N+1}}, \\ |R_M^2(z, s, a)| &\leq \left| \binom{s-1}{M} \right| \left| \left[ \frac{\Gamma(M+1) \Gamma(-\Re s - 1)}{\Gamma(M - \Re s)} {}_1F_1(M+1, \Re s + 2, \Re a) \right. \right. \\ &\quad \left. \left. + \frac{\Gamma(\Re s + 1) {}_1F_1(M - \Re s, -\Re s, \Re a)}{(\Re a)^{\Re s + 1}} \right] \frac{1}{|\log(-z)|^{M+1}} \right|, \end{aligned} \quad (24)$$

where  ${}_1F_1(a, b, z)$  is the Kummer confluent hypergeometric function. Moreover, the  $N$ -expansion in (20) is convergent for  $|z| > 1$ .

**Proof.** Consider first  $z \in (-\infty, -1)$ . For  $\Re s > 0$  and  $\Re a > 0$  we have

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} e^{-ax}}{1 - ze^{-x}} dx. \quad (25)$$

We perform the change of variable  $x \mapsto y \log(-z)$  and divide the integral (25) into two parts:

$$\Phi(z, s, a) = \frac{(\log(-z))^s}{\Gamma(s)} [I_1(z, s, a) + I_2(z, s, a)],$$

where

$$I_1(z, s, a) \equiv \int_0^1 \frac{y^{s-1} e^{-ay \log(-z)}}{1 + e^{(1-y) \log(-z)}} dy, \quad I_2(z, s, a) \equiv \int_1^\infty \frac{y^{s-1} e^{-ay \log(-z)}}{1 + e^{(1-y) \log(-z)}} dy.$$

To obtain a convergent expansion of  $I_1(z, s, a)$  we perform the change of variable  $y \mapsto 1 - t$ , multiply and divide by  $(-z)^{-t}$  and expand  $(1 + (-z)^{-t})^{-1}$  in power series of  $(-z)^{-t}$ ,

$$\begin{aligned} \frac{1}{1 + (-z)^{-t}} &= \sum_{n=0}^{N-1} (-(-z)^{-t})^n + r_N^1((-z)^{-t}), \\ r_N^1((-z)^{-t}) &\equiv \frac{(-1)^N (-z)^{-Nt}}{1 + (-z)^{-t}}. \end{aligned} \quad (26)$$

Interchanging sum and integral and taking into account that

$$\begin{aligned} &\int_0^1 (1-t)^{s-1} e^{-t(n+1-a) \log(-z)} dt \\ &= (-1)^{a-n} (\log(-z))^{-s} \frac{\Gamma(s) - \Gamma(s, (a-n-1) \log(-z))}{z^{n+1-a} (a-n-1)^s}, \end{aligned}$$

we obtain

$$I_1(z, s, a) = (\log(-z))^{-s} \left[ \sum_{n=0}^{N-1} \frac{A_n(z, s, a)}{z^{n+1}} + R_N^1(z, s, a) \right], \quad (27)$$

where the coefficients  $A_n(z, s, a)$  are defined in (21) and

$$R_N^1(z, s, a) \equiv (\log(-z))^s \int_0^1 (1-t)^{s-1} (-z)^{-t-a(1-t)} r_N^1((-z)^{-t}) dt. \quad (28)$$

On the other hand, to obtain an asymptotic expansion of  $I_2(z, s, a)$  for large  $z$  we perform the change of variable  $y \mapsto 1 + t/\log(-z)$  and expand  $(1 + t/\log(-z))^{s-1}$  in power series of  $t/\log(-z)$ ,

$$\left(1 + \frac{t}{\log(-z)}\right)^{s-1} = \sum_{n=0}^{M-1} \binom{s-1}{n} \left(\frac{t}{\log(-z)}\right)^n + r_M^2\left(\frac{t}{\log(-z)}\right), \quad (29)$$

where

$$\begin{aligned} r_M^2\left(\frac{t}{\log(-z)}\right) &= \frac{1}{M!} \frac{d^M}{dx^M} (1+x)^{s-1} \Big|_{x=\xi} \left(\frac{t}{\log(-z)}\right)^M \\ &= \binom{s-1}{M} (1+\xi)^{s-M-1} \left(\frac{t}{\log(-z)}\right)^M \end{aligned} \quad (30)$$

for some  $\xi \in (0, t/\log(-z))$ . Interchanging sum and integral and taking into account that

$$\int_0^\infty \frac{e^{-at} t^n}{1+e^{-t}} dt = \frac{1}{2} \begin{cases} \psi((a+1)/2) - \psi(a/2) & \text{if } n=0, \\ n! 2^{-n} [\zeta(n+1, a/2) - \zeta(n+1, (a+1)/2)] & \text{if } n>0, \end{cases}$$

we obtain

$$I_2(z, s, a) = (-z)^{-a} \left[ \sum_{n=0}^{M-1} \frac{B_n(z, s)}{(\log(-z))^{n+1}} + R_M^2(z, s, a) \right], \quad (31)$$

where the coefficients  $B_n(z, s)$  are defined in (22) and

$$R_M^2(z, s, a) \equiv \frac{1}{\log(-z)} \int_0^\infty \frac{e^{-at}}{1+e^{-t}} r_M^2\left(\frac{t}{\log(-z)}\right) dt. \quad (32)$$

The expansion (20) is the sum of the expansions (27) and (31). The remainder in (20) is the sum of the remainders in (28) and (32).

For  $z < -1$  and  $0 < s < 1$ , both the expansions (26) and (29) verify the error test. Therefore we find, for  $N, M = 1, 2, 3, \dots$ ,

$$|r_N^1((-z)^{-t})| \leq (-z)^{-Nt}, \quad \left| r_M^2\left(\frac{t}{\log(-z)}\right) \right| \leq \left| \binom{s-1}{M} \right| \frac{t^M}{(\log(-z))^M}.$$

Introducing these bounds in (28) and (32), respectively, and after trivial manipulations we obtain (23).

Now we consider  $z \in \mathbb{C}$ . The left-hand side of (20), as defined in (25), is an analytic function of  $z$  for  $z \in \mathbb{C} \setminus [1, \infty)$ . Functions  $A_n(z, s, a)z^{-(n+1)}$  and  $(-z)^{-a}(\log(-z))^{s-n-1}$  in the right-hand side of (20) are analytic functions of  $z$  for  $z \in \mathbb{C} \setminus [0, \infty)$ . The remainders  $R_N^1(z, s, a)$  and  $R_M^2(z, s, a)$  in the right-hand side of (20) and defined by (28) and (32), respectively, are analytic functions of  $z$  for  $z \in \mathbb{C} \setminus [0, \infty)$  and  $|z| > 1$ . The right-hand side of (20) coincides with the left-hand side for  $z \in (-\infty, -1)$ . Therefore, they also coincide in  $\{z \in \mathbb{C}, |z| > 1, z \notin [1, \infty)\}$ .

For any  $z \in \mathbb{C}$ ,  $z \notin [1, \infty)$  and  $|z| > 1$  we consider the explicit expressions of  $r_N^1((-z)^{-t})$  and  $r_M^2(t/\log(-z))$  given by (26) and (30), respectively. They have the following bounds:

$$|r_N^1((-z)^{-t})| \leq |z|^{-Nt}$$

and

$$\left| r_M^2\left(\frac{t}{\log(-z)}\right) \right| \leq \begin{cases} \left| \binom{s-1}{M} \right| (1 + |\frac{t}{\log(-z)}|)^{s-M} |\frac{t}{\log(-z)}|^M & \text{if } s \geq M+1, \\ 1 & \text{if } s < M+1. \end{cases}$$

Introducing these bounds in (28) and (32) and after trivial manipulations we obtain (24).

The coefficients and remainder in the expansion (27) verify  $A_N(z, s, a) = \mathcal{O}(1)$  and  $R_N^1(z, s, a) = \mathcal{O}(z^{-N-1})$  when  $z \rightarrow \infty$  while  $N \leq a - 1$ . Therefore, (27) is an asymptotic expansion for  $z \rightarrow \infty$  while  $N \leq a - 1$ . Moreover, using [1, Eq. (6.5.32)] we see that  $\lim_{N \rightarrow \infty} R_N^1(z, s, a) = 0$  for  $|z| > 1$  and then (27) is a convergent expansion.  $\square$

#### 4. Numerical experiments

Tables 1–9 show numerical experiments about the approximation supplied by Theorems 1–3 and the accuracy of the error bounds.

In all these tables, the second column represents the value of  $\Phi(z, s, a)$ . The third and sixth columns represent, respectively, a first and a second order approximation given by the corresponding theorem. Fourth and seventh columns represent the respective relative errors  $|R_N(z, s, a)/\Phi(z, s, a)|$  or  $(|R_N^1(z, s, a)| + |R_M^2(z, s, a)|)/|\Phi(z, s, a)|$ . Fifth and last columns represent the respective error bounds given by the corresponding theorem.

Table 1  
 $z = -2.5, s = 1.25$

| $a$   | $\Phi(z, s, a)$ | 1st ( $N = 1$ )<br>ord. appr. | Rel.<br>error | Rel. error<br>bound | 2nd ( $N = 2$ )<br>ord. appr. | Rel.<br>error | Rel. error<br>bound |
|-------|-----------------|-------------------------------|---------------|---------------------|-------------------------------|---------------|---------------------|
| 10    | 0.017565        | 0.0160669                     | 0.085         | 0.36                | 0.0175014                     | 0.0036        | 0.05                |
| 100   | 0.000911614     | 0.0009035                     | 0.0089        | 0.039               | 0.00091157                    | 4.0e-5        | 5.6e-4              |
| 500   | 0.000121058     | 0.000120842                   | 0.0018        | 0.008               | 0.000121058                   | 1.7e-6        | 2.25e-5             |
| 1000  | 5.08534e-5      | 5.0808e-5                     | 8.9e-4        | 0.0039              | 5.08533e-5                    | 4.3e-7        | 5.6e-6              |
| 5000  | 6.79668e-6      | 6.79547e-6                    | 1.7e-4        | 8.0e-4              | 6.79668e-6                    | 1.8e-8        | 2.26e-7             |
| 10000 | 2.8574e-6       | 2.85714e-6                    | 8.9e-5        | 4.0e-4              | 2.8574e-6                     | 4.3e-9        | 5.6e-8              |

Table 2  
 $z = -1.5 + i, s = 1.25 - 0.5i$

| $a$   | $\Phi(z, s, a)$               | 1st ( $N = 1$ )<br>ord. appr. | Rel.<br>error | Rel. error<br>bound | 2nd ( $N = 2$ )<br>ord. appr. | Rel.<br>error | Rel. error<br>bound |
|-------|-------------------------------|-------------------------------|---------------|---------------------|-------------------------------|---------------|---------------------|
| 10    | 0.00200079 +<br>0.0223825i    | 0.000814248 +<br>0.020869i    | 0.085         | 0.47                | 0.00191876 +<br>0.0223934i    | 0.0037        | 0.08                |
| 100   | -0.00105842 +<br>0.00052902i  | -0.00105314 +<br>0.00051981i  | 0.009         | 0.05                | -0.00105844 +<br>0.00052898i  | 4.5e-5        | 8.9e-4              |
| 500   | -0.000148034 -<br>5.32356e-5i | -0.000147759 -<br>5.33036e-5i | 1.8e-3        | 0.01                | -0.000148033 -<br>5.32358e-5i | 1.7e-6        | 3.6e-5              |
| 1000  | -5.08776e-5 -<br>4.21875e-5i  | -5.08183e-5 -<br>4.21812e-5i  | 9.0e-4        | 5.0e-3              | -5.08776e-5 -<br>4.21875e-5i  | 4.3e-7        | 9.0e-6              |
| 5000  | -6.47749e-7 -<br>8.81076e-6i  | -6.46771e-7 -<br>8.8095e-6i   | 1.8e-4        | 1.0e-3              | -6.47749e-7 -<br>8.81076e-6i  | 1.7e-8        | 3.6e-7              |
| 10000 | 1.00227e-6 -<br>3.5764e-6i    | 1.00238e-6 -<br>3.57608e-6i   | 9.0e-5        | 5.0e-4              | 1.00227e-6 -<br>3.5764e-6i    | 4.3e-9        | 9.0e-8              |

Table 3

 $z = 0.2 - 1.25i, s = 0.75, \text{Arg}(a) = \pi/4$ 

| $ a $ | $\Phi(z, s, a)$               | 1st ( $N = 1$ )<br>ord. appr.   | Rel.<br>error | Rel. error<br>bound | 2nd ( $N = 2$ )<br>ord. appr.   | Rel.<br>error | Rel. error<br>bound |
|-------|-------------------------------|---------------------------------|---------------|---------------------|---------------------------------|---------------|---------------------|
| 10    | -0.00241498 -<br>0.128186i    | -0.00236462 -<br>0.1198i        | 0.065         | 0.78                | -0.00295554 -<br>0.127443i      | 0.007         | 0.27                |
| 100   | -0.000430345 -<br>0.021441i   | -0.000420495 -<br>0.0213038e-6i | 0.006         | 0.08                | -0.000431003 -<br>0.0214397e-6i | 6.6e-5        | 0.003               |
| 500   | -0.000126378 -<br>0.00637947i | -0.000125757 -<br>0.00637133i   | 0.001         | 0.017               | -0.000126386 -<br>0.00637946i   | 2.6e-6        | 1.2e-4              |
| 1000  | -7.49614e-5 -<br>0.00379083i  | -7.47757e-5 -<br>0.00378841i    | 0.0006        | 0.008               | -7.49626e-5 -<br>0.00379083i    | 6.5e-7        | 3.0e-5              |
| 5000  | -2.23743e-5 -<br>0.00113314i  | -2.23631e-5 -<br>0.001133i      | 1.3e-4        | 1.7e-3              | -2.23743e-5 -<br>0.00113314e-9i | 2.6e-8        | 1.2e-6              |
| 10000 | -1.33005e-5 -<br>0.000673729i | -1.32972e-5 -<br>0.000673686i   | 6.4e-5        | 8.0e-4              | -1.33005e-5 -<br>0.000673729i   | 6.0e-9        | 3.0e-7              |

Table 4

 $z = 1 - 1.5i, s = 0.75$ 

| $a$    | $\Phi(z, s, a)$       | 1st ( $N = 1$ )<br>ord. appr. | Rel.<br>error | Rel. error<br>bound | 2nd ( $N = 2$ )<br>ord. appr. | Rel.<br>error | Rel. error<br>bound |
|--------|-----------------------|-------------------------------|---------------|---------------------|-------------------------------|---------------|---------------------|
| 0.1    | 5.02909 -<br>1.29793i | 5.03413 -<br>1.42109i         | 0.02          | 0.03                | 5.02573 -<br>1.28722i         | 0.0021        | 0.0027              |
| 0.05   | 8.86483 -<br>1.35695i | 8.86813 -<br>1.42109i         | 0.007         | 0.009               | 8.86393 -<br>1.35415i         | 0.00033       | 0.0004              |
| 0.01   | 31.0327 -<br>1.40782i | 31.0335 -<br>1.42109i         | 0.0004        | 0.0005              | 31.0327 -<br>1.40771i         | 3.9e-6        | 4.6e-6              |
| 0.005  | 52.5933 -<br>1.41443i | 52.5937 -<br>1.42109i         | 1.3e-4        | 1.5e-4              | 52.5933 -<br>1.4144i          | 5.8e-7        | 6.7e-7              |
| 0.001  | 177.23 -<br>1.41976i  | 177.23 -<br>1.42109i          | 7.6e-6        | 9.0e-6              | 177.23 -<br>1.41975i          | 7.0e-9        | 8.0e-9              |
| 0.0005 | 298.48 -<br>1.42042i  | 298.48 -<br>1.42109i          | 2.25e-6       | 2.7e-6              | 298.48 -<br>1.42042i          | 1.16e-9       | 1.19e-9             |

Table 5

 $z = 0.8 - 0.5i, s = 1.5, \text{Arg}(a) = \pi/8$ 

| $ a $  | $\Phi(z, s, a)$       | 1st ( $N = 1$ )<br>ord. appr. | Rel.<br>error | Rel. error<br>bound | 2nd ( $N = 2$ )<br>ord. appr. | Rel.<br>error | Rel. error<br>bound |
|--------|-----------------------|-------------------------------|---------------|---------------------|-------------------------------|---------------|---------------------|
| 0.1    | 26.911 -<br>18.486i   | 27.0492 -<br>18.5388i         | 0.0045        | 0.06                | 26.8939 -<br>18.4881i         | 0.0005        | 0.0075              |
| 0.05   | 75.0515 -<br>50.6359i | 75.1247 -<br>50.6619i         | 0.00086       | 0.01                | 75.0471 -<br>50.6365i         | 5.0e-5        | 7.0e-4              |
| 0.01   | 832.21 -<br>556.535i  | 832.225 -<br>556.54i          | 1.6e-5        | 2.0e-4              | 832.21 -<br>556.535 i         | 1.9e-7        | 2.5e-6              |
| 0.005  | 2352.5 -<br>1572.36i  | 2352.51 -<br>1572.36i         | 2.9e-6        | 3.5e-5              | 2352.5 -<br>1572.36i          | 1.7e-8        | 2.0e-7              |
| 0.001  | 26294.1 -<br>17569.6i | 26294.1 -<br>17569.6i         | 5.0e-8        | 6.0e-7              | 26294.1 -<br>17569.6 i        | 6.0e-11       | 8.0e-10             |
| 0.0005 | 74369.7 -<br>49692.7i | 74369.7 -<br>49692.7i         | 9.0e-9        | 1.0e-7              | 74369.7 -<br>49692.7i         | 5.3e-12       | 7.0e-11             |

Table 6

 $a = 2.3, s = 1.5$ 

| $z$     | $\Phi(-z, s, a)$ | 1st ord. appr.<br>( $N = M = 1$ ) | Rel.<br>error | Rel. error<br>bound | 2nd ord. appr.<br>( $N = M = 2$ ) | Rel.<br>error | Rel. error<br>bound |
|---------|------------------|-----------------------------------|---------------|---------------------|-----------------------------------|---------------|---------------------|
| 10      | 0.0496236        | 0.0621341                         | 0.252108      | 0.35                | 0.04472                           | 0.0988        | 0.15                |
| 100     | 0.0064151        | 0.006712                          | 0.046         | 0.05                | 0.006366                          | 0.0077        | 0.01                |
| 1000    | 0.00067          | 0.0006744                         | 0.006         | 0.0068              | 0.006698                          | 0.0005        | 0.0007              |
| 10000   | 6.74144e-5       | 6.74648e-5                        | 7.5e-4        | 7.8e-4              | 6.74123e-5                        | 3.0e-5        | 4.2e-5              |
| 100000  | 6.74604e-6       | 6.74659e-6                        | 8.2e-5        | 8.3e-5              | 6.74603e-6                        | 1.75e-6       | 2.39e-6             |
| 1000000 | 6.74654e-7       | 6.7466e-7                         | 8.56e-6       | 8.66e-6             | 6.74654e-7                        | 9.7e-8        | 1.3e-7              |

Table 7

 $a = 3.2 + 1.2i, s = 3.1 - 0.15i$ 

| $z$     | $\Phi(-z, s, a)$            | 1st ord. appr.<br>( $N = M = 1$ ) | Rel.<br>error | Rel. error<br>bound | 2nd ord. appr.<br>( $N = M = 2$ ) | Rel.<br>error | Rel. error<br>bound |
|---------|-----------------------------|-----------------------------------|---------------|---------------------|-----------------------------------|---------------|---------------------|
| 10      | 0.0013519 -<br>0.0040017i   | 0.0014445 -<br>0.0055717i         | 0.37          | 0.65                | 0.0016019 -<br>0.0033243i         | 0.17          | 0.32                |
| 100     | 9.8022e-5 -<br>5.1738e-4i   | 8.46496e-5 -<br>5.30846e-4i       | 0.036         | 0.098               | 9.99479e-5 -<br>5.19642e-4i       | 0.0056        | 0.016               |
| 1000    | 8.72394e-6 -<br>5.29676e-5i | 8.60328e-6 -<br>5.30886e-5i       | 0.003         | 0.01                | 8.61373e-6 -<br>5.2967e-5i        | 6.3e-5        | 4.0e-4              |
| 10000   | 8.61369e-7 -<br>5.3077e-6i  | 8.6015e-7 -<br>5.30892e-6i        | 3.2e-4        | 1.0e-3              | 8.61373e-7 -<br>5.3077e-6i        | 7.9e-7        | 7.0e-6              |
| 100000  | 8.60274e-8 -<br>5.3088e-7i  | 8.60152e-8 -<br>5.30892e-7i       | 3.2e-5        | 1.0e-4              | 8.60273e-8 -<br>5.3088e-7i        | 7.8e-9        | 1.0e-7              |
| 1000000 | 8.60164e-9 -<br>5.30891e-8i | 8.60152e-9 -<br>5.30892e-8i       | 3.2e-6        | 1.0e-5              | 8.60164e-9 -<br>5.30891e-8i       | 6.65e-11      | 1.4e-9              |

Table 8

 $a = 2.3, s = 1.1, \text{Arg}(z) = -\pi/5$ 

| $ z $   | $\Phi(-z, s, a)$            | 1st ord. appr.<br>( $N = M = 1$ ) | Rel.<br>error | Rel. error<br>bound | 2nd ord. appr.<br>( $N = M = 2$ ) | Rel.<br>error | Rel. error<br>bound |
|---------|-----------------------------|-----------------------------------|---------------|---------------------|-----------------------------------|---------------|---------------------|
| 10      | 0.0527126 +<br>0.0299105i   | 0.0601857 +<br>0.0410424i         | 0.22          | 0.29                | 0.0514345 +<br>0.0257576i         | 0.07          | 0.1                 |
| 100     | 0.0059631 +<br>0.0041668i   | 0.0060599 +<br>0.0438851i         | 0.033         | 0.037               | 0.0059581 +<br>0.0041377i         | 0.004         | 0.0056              |
| 1000    | 6.05129e-4 +<br>4.37488e-4i | 6.06195e-4 +<br>4.40352e-4i       | 0.0041        | 0.0043              | 6.05106e-4 +<br>4.37326e-4i       | 2.18e-4       | 2.97e-4             |
| 10000   | 6.06094e-5 +<br>4.64715e-5i | 6.06205e-5 +<br>4.4043e-5i        | 4.5e-4        | 4.6e-4              | 6.06093e-5 +<br>4.40101e-5i       | 1.1e-5        | 1.5e-5              |
| 100000  | 6.06194e-6 +<br>4.404e-6i   | 6.06206e-6 +<br>4.40434e-6i       | 4.77e-5       | 4.83e-5             | 6.06194e-6 +<br>4.404e-6i         | 5.89e-7       | 7.97e-7             |
| 1000000 | 6.06204e-7 +<br>4.40431e-7i | 6.06206e-7 +<br>4.40434e-7i       | 4.89e-6       | 4.9e-6              | 6.06204e-7 +<br>4.40431e-7i       | 3.0e-8        | 4.08e-8             |

Table 9

 $a = 2.4 - 0.5i$ ,  $s = 3.6$ ,  $\text{Arg}(z) = \pi/2$ 

| $ z $    | $\Phi(-z, s, a)$            | 1st ord. appr.<br>( $N = M = 1$ ) | Rel.<br>error | Rel. error<br>bound | 2nd ord. appr.<br>( $N = M = 2$ ) | Rel.<br>error | Rel. error<br>bound |
|----------|-----------------------------|-----------------------------------|---------------|---------------------|-----------------------------------|---------------|---------------------|
| 100      | 0.00202396 –<br>6.05279e–4i | 0.00218045 –<br>7.71334e–4i       | 0.1           | 0.24                | 0.0019784 –<br>5.6363e–4i         | 0.03          | 0.094               |
| 1000     | 2.21931e–4 –<br>7.95909e–5i | 2.25921e–4 –<br>7.68608e–5i       | 0.02          | 0.047               | 2.21127e–4 –<br>7.69636e–5i       | 0.003         | 0.01                |
| 10000    | 2.25914e–5 –<br>7.8998e–6i  | 2.26534e–5 –<br>7.91478e–6i       | 0.0027        | 0.007               | 2.25862e–5 –<br>7.90473e–6i       | 3.0e–4        | 0.001               |
| 100000   | 2.26465e–6 –<br>7.90974e–7i | 2.26529e–6 –<br>7.90909e–7i       | 2.69e–4       | 8.6e–4              | 2.26466e–6 –<br>7.90926e–7i       | 2.2e–5        | 7.7e–5              |
| 1000000  | 2.2652e–7 –<br>7.90905e–8i  | 2.26526e–7 –<br>7.90893e–8i       | 2.32e–5       | 9.76e–5             | 2.2652e–7 –<br>7.90906e–8i        | 1.45e–6       | 5.2e–6              |
| 10000000 | 2.26525e–8 –<br>7.90896e–9i | 2.26525e–8 –<br>7.90895e–9i       | 2.0e–6        | 1.0e–5              | 2.26525e–8 –<br>7.90895e–9i       | 8.8e–8        | 3.2e–7              |

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