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Asymptotic relations between the Hahn-type polynomials and Meixner–Pollaczek, Jacobi, Meixner and Krawtchouk polynomials

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Abstract

It has been shown in Ferreira et al. [Asymptotic relations in the Askey scheme for hypergeometric orthogonal polynomials, Adv. in Appl. Math. 31(1) (2003) 61–85], López and Temme [Approximations of orthogonal polynomials in terms of Hermite polynomials, Methods Appl. Anal. 6 (1999) 131–146; The Askey scheme for hypergeometric orthogonal polynomials viewed from asymptotic analysis, J. Comput. Appl. Math. 133 (2001) 623–633] that the three lower levels of the Askey table of hypergeometric orthogonal polynomials are connected by means of asymptotic relations. In Ferreira et al. [Limit relations between the Hahn polynomials and the Hermite, Laguerre and Charlier polynomials, submitted for publication] we have established new asymptotic connections between the fourth level and the two lower levels. In this paper, we continue with that program and obtain asymptotic expansions between the fourth level and the third level: we derive 16 asymptotic expansions of the Hahn, dual Hahn, continuous Hahn and continuous dual Hahn polynomials in terms of Meixner–Pollaczek, Jacobi, Meixner and Krawtchouk polynomials. From these expansions, we also derive three new limits between those polynomials. Some numerical experiments show the accuracy of the approximations and, in particular, the accuracy in the approximation of the zeros of those polynomials.

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Keywords: Askey scheme of hypergeometric orthogonal polynomials; Asymptotic expansions; Limits between polynomials

1. Introduction

It is well known the existence of several limit relations between polynomials of the Askey scheme of hypergeometric orthogonal polynomials [6]. For example, the limit

$$\lim_{\alpha\to\infty} \alpha^{-n/2} L_n^{\alpha}\left(x\sqrt{2\alpha}+\alpha\right) = \frac{(-1)^n 2^{-n/2}}{n!} H_n(x), \tag{1}$$

shows that, when the variable is properly scaled, the Laguerre polynomials become the Hermite polynomials for large values of the order parameter. Moreover, this limit gives insight in the location of the zeros of the Laguerre polynomials for large values of the order parameter.

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It has been recently pointed out that this limit may be obtained from an asymptotic expansion of the Laguerre polynomials in terms of the Hermite polynomials for large α [7]:

$$L_n^{\alpha}(x) = (-1)^n B^n \sum_{k=0}^{n} \frac{c_k}{B^k} \frac{H_{n-k}(X)}{(n-k)!}, \tag{2}$$

where the values of X and B and other details are given in [7]. This expansion has an asymptotic character for large of α. The limit (1) is obtained from the first order approximation of this expansion.

The asymptotic method from which expansions like (2) are obtained was introduced and developed in [1,7–9]. The method to approximate orthogonal polynomials in terms of Hermite, Laguerre and Charlier polynomials is described in [7,9,1], respectively. Based on those methods, asymptotic expansions of Laguerre and Jacobi polynomials in terms of Hermite polynomials are given in [7]. Asymptotic expansions of Meixner–Pollaczek, Jacobi, Meixner and Krawtchouk polynomials in terms of Laguerre polynomials are given in [9]. Asymptotic expansions of Meixner–Pollaczek, Jacobi, Meixner and Krawtchouk polynomials in terms of Hermite and Charlier polynomials are given in [1]. Asymptotic expansions of the four Hahn polynomials in terms of Hermite, Laguerre and Charlier polynomials are given in [2].

This asymptotic method is based on the availability of a generating function for the polynomials and it is different from the techniques described in [4,5], which are based on a connection problem. Our method is also different from the sophisticated uniform methods considered for example in [3,10] which consider large values of the degree n and fixed values of the variable x and the parameters. Another different method, based on a representation of the Hahn polynomials in terms of a difference operator is considered in [11], in which an asymptotic formula for the Hahn polynomials $Q_n(x;\alpha,\beta,N)$ in terms of Jacobi polynomials is derived for large values of N. In our method we keep the degree n fixed and let some parameter(s) of the polynomial go to infinity. The purpose of this paper is the continuation of the asymptotic program developed in [1,2,7–9] and derive asymptotic expansions (and limits when it is possible) between the fourth and third levels of the Askey tableau.

In the following section we summarize the asymptotic expansions and the limit relations obtained in this paper. In Section 3 we briefly explain the principles of the Hermite-type, Laguerre-type and Charlier-type asymptotic approximations introduced in [1,7,9]. In Section 4 we prove the formulas presented in Section 2. Some numerical experiments illustrating the accuracy of the approximations are given in Section 5.

2. Descending asymptotic expansions and limits

Throughout this paper, we will use the notation and the definitions of the hypergeometric orthogonal polynomials of the third and fourth levels of the Askey-tableau considered in [6] (Fig. 1):

Continuous dual Hahn: $S_n(x^2;a,b,c)$

$$\frac{S_n(x^2;a,b,c)}{(a+b)_n(a+c)_n} = {}_3F_2\left(\left.\begin{array}{c} -n, a+\mathrm{i}x, a-\mathrm{i}x \\ a+b, a+c \end{array}\right| 1\right).$$

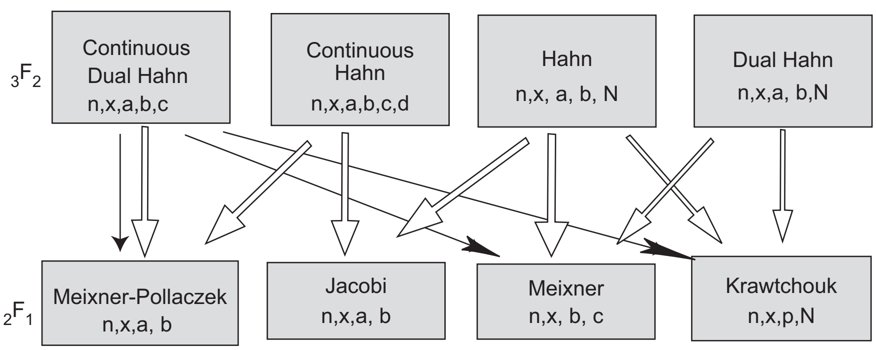

Fig. 1. Thick arrows indicate known limits, whereas thin arrows indicate new limits derived in this paper.

Continuous Hahn: $p_n(x; \alpha, \beta, \overline{\alpha}, \overline{\beta})$, $\alpha, \beta \in \mathbb{C}$ ($\alpha = a + \mathrm{i}c$, $\beta = b + \mathrm{i}d$, $a, b, c, d \in \mathbb{R}$)

$$p_n(x; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = \mathrm{i}^n \frac{(\alpha+\overline{\alpha})_n(\alpha+\overline{\beta})_n}{n!} {}_3F_2\left(\left. \begin{matrix} -n, n+\alpha+\beta+\overline{\alpha}+\overline{\beta}-1, \alpha+\mathrm{i}x \\ \alpha+\overline{\alpha}, \alpha+\overline{\beta} \end{matrix} \right| 1 \right).$$

Hahn: $Q_n(x; a, b, N)$

$$Q_n(x; a, b, N) = {}_3F_2\left(\left. \begin{matrix} -n, n+a+b+1, -x \\ a+1, -N \end{matrix} \right| 1 \right), \quad n = 0, 1, 2, \ldots, N.$$

Dual Hahn: $R_n(\lambda(x); a, b, N)$, with $\lambda(x) = x(x+a+b+1)$

$$R_n(\lambda(x); a, b, N) = {}_3F_2\left(\left. \begin{matrix} -n, -x, x+a+b+1 \\ a+1, -N \end{matrix} \right| 1 \right), \quad n = 0, 1, 2, \ldots, N.$$

Meixner–Pollaczek: $P_n^{(\lambda)}(x; \phi)$

$$P_n^{(\lambda)}(x; \phi) = \frac{(2\lambda)_n}{n!} \mathrm{e}^{\mathrm{i}n\phi} {}_2F_1\left(\left. \begin{matrix} -n, \lambda+\mathrm{i}x \\ 2\lambda \end{matrix} \right| 1-\mathrm{e}^{-2\mathrm{i}\phi} \right).$$

Jacobi: $P_n^{(\alpha,\beta)}(x)$

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1\left(\left. \begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \right| \frac{1-x}{2} \right).$$

Meixner: $M_n(x; \beta, c)$

$$M_n(x; \beta, c) = {}_2F_1\left(\left. \begin{matrix} -n, -x \\ \beta \end{matrix} \right| 1-\frac{1}{c} \right).$$

Krawtchouk: $K_n(x; p, N)$

$$K_n(x; p, N) = {}_2F_1\left(\left. \begin{matrix} -n, -x \\ -N \end{matrix} \right| \frac{1}{p} \right), \quad n = 0, 1, 2, \ldots, N.$$

The orthogonality property of the polynomials of the Askey table only holds when the variable x and other parameters which appear in the polynomials are restricted to certain real intervals [6]. The expansions that we resume below are valid for larger domains of the variable and the parameters and for any $n \in \mathbb{N}$. Nevertheless, for the sake of clearness, we will consider that the variable and the parameters are restricted to the orthogonality intervals given in [6]. All the square roots that appear in what follows assume real positive values for real positive argument. The coefficients c_k given below are the coefficients of the Taylor expansion at $w = 0$ of the given functions $f(x, w)$:

$$c_k = \frac{1}{k!} \left. \frac{\partial^k f(x, w)}{\partial w^k} \right|_{w=0}. \tag{3}$$

The first three coefficients c_k are $c_0 = 1$, $c_1 = c_2 = 0$. Higher coefficients c_k, $k \geqslant 3$ can be obtained recurrently from a differential equation satisfied by $f(x, w)$ or directly from their definition (3) (using computer algebra programs like *Mathematica*, *Matlab*, etc.). In our previous works about polynomials located in the first levels of the Askey-tableau [1,2], we have given recurrent formulas for c_k using a differential equation satisfied by $f(x, w)$. However, the functions $f(x, w)$ involved in this paper are more complicated and analytic formulas for c_k are too cumbersome to be written down here.

2.1. Continuous dual Hahn to Meixner–Pollaczek

2.1.1. Asymptotic expansion for large a and b

$$\frac{S_n(x^2; a, b, c)}{(a+b)_n n!} = \sum_{k=0}^{n} c_k P_{n-k}^{(C)}(X; A), \tag{4}$$

$A \neq m\pi, m \in \mathbb{Z}$ is an arbitrary constant,

$$C = p_1(x)\cos A + \frac{1}{2}p_1(x)^2 - p_2(x), \quad X = -\frac{p_1(x)\cos(2A) + [p_1(x)^2 - 2p_2(x)]\cos A}{2\sin A}, \tag{5}$$

where

$$p_1(x) = c + \frac{ab - x^2}{a+b},$$
$$p_2(x) = \frac{c(c+1)}{2} + \frac{abc}{a+b} + \frac{ab(1+a)(1+b) - [c + 2ab + (1+c)(1+2(a+b))]x^2 + x^4}{2(a+b)(1+a+b)} \tag{6}$$

and

$$f(x,w) = (1 - e^{iA}w)^{C-iX}(1 - e^{-iA}w)^{C+iX}(1-w)^{-c+ix}\,{}_2F_1\left(\left.\begin{matrix} a+ix, b+ix \\ a+b \end{matrix}\right| w\right).$$

2.1.2. Asymptotic property

$$c_k P_{n-k}^{(C)}(X; A) = \mathcal{O}(a^{n+[k/3]-k}) \tag{7}$$

when $a,\ b \to \infty$ uniformly in c with $a \sim b$ and c/a bounded.

2.1.3. Limit

$$\lim_{a\to\infty} \frac{S_n(\tilde{x}; a, a, \tilde{c})}{(2a)^n n!} = P_n^{(c)}(x; A), \tag{8}$$

where

$$\tilde{x} = -a^2 - 4a\sqrt{2a\sin\left(\frac{A}{2}\right)\left[c\sin\left(\frac{3A}{2}\right) - x\cos\left(\frac{3A}{2}\right)\right]},$$
$$\tilde{c} = -a + 2c\cos A - 2x\sin A + 2\sqrt{2a\sin\left(\frac{A}{2}\right)\left[c\sin\left(\frac{3A}{2}\right) - x\cos\left(\frac{3A}{2}\right)\right]}. \tag{9}$$

2.2. Dual Hahn to Meixner–Pollaczek

2.2.1. Asymptotic expansion for large a and N

$$\frac{(-N)_n R_n(\lambda(x); a, b, N)}{n!} = \sum_{k=0}^{n} c_k P_{n-k}^{(C)}(X; A), \quad \lambda(x) \equiv x(x+a+b+1), \tag{10}$$

$A \neq m\pi, m \in \mathbb{Z}$ is an arbitrary constant, X and C are given in (5) with

$$p_1(x) = \frac{\lambda(x)}{a+1} - N,$$
$$p_2(x) = \frac{(N-x)(N-x-1)}{2} + \frac{x(b+x)[2(2+a)(x-N) + (b+x-1)(x-1)]}{2(a+1)(2+a)} \tag{11}$$

and

$$f(x,w) = (1 - e^{iA}w)^{C-iX}(1 - e^{-iA}w)^{C+iX}(1-w)^{N-x}\,{}_2F_1\left(\left.\begin{matrix} -x, -b-x \\ a+1 \end{matrix}\right| w\right).$$

2.2.2. Asymptotic property

$$c_k P_{n-k}^{(C)}(X; A) = \mathcal{O}(N^{n+[k/3]-k}) \quad \text{when } a, N \to \infty \text{ with } a \sim N. \tag{12}$$

2.3. Continuous Hahn to Meixner–Pollaczek

2.3.1. Asymptotic expansion for large a and b

$$\frac{(2a+2b-1)_n\, p_n(x;\alpha,\beta,\overline{\alpha},\overline{\beta})}{(2a)_n(a+b+\mathrm{i}(c-d))_n i^n} = \sum_{k=0}^{n} c_k P_{n-k}^{(C)}(X;A), \tag{13}$$

$A \neq m\pi$, $m \in \mathbb{Z}$ is an arbitrary constant, X and C are given in (5) with

$$p_1(x) = \frac{\mathrm{i}[1-2(a+b)][b(c+x)+a(d+x)]}{a[(a+b)+\mathrm{i}(c-d)]},$$

$$\begin{aligned} p_2(x) = {} & \frac{p_1(x)}{\mathrm{i}(1+2a)[1+a+b+\mathrm{i}(c-d)][b(c+x)+a(d+x)]} \\ & \times \{(a+b)[(1+2b)(c+x)(d+x+b(c+x))+a^2(b-2(d+x)^2) \\ & + a[b^2-(d+x)(2c+d+3x)-b(-1+4dx+4x^2+4c(d+x))]]\} \end{aligned} \tag{14}$$

and

$$\begin{aligned} f(x,w) = {} & (1-\mathrm{e}^{\mathrm{i}A}w)^{C-\mathrm{i}X}(1-\mathrm{e}^{-\mathrm{i}A}w)^{C+\mathrm{i}X}(1-w)^{1-2(a+b)} \\ & \times {}_3F_2\left(\left. \begin{matrix} a+b-\frac{1}{2}, a+b, a+\mathrm{i}(c+x) \\ 2a, a+c+\mathrm{i}(b-d) \end{matrix} \right| -\frac{4w}{(1-w)^2} \right). \end{aligned}$$

2.3.2. Asymptotic property

$$c_k P_{n-k}^{(C)}(X;A) = \mathcal{O}(a^{n+[k/3]-k}) \quad \text{when } a,\ b \to \infty \text{ with } a \sim b. \tag{15}$$

2.4. Hahn to Meixner–Pollaczek

2.4.1. Asymptotic expansion for large a, b and N

$$\frac{(a+b+1)_n}{n!} Q_n(x;a,b,N) = \sum_{k=0}^{n} c_k P_{n-k}^{(C)}(X;A), \tag{16}$$

$A \neq m\pi$, $m \in \mathbb{Z}$ is an arbitrary constant, X and C are given in (5) with

$$p_1(x) = \frac{(1+a+b)[(1+a)N-(2+a+b)x]}{(1+a)N},$$

$$\begin{aligned} p_2(x) = {} & \frac{(2+3(a+b)+(a+b)^2)}{2N(N-1)(a+2)(a+1)} \times \{N^2(2+3a+a^2)-N(2+a)[1+a+(6+2a+2b)x] \\ & + (3+a+b)x[a-b+x(4+a+b)]\} \end{aligned} \tag{17}$$

and

$$\begin{aligned} f(x,w) = {} & (1-\mathrm{e}^{\mathrm{i}A}w)^{C-\mathrm{i}X}(1-\mathrm{e}^{-\mathrm{i}A}w)^{C+\mathrm{i}X}(1-w)^{-1-a-b} \\ & \times {}_3F_2\left(\left. \begin{matrix} (a+b+1)/2, (a+b+2)/2, -x \\ a+1, -N \end{matrix} \right| -\frac{4w}{(1-w)^2} \right). \end{aligned}$$

2.4.2. Asymptotic properties

$$c_k P_{n-k}^{(C)}(X;A) = \mathcal{O}(a^{n+[k/3]-k}) \quad \text{when } a, b, N \to \infty \text{ with } a \sim b \sim N. \tag{18}$$

2.5. Continuous dual Hahn to Jacobi

2.5.1. Asymptotic expansion for large a and b

$$\frac{S_n(x^2;a,b,c)}{(a+b)_n n!} = \sum_{k=0}^{n} c_k P_{n-k}^{(A,C)}(X), \tag{19}$$

where $X \neq \pm 1$ is an arbitrary constant,

$$A = \frac{1}{X+1}[2p_1(x)^2 + p_1(x) + 3p_1(x)X - 4p_2(x) + X^2 - X - 2],$$
$$C = \frac{1}{X-1}[2p_1(x)^2 - p_1(x) + 3p_1(x)X - 4p_2(x) + X^2 + X - 2], \tag{20}$$

with $p_1(x)$ and $p_2(x)$ given in (6) and

$$f(x,w) = \frac{R(1+R-w)^A(1+R+w)^C}{2^{A+C}}(1-w)^{-c+\mathrm{i}x}\,{}_2F_1\left(\left.\begin{matrix} a+\mathrm{i}x, b+\mathrm{i}x \\ a+b \end{matrix}\right| w\right),$$

with $R = \sqrt{1-2Xw+w^2}$.

2.5.2. Asymptotic property

$$c_k P_{n-k}^{(A,C)}(X) = \mathcal{O}(a^{n+[k/3]-k}) \quad \text{when } a, b \to \infty \text{ uniformly in } c \text{ with } a \sim b \text{ and } c/a \text{ bounded.} \tag{21}$$

2.6. Dual Hahn to Jacobi

2.6.1. Asymptotic expansion for large a and N

$$\frac{(-N)_n R_n(\lambda(x);a,b,N)}{n!} = \sum_{k=0}^{n} c_k P_{n-k}^{(A,C)}(X), \quad \lambda(x) \equiv x(x+a+b+1), \tag{22}$$

where $X \neq \pm 1$ is an arbitrary constant, A and C are given in (20) and

$$f(x,w) = \frac{R(1+R-w)^A(1+R+w)^C}{2^{A+C}}(1-w)^{N-x}\,{}_2F_1\left(\left.\begin{matrix} -x, -b-x \\ a+1 \end{matrix}\right| w\right).$$

2.6.2. Asymptotic property

$$c_k P_{n-k}^{(A,C)}(X) = \mathcal{O}(N^{n+[k/3]-k}) \quad \text{when } a,\ N \to \infty \text{ with } a \sim N. \tag{23}$$

2.7. Continuous Hahn to Jacobi

2.7.1. Asymptotic expansion for large a and b

$$\frac{(2a+2b-1)_n p_n(x;\alpha,\beta,\overline{\alpha},\overline{\beta})}{(2a)_n(a+b+\mathrm{i}(c-d))_n i^n} = \sum_{k=0}^{n} c_k P_{n-k}^{(A,C)}(X), \tag{24}$$

where $X \neq \pm 1$ is an arbitrary constant, A and C are given in (20) and

$$f(x,w) = \frac{R(1+R-w)^A(1+R+w)^C}{2^{A+C}(1-w)^{2(a+b)-1}}\,{}_3F_2\left(\left.\begin{matrix} a+b-\frac{1}{2}, a+b, a+\mathrm{i}(c+x) \\ 2a, a+c+\mathrm{i}(b-d) \end{matrix}\right| -\frac{4w}{(1-w)^2}\right).$$

2.7.2. Asymptotic property

$$c_k P_{n-k}^{(A,C)}(X) = \mathcal{O}(a^{n+[k/3]-k}) \quad \text{when } a,\ b \to \infty \text{ with } a \sim b. \tag{25}$$

2.8. Hahn to Jacobi

2.8.1. Asymptotic expansion for large a, b, N

$$\frac{(a+b+1)_n}{n!} Q_n(x; a, b, N) = \sum_{k=0}^{n} c_k P_{n-k}^{(A,C)}(X), \tag{26}$$

where $X \neq \pm 1$ is an arbitrary constant, A and C are given in (20) and

$$f(x, w) = \frac{R(1+R-w)^A (1+R+w)^C}{2^{A+C}(1-w)^{1+a+b}} {}_3F_2\left(\left. \begin{matrix} (a+b+1)/2, (a+b+2)/2, -x \\ a+1, -N \end{matrix} \right| -\frac{4w}{(1-w)^2} \right).$$

2.8.2. Asymptotic property

$$c_k P_{n-k}^{(A,C)}(X) = \mathcal{O}(a^{n+[k/3]-k}) \quad \text{when } a,\ b,\ N \to \infty \text{ with } a \sim b \sim N. \tag{27}$$

2.9. Continuous dual Hahn to Meixner

2.9.1. Asymptotic expansion for large a, b **y** c

$$\frac{S_n(x^2; a, b, c)}{(a+b)_n n!} = \sum_{k=0}^{n} c_k \frac{(A)_{n-k}}{(n-k)!} M_{n-k}(X; A, C), \tag{28}$$

where $C \neq 0, 1$ is an arbitrary constant,

$$X = \frac{C^2}{1-C}[p_1(x)^2 + p_1(x) - 2p_2(x)], \quad A = (1+C)p_1(x) + Cp_1(x)^2 - 2Cp_2(x), \tag{29}$$

with $p_1(x)$ and $p_2(x)$ given in (6) and

$$f(x, w) = \left(1 - \frac{w}{C}\right)^{-X} (1-w)^{X+A} (1-w)^{-c+\mathrm{i}x} {}_2F_1\left(\left. \begin{matrix} a+\mathrm{i}x, b+\mathrm{i}x \\ a+b \end{matrix} \right| w \right).$$

2.9.2. Asymptotic property

$$c_k \frac{(A)_{n-k}}{(n-k)!} M_{n-k}(X; A, C) = \mathcal{O}(a^{n+[k/3]-k}) \tag{30}$$

when $a,\ b \to \infty$ uniformly in c with $a \sim b$ and c/a bounded.

2.9.3. Limit

$$\lim_{a \to \infty} \left[\frac{S_n(\tilde{x}; a, a, \tilde{c})}{\left(c + \frac{C-1}{C}\frac{x}{1+2a}\right)_n (2a)_n} \right] = M_n(x; c, C), \tag{31}$$

where

$$\tilde{x} = -a^2 - 2\frac{a}{C}\sqrt{2ax(1-C)}, \quad \tilde{c} = -a + c - \frac{1-C}{C}x - \frac{2}{C}\sqrt{2ax(1-C)}. \tag{32}$$

2.10. Dual Hahn to Meixner

2.10.1. Asymptotic expansion for large a and N

$$\frac{(-N)_n R_n(\lambda(x); a, b, N)}{n!} = \sum_{k=0}^{n} c_k \frac{(A)_{n-k}}{(n-k)!} M_{n-k}(X; A, C), \quad \lambda(x) \equiv x(x+a+b+1), \tag{33}$$

where $C \neq 0, 1$ is an arbitrary constant, X and A are given in (29), $p_1(x)$ and $p_2(x)$ in (11) and

$$f(x,w) = \left(1 - \frac{w}{C}\right)^{-X} (1-w)^{X+A} (1-w)^{N-x} {}_2F_1\left(\left.\begin{matrix} -x, -b-x \\ a+1 \end{matrix}\right| w\right).$$

2.10.2. Asymptotic property

$$c_k \frac{(A)_{n-k}}{(n-k)!} M_{n-k}(X; A, C) = \mathcal{O}(N^{n-k-1}) \quad \text{when } a,\ N \to \infty \text{ with } a \sim N. \tag{34}$$

2.11. Continuous Hahn to Meixner

2.11.1. Asymptotic expansions for large a and b

$$\frac{(2a+2b-1)_n p_n(x; \alpha, \beta, \overline{\alpha}, \overline{\beta})}{(2a)_n (a+b+\mathrm{i}(c-d))_n i^n} = \sum_{k=0}^{n} c_k \frac{(A)_{n-k}}{(n-k)!} M_{n-k}(X; A, C), \tag{35}$$

where $C \neq 0, 1$ is an arbitrary constant, X and A are given in (29), $p_1(x)$ and $p_2(x)$ in (14) and

$$f(x,w) = \left(1 - \frac{w}{C}\right)^{-X} (1-w)^{X+A} (1-w)^{1-2(a+b)} {}_3F_2\left(\left.\begin{matrix} a+b-\frac{1}{2}, a+b, a+\mathrm{i}(c+x) \\ 2a, a+c+\mathrm{i}(b-d) \end{matrix}\right| -\frac{4w}{(1-w)^2}\right).$$

2.11.2. Asymptotic property

$$c_k \frac{(A)_{n-k}}{(n-k)!} M_{n-k}(X; A, C) = \mathcal{O}(a^{n+[k/3]-k}) \quad \text{when } a,\ b \to \infty \text{ with } a \sim b. \tag{36}$$

2.12. Hahn to Meixner

2.12.1. Asymptotic expansions for large a, b and N

$$\frac{(a+b+1)_n}{n!} Q_n(x; a, b, N) = \sum_{k=0}^{n} c_k \frac{(A)_{n-k}}{(n-k)!} M_{n-k}(X; A, C), \tag{37}$$

where $C \neq 0, 1$ is an arbitrary constant, X and A are given in (29), $p_1(x)$ and $p_2(x)$ in (17) and

$$f(x,w) = \left(1 - \frac{w}{C}\right)^{-X} (1-w)^{X+A} (1-w)^{-1-a-b} {}_3F_2\left(\left.\begin{matrix} (a+b+1)/2, (a+b+2)/2, -x \\ a+1, -N \end{matrix}\right| -\frac{4w}{(1-w)^2}\right).$$

2.12.2. Asymptotic property

$$c_k \frac{(A)_{n-k}}{(n-k)!} M_{n-k}(X; A, C) = \mathcal{O}(a^{n-k}) \quad \text{when } a,\ b,\ N \to \infty \text{ with } a \sim b \sim N. \tag{38}$$

2.13. Continuous dual Hahn to Krawtchouk

2.13.1. Asymptotic expansion for large a and b

$$\frac{S_n(x^2; a, b, c)}{(a+b)_n n!} = \sum_{k=0}^{n} \binom{C}{n-k} c_k K_{n-k}(X; A, C), \tag{39}$$

where $A \neq 0, 1$ is an arbitrary constant,

$$X = \frac{A^2}{1-A}[p_1(x)^2 - p_1(x) - 2p_2(x)], \quad C = p_1(x) + \frac{X}{A}, \tag{40}$$

with $p_1(x)$ and $p_2(x)$ given in (6) and

$$f(x,w) = \left(1 - \frac{1-A}{A} w\right)^{-X} (1+w)^{X-C} (1-w)^{-c+\mathrm{i}x} {}_2F_1\left(\left.\begin{matrix} a+\mathrm{i}x, b+\mathrm{i}x \\ a+b \end{matrix}\right| w\right).$$

2.13.2. Asymptotic property

$$\binom{C}{n-k} c_k K_{n-k}(X; A, C) = \mathcal{O}(a^{n+[k/3]-k}) \tag{41}$$

when $a,\ b \to \infty$ uniformly in c with $a \sim b$ and c/a bounded.

2.13.3. Limit

$$\lim_{a\to\infty} \frac{S_n(\tilde{x}; a, a, \tilde{c})}{\binom{\tilde{C}}{n}(2a)_n n!} = K_n(x; A, c), \tag{42}$$

where

$$\tilde{x} = -a^2 - 2\frac{a}{A}\sqrt{2a(2A^2c - 3Ax + x)}, \quad \tilde{C} = \frac{A^2(3+2a)c + x - A[(2a+1)c + 3x]}{A(A-1)(2a+1)},$$

$$\tilde{c} = -a + c - \frac{1}{A}\left[x + 2\sqrt{2a(2A^2c - 3Ax + x)}\right]. \tag{43}$$

2.14. Dual Hahn to Krawtchouk

2.14.1. Asymptotic expansion for large a and N

$$\frac{(-N)_n R_n(\lambda(x); a, b, N)}{n!} = \sum_{k=0}^{n} \binom{C}{n-k} c_k K_{n-k}(X; A, C), \quad \lambda(x) \equiv x(x+a+b+1), \tag{44}$$

where $A \neq 0, 1$ is an arbitrary constant, X and C are given in (40), $p_1(x)$ and $p_2(x)$ in (11) and

$$f(x, w) = \left(1 - \frac{1-A}{A}w\right)^{-X} (1+w)^{X-C}(1-w)^{N-x}\, {}_2F_1\left(\left.\begin{matrix} -x, -b-x \\ a+1 \end{matrix}\right| w\right).$$

2.14.2. Asymptotic property

$$\binom{C}{n-k} c_k K_{n-k}(X; A, C) = \mathcal{O}(N^{n+[k/3]-k}) \quad \text{when } a,\ N \to \infty \text{ with } a \sim N. \tag{45}$$

2.15. Continuous Hahn to Krawtchouk

2.15.1. Asymptotic expansions for large a and b

$$\frac{(2a+2b-1)_n p_n(x; \alpha, \beta, \overline{\alpha}, \overline{\beta})}{(2a)_n (a+b+\mathrm{i}(c-d))_n i^n} = \sum_{k=0}^{n} \binom{C}{n-k} c_k K_{n-k}(X; A, C), \tag{46}$$

where $A \neq 0, 1$ is an arbitrary constant, X and C are given in (40), $p_1(x)$ and $p_2(x)$ in (14) and

$$f(x, w) = \left(1 - \frac{1-A}{A}w\right)^{-X} (1+w)^{X-C}(1-w)^{N-x}(1-w)^{1-2(a+b)}$$
$$\times\, {}_3F_2\left(\left.\begin{matrix} a+b-\frac{1}{2}, a+b, a+\mathrm{i}(c+x) \\ 2a, a+c+\mathrm{i}(b-d) \end{matrix}\right| -\frac{4w}{(1-w)^2}\right).$$

2.15.2. Asymptotic property

$$\binom{C}{n-k} c_k K_{n-k}(X; A, C) = \mathcal{O}(a^{n+[k/3]-k}) \quad \text{when } a,\ b \to \infty \text{ with } a \sim b. \tag{47}$$

2.16. Hahn to Krawtchouk

2.16.1. Asymptotic expansions for large a, b and N

$$\frac{(a+b+1)_n}{n!} Q_n(x; a, b, N) = \sum_{k=0}^{n} \binom{C}{n-k} c_k K_{n-k}(X; A, C), \tag{48}$$

where $A \neq 0, 1$ is an arbitrary constant, X and C are given in (40), $p_1(x)$ and $p_2(x)$ in (17) and

$$f(x,w) = \left(1 - \frac{1-A}{A} w\right)^{-X} (1+w)^{X-C} (1-w)^{N-x} (1-w)^{-1-a-b} \\ \times {}_3F_2\left(\left. \begin{matrix} (a+b+1)/2, (a+b+2)/2, -x \\ a+1, -N \end{matrix} \right| -\frac{4w}{(1-w)^2} \right).$$

2.16.2. Asymptotic property

$$\binom{C}{n-k} c_k K_{n-k}(X; A, C) = \mathcal{O}(a^{n+[k/3]-k}) \quad \text{when } a,\ b,\ N \to \infty \text{ with } a \sim b \sim N. \tag{49}$$

3. Principles of the asymptotic approximations

The asymptotic expansions of polynomials in terms of polynomials listed above follow from an asymptotic principle based on the "matching" of their generating functions [7]. We give below details for the case in which the basic approximant are the Meixner–Pollaczek polynomials and resume the main formulas for the remaining cases (Jacobi, Meixner and Krawtchouk).

3.1. Expansions in terms of Meixner–Pollaczek polynomials

The Meixner–Pollaczek polynomials $P_n^{(\lambda)}(x; \phi)$ for $n \in \mathbb{N}$ follow from the generating function [6, (1.7.11)]

$$(1 - e^{i\phi} w)^{-\lambda + ix} (1 - e^{-i\phi} w)^{-\lambda - ix} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(x; \phi) w^n. \tag{50}$$

This formula gives the following Cauchy-type integral for the Meixner–Pollaczek polynomials

$$P_n^{(\lambda)}(x; \phi) = \frac{1}{2\pi i} \int_{\mathscr{C}} (1 - e^{i\phi} w)^{-\lambda + ix} (1 - e^{-i\phi} w)^{-\lambda - ix} dw / w^{n+1}, \tag{51}$$

where $\mathscr{C}$ is a circle around the origin and the integration is in the positive direction.

All of the polynomials $p_n(x)$ of the Askey table have a generating function of the form

$$F(x, w) = \sum_{n=0}^{\infty} p_n(x) w^n, \tag{52}$$

where $F(x, w)$ is analytic with respect to w in a domain that contains the origin $w = 0$, and $p_n(x)$ is independent of w.

The relation (52) gives for $p_n(x)$ the Cauchy-type integral

$$p_n(x) = \frac{1}{2\pi i} \int_{\mathscr{C}} F(x, w) \frac{dw}{w^{n+1}},$$

where $\mathscr{C}$ is a circle around the origin inside the domain where $F(x, w)$ is analytic as a function of w. We define $f(x, w)$ by means of the formula:

$$F(x, w) = (1 - e^{iA} w)^{-C + iX} (1 - e^{-iA} w)^{-C - iX} f(x, w), \tag{53}$$

where $A \neq m\pi$, $m \in \mathbb{Z}$ is an arbitrary constant, C and X do not depend on w and can be chosen arbitrarily. This gives

$$p_n(x) = \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{C}} (1 - \mathrm{e}^{\mathrm{i}A} w)^{-C+\mathrm{i}X} (1 - \mathrm{e}^{-\mathrm{i}A} w)^{-C-\mathrm{i}X} \frac{f(x, w)}{w^{n+1}} \, \mathrm{d}w. \tag{54}$$

Since $f(x, w)$ is also analytic as a function of w at $w = 0$, we can expand $f(x, w) = \sum_{k=0}^{\infty} c_k w^k$. Substituting this expansion in (54) and taking into account (51), we obtain

$$p_n(x) = \sum_{k=0}^{n} c_k P_{n-k}^{(C)}(X; A). \tag{55}$$

The choice of C and X is based on our requirement that $c_1 = c_2 = 0$. This happens if we take

$$C = p_1(x) \cos A + \frac{1}{2} p_1(x)^2 - p_2(x), \quad X = \frac{-1}{2 \sin A} [p_1(x) \cos 2A + (p_1(x)^2 - 2p_2(x)) \cos A]$$

and we assume that $F(x, 0) = p_0(x) = 1$ (which implies $c_0 = 1$). The special choice of X and C is crucial for obtaining asymptotic properties.

3.2. Expansions in terms of Jacobi polynomials

If $F(x, w)$ is the generating function of the polynomials $p_n(x)$ then

$$p_n(x) = \sum_{k=0}^{n} c_k P_{n-k}^{(A,C)}(X), \tag{56}$$

where $P_n^{(A,C)}(X)$ are the Jacobi polynomials, $X \neq \pm 1$ is an arbitrary constant and the coefficients c_k follow from

$$f(x, w) = \sum_{k=0}^{\infty} c_k w^k \quad \text{with } f(x, w) = \frac{R(1 + R - w)^A (1 + R + w)^C}{2^{A+C}} F(x, w),$$

$$R = \sqrt{1 - 2Xw + w^2}, \quad c_1 = c_2 = 0,$$

$$A = \frac{1}{X + 1} [2p_1(x)^2 + p_1(x) + 3p_1(x)X - 4p_2(x) + X^2 - X - 2],$$

$$C = \frac{1}{X - 1} [2p_1(x)^2 - p_1(x) + 3p_1(x)X - 4p_2(x) + X^2 + X - 2],$$

and we assume that $F(x, 0) = p_0(x) = 1$ (which implies $c_0 = 1$).

3.3. Expansions in terms of Meixner polynomials

If $F(x, w)$ is the generating function of the polynomials $p_n(x)$ then

$$p_n(x) = \sum_{k=0}^{n} \frac{c_k (A)_{n-k}}{(n - k)!} M_{n-k}(X; A, C), \tag{57}$$

where $M_n(x; A, C)$ are the Meixner polynomials, $n \in \mathbb{N}$ and the coefficients c_k follow from

$$f(x, w) = \sum_{k=0}^{\infty} c_k w^k \quad \text{with } f(x, w) = \left(1 - \frac{w}{C}\right)^{-X} (1 - w)^{X+A} F(x, w),$$

where $C \neq 0, 1$ is an arbitrary constant. The choice of A and X is based on our requirement that $c_1 = c_2 = 0$. This happens if we take

$$A = (1+C)p_1(x) + Cp_1(x)^2 - 2Cp_2(x), \quad X = \frac{C^2}{1-C}[p_1(x)^2 + p_1(x) - 2p_2(x)]$$

and we assume that $F(x,0) = p_0(x) = 1$ (which implies $c_0 = 1$).

3.4. Expansions in terms of Krawtchouk polynomials

If $F(x,w)$ is the generating function of the polynomials $p_n(x)$ then

$$p_n(x) = \sum_{k=0}^{n} \binom{C}{n-k} c_k K_{n-k}(X; A, C), \tag{58}$$

where $K_n(X; A, C)$ are the Krawtchouk polynomials, $A \neq 0, 1$ is an arbitrary constant and the coefficients c_k follow from

$$f(x,w) = \sum_{k=0}^{\infty} c_k w^k \quad \text{with } f(x,w) = \left(1 - \frac{1-A}{A} w\right)^{-X} (1+w)^{X-C} F(x,w),$$

$$X = \frac{A^2}{1-A}[p_1(x)^2 - p_1(x) - 2p_2(x)], \quad C = p_1(x) + \frac{A}{1-A}[p_1(x)^2 - p_1(x) - 2p_2(x)],$$

$c_1 = c_2 = 0$ and we assume that $F(x,0) = p_0(x) = 1$ (which implies $c_0 = 1$).

3.5. Asymptotic properties of the coefficients c_k

The asymptotic nature of the expansions (55)–(58) for large values of some of the parameters of the polynomial $p_n(x)$ depends on the asymptotic behaviour of the coefficients c_k. To prove the asymptotic character of the expansions given in Section 2 we will need the following lemma proved in [9]:

Lemma 1. *Let $\phi(w)$ be an analytic function at $w = 0$, with Maclaurin expansion of the form*

$$\phi(w) = \mu^s \omega^m (a_0 + a_1 w + a_2 w^2 + a_3 w^3 + \cdots),$$

where m is a positive integer, s is an integer number, and a_k are complex numbers that satisfy $a_k = \mathcal{O}(1)$ when $\mu \to \infty$, $a_0 \neq 0$. Let c_k denote the coefficients of the power series of $f(w) = \mathrm{e}^{\phi(w)}$, that is,

$$f(w) = \mathrm{e}^{\phi(w)} = \sum_{k=0}^{\infty} c_k w^k.$$

Then $c_0 = 1$, $c_k = 0$, $k = 1, 2, \ldots, m-1$ and $c_k = \mathcal{O}(\mu^{[sk/m]})$ if $s > 0$, $c_k = \mathcal{O}(\mu^s)$ if $s \leqslant 0$ when $\mu \to \infty$.

4. Proofs of formulas given in Section 2

In this section, we give the proofs of the derivation of the formulas obtained in Section 2. We detail the proofs of the formulas in Section 2.1 and give the more important ideas for the other cases.

4.1. Proofs of formulas of Section 2.1

We use the method of Section 3.1 for obtaining an asymptotic representation of the continuous dual Hahn polynomials in terms of the Meixner–Pollazcek polynomials. The continuous dual Hahn polynomials have the generating function [6, (1.3.12)]

$$F(x,w) = (1-w)^{-c+\mathrm{i}x}\,{}_2F_1\left(\left.\begin{matrix} a+\mathrm{i}x, b+\mathrm{i}x \\ a+b \end{matrix}\right| w\right) = \sum_{n=0}^{\infty} \frac{S_n(x^2;a,b,c)}{(a+b)_n n!} w^n.$$

This is the explicit form of Eq. (52) for the example of the continuous dual Hahn polynomials with $p_n(x) = S_n(x^2; a, b, c)/$
$(a+b)_n n!$. From (53) we have

$$f(x,w) = (1-\mathrm{e}^{\mathrm{i}A}w)^{C-\mathrm{i}X}(1-\mathrm{e}^{-\mathrm{i}A}w)^{C+\mathrm{i}X}(1-w)^{-c+\mathrm{i}x}\,{}_2F_1\left(\left.\begin{matrix} a+\mathrm{i}x, b+\mathrm{i}x \\ a+b \end{matrix}\right| w\right).$$

Then, formula (55) becomes formula (4) in Section 2.1.1:

$$\frac{S_n(x^2;a,b,c)}{(a+b)_n n!} = \sum_{k=0}^{n} c_k P_{n-k}^{(C)}(X;A),$$

with A, C and X given by (5) and c_k defined in (3).

The asymptotic property in Section 2.1.2 follows from the fact that the function $f(x,w)$ can be written as $f(x,w) = \mathrm{e}^{\phi(w)}$ with

$$\phi(w) = (C-\mathrm{i}X)\log(1-\mathrm{e}^{\mathrm{i}A}w) + (C+\mathrm{i}X)\log(1-\mathrm{e}^{-\mathrm{i}A}w) + (-c+\mathrm{i}x)\log(1-w)$$
$$+ \log{}_2F_1\left(\left.\begin{matrix} a+\mathrm{i}x, b+\mathrm{i}x \\ a+b \end{matrix}\right| w\right).$$

The function

$$y(w) = \log{}_2F_1\left(\left.\begin{matrix} a+\mathrm{i}x, b+\mathrm{i}x \\ a+b \end{matrix}\right| w\right)$$

satisfies the following differential equation in the variable w :

$$w(1-w)(y'' + y'^2) + [a+b-(a+b+2\mathrm{i}x+1)w]y' - (a+\mathrm{i}x)(b+\mathrm{i}x) = 0.$$

Substituting the Maclaurin series of $y(w) = \sum_{k=1}^{\infty} b_k w^k$ into this differential equation, we obtain

$$b_1 = \frac{(a+\mathrm{i}x)(b+\mathrm{i}x)}{a+b}, \quad b_2 = (a+\mathrm{i}x)(b+\mathrm{i}x)\frac{(a+b)(a+b+2\mathrm{i}x+1)-(a+\mathrm{i}x)(b+\mathrm{i}x)}{2(a+b)^2(a+b+1)}$$

and

$$b_{k+1} = \frac{1}{(k+1)(k+a+b)}\Bigg\{k(k+a+b+2\mathrm{i}x)b_k - kb_kb_1$$
$$+ \sum_{j=0}^{k-2}(j+1)b_{j+1}[(k-j-1)b_{k-j-1} - (k-j)b_{k-j}]\Bigg\}.$$

Then, $b_1 = \mathcal{O}(a)$, $b_2 = \mathcal{O}(a)$ and using the above recurrence we can show by induction over k that $b_k = \mathcal{O}(a)$ for $k > 2$. We trivially have $C = \mathcal{O}(a)$ and $X = \mathcal{O}(a)$. Therefore, the function $\phi(w)$ verifies Lemma 1 with $\mu = a$, $s = 1$ and $m = 3$: $\phi(w) = a\omega^3(a_0 + a_1 w + a_2 w^2 + \cdots)$ with $a_0 \neq 0$ and $a_k = \mathcal{O}(1)$, $a \to \infty$. Hence, we have $c_k = \mathcal{O}(a^{[k/3]})$. On the other hand, $P_0^{(C)}(X;A) = 1 = \mathcal{O}(a^0)$, $P_1^{(C)}(X;A) = p_1(x) = \mathcal{O}(a)$, and using the following recurrence relation [6, Eq. 1.7.3],

$$(n+1)P_{n+1}^{(C)}(X;A) - [2X\sin A + (n+C)\cos A]P_n^{(C)}(X;A) + (n+2C-1)P_{n-1}^{(C)}(X;A) = 0,$$

we can show by induction over n that $P_{n-k}^{(C)}(X;A) = \mathcal{O}(a^{n-k})$ and we obtain 2.1.2.

Considering the first term of the expansion (4), solving $x(X,C)$ and $c(X,C)$ and putting $a = b$ we obtain the limit (8).

4.2. Proofs of formulas of Section 2.2

Substitute:

$$F(x,w)=(1-w)^{N-x}{}_2F_1\left(\left.\begin{matrix}-x,-b-x\\ a+1\end{matrix}\right|w\right)\quad\text{and}\quad p_n(x)=\frac{(-N)_nR_n(\lambda(x);a,b,N)}{n!}$$

in the formulas of Section 3.1 to obtain (10).

We have $C=\mathcal{O}(N)$ and $X=\mathcal{O}(N)$. Therefore the function $\phi(w)=\log f(x,w)$ verifies Lemma 1 with $\mu=N$, $s=1$ and $m=3$ and we have $c_k=\mathcal{O}(N^{[k/3]})$. On the other hand, using the recurrence of the previous subsection, we can show by induction over n that $P_{n-k}^{(C)}(X;A)=\mathcal{O}(N^{n-k})$ and we obtain 2.2.2.

4.3. Proofs of formulas of Section 2.3

Substitute:

$$F(x,w)=(1-w)^{1-2(a+b)}{}_3F_2\left(\left.\begin{matrix}a+b-\frac{1}{2},a+b,a+\mathrm{i}(b+x)\\ 2a,a+c+\mathrm{i}(b+d)\end{matrix}\right|-\frac{4w}{(1-w)^2}\right)$$

and

$$p_n(x)=\frac{(2a+2b-1)_n}{(2a)_n(a+b+\mathrm{i}(c-b))_n}p_n(x;\alpha,\beta,\overline{\alpha},\overline{\beta})$$

in the formulas of Section 3.1 to obtain (13).

Substitute the Maclaurin series of $y(w)=\log{}_3F_2$ into its differential equation [12]. From a similar argument to that of Section 4.1 we have that the Taylor coefficients at $w=0$ of $y(w)$ are of the order $\mathcal{O}(a)$. We trivially have $C=\mathcal{O}(a)$ and $X=\mathcal{O}(a)$. Therefore, the function $\phi(w)=\log f(x,w)$ verifies Lemma 1 with $\mu=a$, $s=1$ and $m=3$ and we have $c_k=\mathcal{O}(a^{[k/3]})$. Using the recurrence of Section 4.1, we can show by induction over n that $P_{n-k}^{(C)}(X;A)=\mathcal{O}(a^{n-k})$ and we obtain 2.3.2.

4.4. Proofs of formulas of Section 2.4

Substitute:

$$F(x,w)=(1-w)^{-a-b-1}{}_3F_2\left(\left.\begin{matrix}(a+b+1)/2,(a+b+2)/2,-x\\ a+1,-N\end{matrix}\right|-\frac{4w}{(1-w)^2}\right)$$

and

$$p_n(x)=\frac{(-N)_nQ_n(x;a,b,N)}{(b+1)_nn!}$$

in the formulas of Section 3.1 to obtain (16).

We have $C=\mathcal{O}(a)$ and $X=\mathcal{O}(a)$. Therefore, the function $\phi(w)=\log f(x,w)$ verifies Lemma 1 with $\mu=a$, $s=1$ and $m=3$ and we have $c_k=\mathcal{O}(a^{[k/3]})$. Using the recurrence of Section 4.1, we can show by induction over n that $P_{n-k}^{(C)}(X;A)=\mathcal{O}(a^{n-k})$ and we obtain 2.4.2.

4.5. Proofs of formulas of Section 2.5

Substitute:

$$F(x,w)=(1-w)^{-c+\mathrm{i}x}{}_2F_1\left(\left.\begin{matrix}a+\mathrm{i}x,b+\mathrm{i}x\\ a+b\end{matrix}\right|w\right)\quad\text{and}\quad p_n(x)=\frac{S_n(x^2;a,b,c)}{(a+b)_nn!}$$

in the formulas of Section 3.2 to obtain (19).

From Section 4.1 we have that the coefficients of the Taylor expansion at $w=0$ of the logarithm of the above ${}_2F_1$ function are of the order $\mathcal{O}(a)$. We have $A=\mathcal{O}(a)$ and $C=\mathcal{O}(a)$. Therefore, the function $\phi(w)=\log f(x,w)$ verifies

Lemma 1 with $\mu=a$, $s=1$ and $m=3$ and we have $c_k=\mathcal{O}(a^{[k/3]})$. On the other hand, $P_0^{(A,C)}(X)=1=\mathcal{O}(a^0)$, $P_1^{(A,C)}(X)=p_1(x)=\mathcal{O}(a)$, and using the following recurrence relation [6]:

$$\frac{2(n+1)(n+A+C+1)}{(2n+A+C+1)(2n+A+C+2)}P_{n+1}^{(A,C)}(X)+\left[\frac{C^2-A^2}{(2n+A+C)(2n+A+C+2)}-X\right]P_n^{(A,C)}(X)$$
$$+\frac{2(n+A)(n+C)}{(2n+A+C)(2n+A+C+1)}P_{n-1}^{(A,C)}(X)=0,$$

we can show by induction over n that $P_{n-k}^{(A,C)}(X)=\mathcal{O}(a^{n-k})$ and we obtain 2.5.2.

4.6. Proofs of formulas of Section 2.6

Substitute:

$$F(x,w)=(1-w)^{N-x}{}_2F_1\left(\left.\begin{matrix}-x,-b-x\\ a+1\end{matrix}\right| w\right)\quad\text{and}\quad p_n(x)=\frac{(-N)_nR_n(\lambda(x);a,b,N)}{n!}$$

in the formulas of Section 3.2 to obtain (22).

We have $A=\mathcal{O}(N)$ and $C=\mathcal{O}(N)$. Therefore the function $\phi(w)=\log f(x,w)$ verifies Lemma 1 with $\mu=N$, $s=1$ and $m=3$ and we have $c_k=\mathcal{O}(N^{[k/3]})$. On the other hand, using the recurrence of the previous subsection, we can show by induction over n that $P_{n-k}^{(A,C)}(X)=\mathcal{O}(N^{n-k})$ and we obtain 2.6.2.

4.7. Proofs of formulas of Section 2.7

Substitute:

$$F(x,w)=(1-w)^{1-2(a+b)}{}_3F_2\left(\left.\begin{matrix}a+b-\frac{1}{2},a+b,a+\mathrm{i}(b+x)\\ 2a,a+c+\mathrm{i}(b+d)\end{matrix}\right| -\frac{4w}{(1-w)^2}\right)$$

and

$$p_n(x)=\frac{(2a+2b-1)_n}{(2a)_n(a+b+\mathrm{i}(c-b))_n}p_n(x;\alpha,\beta,\overline{\alpha},\overline{\beta})$$

in the formulas of Section 3.2 to obtain (24).

From Section 4.3, the coefficients of the Taylor expansion at $w=0$ of the logarithm of the above ${}_3F_2$ function are of the order $\mathcal{O}(a)$. We trivially have $A=\mathcal{O}(a)$ and $C=\mathcal{O}(a)$. Therefore, the function $\phi(w)=\log f(x,w)$ verifies Lemma 1 with $\mu=a$, $s=1$ and $m=3$ and we have $c_k=\mathcal{O}(a^{[k/3]})$. On the other hand, using the recurrence of Section 4.5, we can show by induction over n that $P_{n-k}^{(A,C)}(X)=\mathcal{O}(a^{n-k})$ and we obtain 2.7.2.

4.8. Proofs of formulas of Section 2.8

Substitute:

$$F(x,w)=(1-w)^{-a-b-1}{}_3F_2\left(\left.\begin{matrix}(a+b+1)/2,(a+b+2)/2,-x\\ a+1,-N\end{matrix}\right| -\frac{4w}{(1-w)^2}\right)$$

and

$$p_n(x)=\frac{(-N)_nQ_n(x;a,b,N)}{(b+1)_nn!}$$

in the formulas of Section 3.2 to obtain (26).

We have $A=\mathcal{O}(a)$ and $C=\mathcal{O}(a)$. Therefore, the function $\phi(w)=\log f(x,w)$ verifies Lemma 1 with $\mu=a$, $s=1$ and $m=3$ and we have $c_k=\mathcal{O}(a^{[k/3]})$. Using the recurrence of Section 4.1, we can show by induction over n that $P_{n-k}^{(A,C)}(X)=\mathcal{O}(N^{n-k})$ and we obtain 2.8.2.

4.9. Proofs of formulas of Section 2.9

Substitute:

$$F(x,w)=(1-w)^{-c+\mathrm{i}x}{}_2F_1\left(\left.\begin{matrix} a+\mathrm{i}x, b+\mathrm{i}x \\ a+b\end{matrix}\right| w\right) \quad \text{and} \quad p_n(x)=\frac{S_n(x^2;a,b,c)}{(a+b)_n n!}$$

in the formulas of Section 3.3 to obtain (28).

From Section 4.1 we have that the coefficients of the Taylor expansion at $w=0$ of the logarithm of the above ${}_2F_1$ function are of the order $\mathcal{O}(a)$. We have $A=\mathcal{O}(a)$ and $X=\mathcal{O}(a)$. Therefore, the function $\phi(w)=\log f(x,w)$ verifies Lemma 1 with $\mu=a$, $s=1$ and $m=3$ and we have $c_k=\mathcal{O}(a^{[k/3]})$. On the other hand, $M_0(X;A,C)=\mathcal{O}(a^0)$, $M_1(X;A,C)=p_1(x)/[p_1(x)+Cp_1(x)+Cp_1(x)^2-2Cp_2(x)^2]=\mathcal{O}(a^0)$, and using the following recurrence relation [6],

$$C(n+A)M_{n+1}(X;A,C)-[(C-1)x+(n+C(n+A))]M_n(X;A,C)+nM_{n-1}(X;A,C)=0,$$

we can show by induction over n that $M_{n-k}(X;A,C)=\mathcal{O}(a^0)$, and we obtain 2.9.2. The limit (31) follows from the first term of the expansion (28) after obtaining $x(X,A)$ and $c(X,A)$ and putting $b=a$.

4.10. Proofs of formulas of Section 2.10

Substitute:

$$F(x,w)=(1-w)^{N-x}{}_2F_1\left(\left.\begin{matrix} -x,-b-x \\ a+1\end{matrix}\right| w\right) \quad \text{and} \quad p_n(x)=\frac{(-N)_n R_n(\lambda(x);a,b,N)}{n!}$$

in the formulas of Section 3.3 to obtain (33).

We have $A=\mathcal{O}(N)$ and $X=\mathcal{O}(N^{-1})$. Therefore the function $\phi(w)=\log f(x,w)$ verifies Lemma 1 with $\mu=N$, $s=-1$ and $m=3$ and we have $c_k=\mathcal{O}(N^{-1})$. On the other hand, from the recurrence of Section 4.9, we can show by induction over n that $M_{n-k}(X;A,C)=\mathcal{O}(N^0)$ and we obtain 2.10.2.

4.11. Proofs of formulas of Section 2.11

Substitute:

$$F(x,w)=(1-w)^{1-2(a+b)}{}_3F_2\left(\left.\begin{matrix} a+b-\frac{1}{2}, a+b, a+\mathrm{i}(b+x) \\ 2a, a+c+\mathrm{i}(b+d)\end{matrix}\right| -\frac{4w}{(1-w)^2}\right)$$

and

$$p_n(x)=\frac{(2a+2b-1)_n}{(2a)_n(a+b+\mathrm{i}(c-b))_n}p_n(x;\alpha,\beta,\overline{\alpha},\overline{\beta})$$

in the formulas of Section 3.3 to obtain (35).

From Section 4.3, the coefficients of the Taylor expansion at $w=0$ of the logarithm of the above ${}_3F_2$ function are of the order $\mathcal{O}(a)$. We trivially have $A=\mathcal{O}(a)$ and $X=\mathcal{O}(a)$. Therefore, the function $\phi(w)=\log f(x,w)$ verifies Lemma 1 with $\mu=a$, $s=1$ and $m=3$ and we have $c_k=\mathcal{O}(a^{[k/3]})$. From the recurrence of Section 4.9, we can show by induction over n that $M_{n-k}(X;A,C)=\mathcal{O}(a^0)$, and we obtain 2.11.2.

4.12. Proofs of formulas of Section 2.12

Substitute:

$$F(x,w)=(1-w)^{-a-b-1}{}_3F_2\left(\left.\begin{matrix} (a+b+1)/2, (a+b+2)/2, -x \\ a+1, -N\end{matrix}\right| -\frac{4w}{(1-w)^2}\right)$$

and

$$p_n(x)=\frac{(-N)_n Q_n(x;a,b,N)}{(b+1)_n n!}$$

in the formulas of Section 3.3 to obtain (37).

We have $A=\mathcal{O}(a)$ and $X=\mathcal{O}(a^0)$. Therefore, the function $\phi(w)=\log f(x,w)$ verifies Lemma 1 with $\mu=a$, $s=0$ and $m=3$ and we have $c_k=\mathcal{O}(a^0)$. From the recurrence of Section 4.1, we can show by induction over n that $M_{n-k}(X;A,C)=\mathcal{O}(a^0)$ and we obtain 2.12.2.

4.13. Proofs of formulas of Section 2.13

Substitute:

$$F(x,w)=(1-w)^{-c+\mathrm{i}x}{}_2F_1\left(\left.\begin{matrix}a+\mathrm{i}x,b+\mathrm{i}x\\ a+b\end{matrix}\right| w\right)\quad\text{and}\quad p_n(x)=\frac{S_n(x^2;a,b,c)}{(a+b)_n n!}$$

in the formulas of Section 3.4 to obtain (39).

In Section 4.1 we have shown that the coefficients of the Taylor expansion at $w=0$ of the logarithm of the above ${}_2F_1$ function are of the order $\mathcal{O}(a)$. We have $C=\mathcal{O}(a)$ and $X=\mathcal{O}(a)$. Therefore, the function $\phi(w)=\log f(x,w)$ verifies Lemma 1 with $\mu=a$, $s=1$ and $m=3$ and we have $c_k=\mathcal{O}(a^{[k/3]})$. On the other hand, $K_0(X;A,C)=\mathcal{O}(a^0)$, $K_1(X;A,C)=[p_1(x)-Ap_1(x)]/[p_1(x)-2Ap_1(x)+Ap_1(x)^2-2Ap_2(x)]=\mathcal{O}(a^0)$, and using the following recurrence relation [6],

$$A(C-n)K_{n+1}(X;A,C)+[X-A(C-n)-n(1-A)]K_n(X;A,C)+n(1-A)K_{n-1}(X;A,C)=0,$$

we can show by induction over n that $K_{n-k}(X;A,C)=\mathcal{O}(a^0)$ and we obtain 2.13.2. Replacing x and c in the expansion (39) by $\tilde{x}$ and $\tilde{c}$ given in (43), respectively, and putting $b=a$ we obtain (42).

4.14. Proofs of formulas of Section 2.14

Substitute:

$$F(x,w)=(1-w)^{N-x}{}_2F_1\left(\left.\begin{matrix}-x,-b-x\\ a+1\end{matrix}\right| w\right)\quad\text{and}\quad p_n(x)=\frac{(-N)_n R_n(\lambda(x);a,b,N)}{n!}$$

in the formulas of Section 3.4 to obtain (44).

We have $C=\mathcal{O}(N)$ and $X=\mathcal{O}(N)$. Therefore the function $\phi(w)=\log f(x,w)$ verifies Lemma 1 with $\mu=N$, $s=1$ and $m=3$ and we have $c_k=\mathcal{O}(N^{[k/3]})$. From the recurrence of Section 4.13, we can show by induction over n that $K_{n-k}(X;A,C)=\mathcal{O}(N^0)$ and we obtain 2.14.2.

4.15. Proofs of formulas of Section 2.15

Substitute:

$$F(x,w)=(1-w)^{1-2(a+b)}{}_3F_2\left(\left.\begin{matrix}a+b-\frac{1}{2},a+b,a+\mathrm{i}(b+x)\\ 2a,a+c+\mathrm{i}(b+d)\end{matrix}\right| -\frac{4w}{(1-w)^2}\right)$$

and

$$p_n(x)=\frac{(2a+2b-1)_n}{(2a)_n(a+b+\mathrm{i}(c-b))_n}p_n(x;\alpha,\beta,\overline{\alpha},\overline{\beta})$$

in the formulas of Section 3.4 to obtain (46).

In Section 4.3 we have shown that the coefficients of the Taylor expansion at $w=0$ of the logarithm of the above ${}_3F_2$ function are of the order $\mathcal{O}(a)$. We trivially have $C=\mathcal{O}(a)$ and $X=\mathcal{O}(a)$. Therefore, the function $\phi(w)=\log f(x,w)$ verifies Lemma 1 with $\mu=a$, $s=1$ and $m=3$ and we have $c_k=\mathcal{O}(a^{[k/3]})$. From the recurrence of Section 4.13, we can show by induction over n that $K_{n-k}(X;A,C)=\mathcal{O}(a^0)$ and we obtain 2.15.2.

4.16. Proofs of formulas of Section 2.16

Substitute:

$$F(x,w) = (1-w)^{-a-b-1}\,{}_3F_2\left(\left.\begin{matrix}(a+b+1)/2,\,(a+b+2)/2,\,-x\\ a+1,\,-N\end{matrix}\,\right|\,-\frac{4w}{(1-w)^2}\right)$$

and

$$p_n(x) = \frac{(-N)_n Q_n(x;a,b,N)}{(b+1)_n n!}$$

in the formulas of Section 3.4 to obtain (48).

We have $C = \mathcal{O}(a)$ and $X = \mathcal{O}(a)$. Therefore, the function $\phi(w) = \log f(x,w)$ verifies Lemma 1 with $\mu = a$, $s = 1$ and $m = 3$ and we have $c_k = \mathcal{O}(a^{[k/3]})$. From the recurrence of Section 4.13, we can show by induction over n that $K_{n-k}(X; A, C) = \mathcal{O}(a^0)$ and we obtain 2.16.2.

5. Numerical experiments

In this section, we include Figs. 2–9 and Tables 1–8 that give an idea about the accuracy of the formulas presented in Section 2. In all of the graphics, the degree of the polynomials is $n = 6$, dashed lines represent the exact polynomial

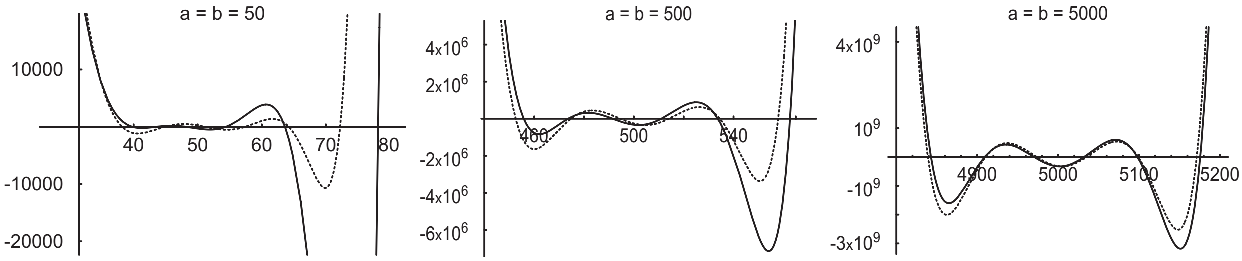

Fig. 2. Expansion (4) for $c = 1$. $S_6(x^2; a, b, 1)/[(a+b)_6 6!]$ versus $P_6^{(C)}(X; A)$.

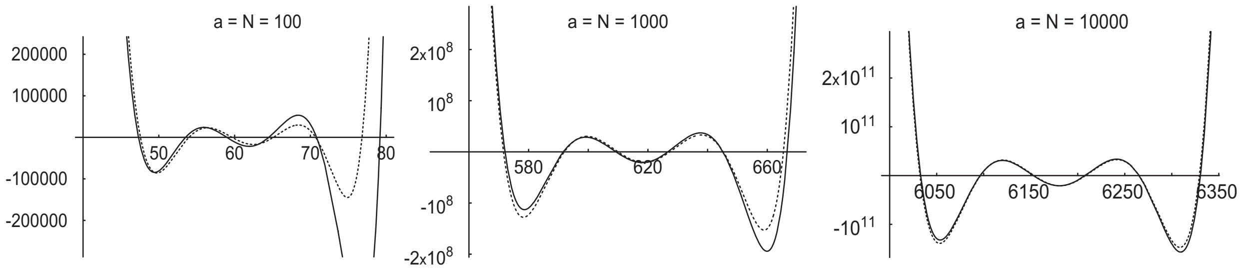

Fig. 3. Expansion (10) for $b = 8$. $(-N)_6 R_6(\lambda(x); a, 8, N)/6!$ versus $P_6^{(C)}(X; A)$.

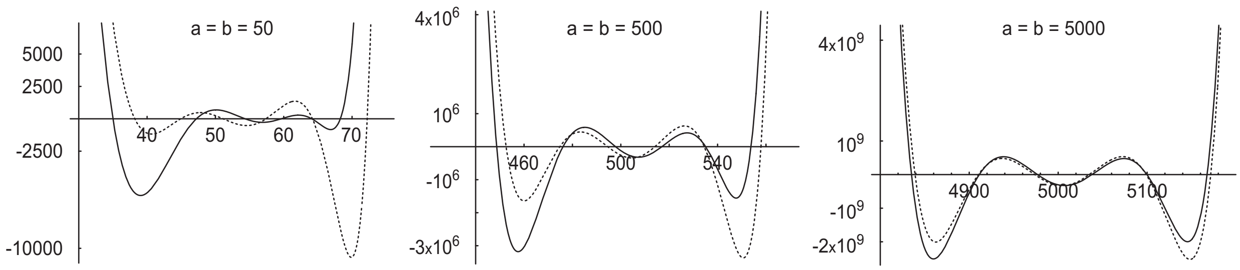

Fig. 4. Expansion (19) for $c = 1$. $S_6(x^2; a, b, 1)/[(a+b)_6 6!]$ versus $P_6^{(A,C)}(X)$.

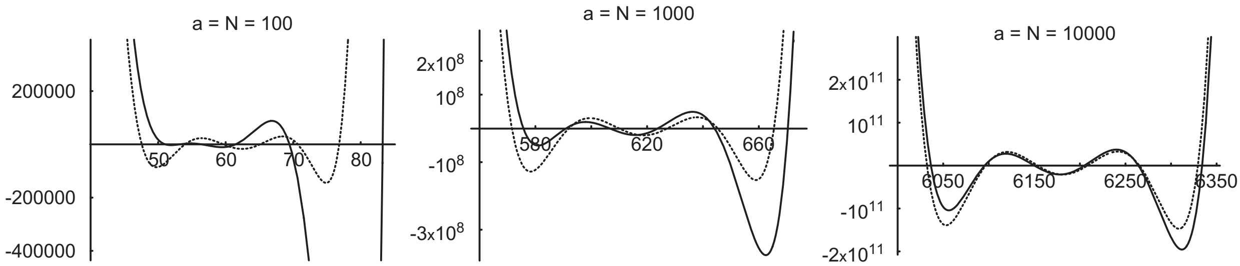

Fig. 5. Expansion (22) for $b=1$. $(-N)_6 R_6(\lambda(x); a, 1, N)/6!$ versus $P_6^{(A,C)}(X)$.

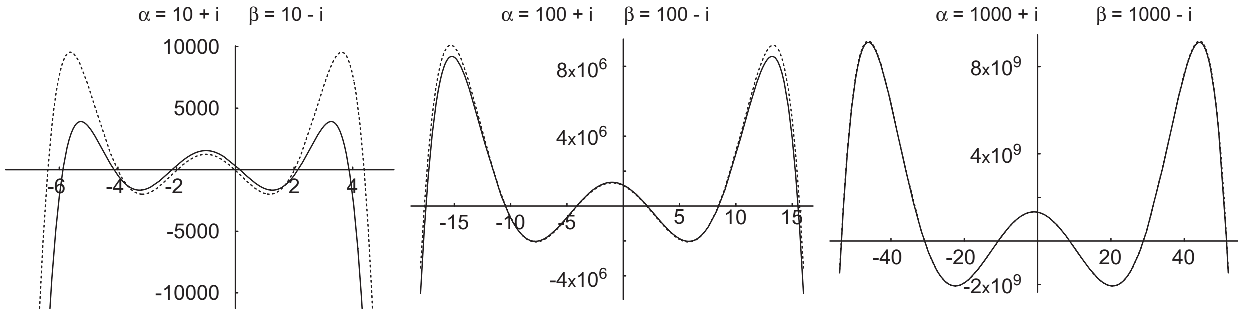

Fig. 6. Expansion (35). $-(2a+2b-1)_6 p_6(x; \alpha, \beta, \overline{\alpha}, \overline{\beta})/[(2a)_6(a+b+\mathrm{i}(c-d))_6]$ versus $(A)_6 M_6(X; A, C)/6!$.

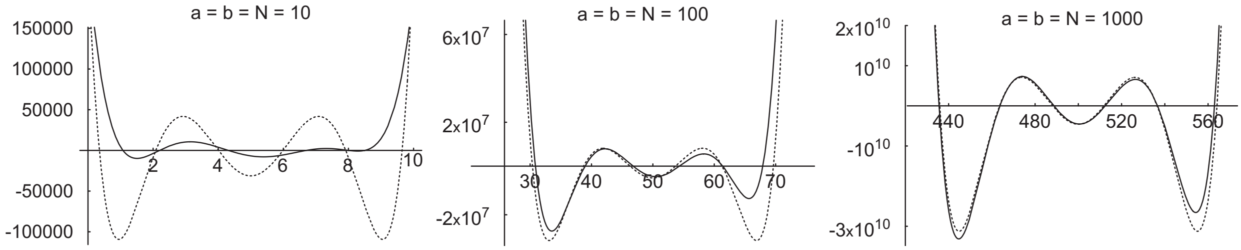

Fig. 7. Expansion (37). $(a+b+1)_6 Q_6(x; a, b, N)$ versus $(A)_6 M_6(X; A, C)$.

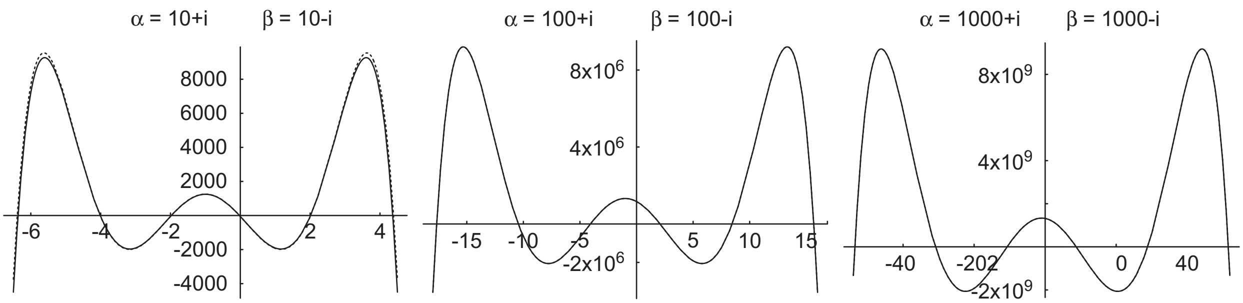

Fig. 8. Expansion (46). $-(2a+2b-1)_6 p_6(x; \alpha, \beta, \overline{\alpha}, \overline{\beta})/[(2a)_6(a+b+\mathrm{i}(c-d))_6]$ versus $\binom{C}{6} K_6(X; A, C)$.

and continuous lines represent the first order approximation given by the corresponding expansion. The first four tables show, for $n=6$, the relative accuracy in the first order approximation of four of the expansions of Section 2 for several values of x. For the same expansions and $n=6$, the next four tables show the relative error in the approximation of the zeros of the four Hahn-type polynomials in terms of the zeros of the corresponding first order polynomial approximations.

Table 1
Numerical experiment about the relative error in the first order approximation (4) of the continuous dual Hahn polynomials by the Meixner–Pollaczek polynomials for $c=1$ and $A=\pi/2$

	$a=b=50$ $k=2$	$a=b=500$ $k=3$	$a=b=5000$ $k=4$	$a=b=50\,000$ $k=5$
$x=0.49\times 10^k$	1.35486	0.704932	0.19936	0.0260865
$x=0.5\times 10^k$	2.28642	0.258402	0.0212764	0.0209069
$x=0.52\times 10^k$	1.52241	0.764548	0.120396	0.000304775

Table 2
Numerical experiment about the relative error in the first order approximation (22) of the dual Hahn polynomials by the Jacobi polynomials for $b=1$ and $X=2$

	$a=N=100$ $k=2$	$a=N=1000$ $k=3$	$a=N=10\,000$ $k=4$	$a=N=100\,000$ $k=5$
$x=0.6\times 10^k$	1.89194	0.419582	0.0911921	0.000306352
$x=0.62\times 10^k$	1.41067	0.296108	0.35991	0.0409521
$x=0.64\times 10^k$	5.03819	0.284283	0.0363085	0.000193346

Table 3
Numerical experiment about the relative error in the first order approximation (35) of the continuous Hahn polynomials by the Meixner polynomials for $c=d=1$ and $C=2$

	$a=b=10$ $k=1$	$a=b=100$ $k=2$	$a=b=1000$ $k=2$	$a=b=10\,000$ $k=3$
$x=0$	7.67706	0.0303631	0.00266215	0.000262868
$x=0.1\times 10^k$	0.20507	0.0500293	0.0146385	0.000724931
$x=0.2\times 10^k$	5.53093	0.026321	0.00182511	0.000337274

Table 4
Numerical experiment about the relative error in the first order approximation (48) of the Hahn polynomials by the Krawtchouk polynomials for $A=\frac{1}{2}$

	$a=b=N=10$ $k=1$	$a=b=N=100$ $k=2$	$a=b=N=1000$ $k=3$	$a=b=N=10\,000$ $k=4$
$x=0.47\times 10^k$	0.737035	0.0570824	0.023382	0.00121746
$x=0.5\times 10^k$	0.744631	0.113814	0.0119359	0.00119936
$x=0.52\times 10^k$	0.741444	0.102285	0.0197468	0.00412202

Table 5
Relative error in the approximation of the zeros of $S_6(x^2;a,b,1)$ by the zeros of $P_6^{(C)}(X;\pi/2)$ (first order approximation in (4))

	$a=b=50$	$a=b=500$	$a=b=5000$
1	0.41e − 1	0.71e − 2	0.78e − 3
2	0.29e − 1	0.22e − 2	0.20e − 3
3	0.58e − 1	0.62e − 2	0.61e − 3
4	0.52e − 1	0.59e − 2	0.61e − 3
5	0.92e − 2	0.15e − 2	0.17e − 3
6	0.82e − 1	0.84e − 2	0.82e − 3

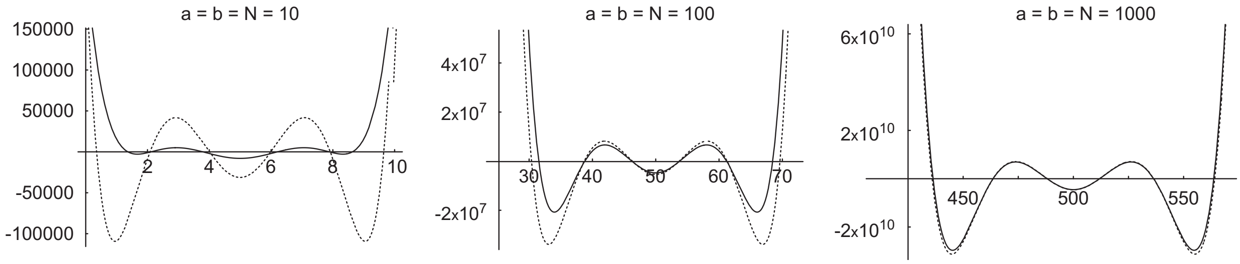

Fig. 9. Expansion (48). $(a+b+1)_6 Q_6(x; a, b, N)$ versus $(C-5)_6 K_6(X; A, C)$.

Table 6
Relative error in the approximation of the zeros of $R_6(\lambda(x); a, 1, N)$ by the zeros of $P_6^{(A,C)}(2)$ (first order approximation in (22))

	$a = N = 100$	$a = N = 1000$	$a = N = 10\,000$
1	0.73e − 1	0.72e − 2	0.72e − 3
2	0.14e − 1	0.16e − 2	0.17e − 3
3	0.53e − 1	0.55e − 2	0.55e − 3
4	0.56e − 1	0.55e − 2	0.55e − 3
5	0.17e − 1	0.17e − 2	0.17e − 3
6	0.83e − 1	0.74e − 2	0.73e − 3

Table 7
Relative error in the approximation of the zeros of $p_6(x; a+\mathrm{i}, b+\mathrm{i}, a-\mathrm{i}, b-\mathrm{i})$ by the zeros of $(A)_6 M_6(X; A, 2)$ (first order approximation in (35))

	$a = b = 10$	$a = b = 100$	$a = b = 1000$
1	0.43e − 1	0.44e − 2	0.46e − 3
2	0.94e − 1	0.10e − 2	0.11e − 3
3	0.24e − 1	0.23e − 3	0.26e − 4
4	0.21e − 0	0.45e − 3	0.32e − 4
5	0.18e − 1	0.13e − 2	0.12e − 3
6	0.68e − 1	0.50e − 2	0.48e − 3

Table 8
Relative error in the approximation of the zeros of $Q_6(x; a, b, N)$ by the zeros of $(C-5)_6 K_6(X; 1/2, C)$ (first order approximation in (48))

	$a = b = N = 10$	$a = b = N = 100$	$a = b = N = 1000$
1	0.28e + 1	0.38e − 1	0.10e − 2
2	0.17e − 1	0.21e − 2	0.62e − 4
3	0.54e − 1	0.14e − 2	0.45e − 4
4	0.36e − 1	0.12e − 2	0.43e − 4
5	0.45e − 2	0.13e − 2	0.54e − 4
6	0.10e − 1	0.16e − 1	0.78e − 3

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