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The Gauss hypergeometric function $F(a, b; c; z)$ for large c

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Abstract

We consider the Gauss hypergeometric function $F(a, b+1; c+2; z)$ for $a, b, c \in \mathbb{C}$, $c \neq -2, -3-4, \ldots$ and $|\arg(1-z)| < \pi$. We derive a convergent expansion of $F(a, b+1; c+2; z)$ in terms of rational functions of a, b, c and z valid for $|b||z| < |c-bz|$ and $|c-b||z| < |c-bz|$. This expansion has the additional property of being asymptotic for large c with fixed a uniformly in b and z (with bounded b/c). Moreover, the asymptotic character of the expansion holds for a larger set of b, c and z specified below.

MSC: 35C20; 41A60

Keywords: Gauss hypergeometric function; Asymptotic expansions

1. Introduction

The asymptotic behaviour of the Gauss hypergeometric function $F(a, b; c; z)$ when different combinations of a, b, c and z are large is a subject of recent interest [4,6,9,12]. The hypergeometric function is defined by the power series

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad |z| < 1, \quad c \neq 0, -1, -2, \ldots . \tag{1}$$

This is an asymptotic expansion of $F(a, b; c; z)$ for $z \to 0$ and/or $c \to \infty$. The condition $|z| < 1$ may be relaxed still keeping the asymptotic character of the expansion for large c [11].

A translation formula for $F(a, b; c; z)$ [8, p. 113, Eq. (5.11)], can be used to obtain an asymptotic representation of $F(a, b; c; z)$ for large values of z with $|\arg(-z)| < \pi$ [8, p. 127]:

$$\begin{aligned} F(a, b; c; z) = {} & \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} \sum_{n=0}^{\infty} \frac{(a)_n (1-c+a)_n}{(1-b+a)_n n!} \frac{1}{z^n} \\ & + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} \sum_{n=0}^{\infty} \frac{(b)_n (1-c+b)_n}{(1-a+b)_n n!} \frac{1}{z^n}. \end{aligned}$$

But when one or several of the parameters a, b, c or z are large (except when only c or z are large), the asymptotic study is more difficult. Some authors have obtained asymptotic expansions of $F(a,b;c;z)$ with certain restrictions on the parameters. Wagner provides in [12] an asymptotic expansion of $F(a,b;c;z)$ when $c\to\infty$ with $a^2=o(c)$ and $b^2=o(c)$. This result is obtained from an integral representation of $F(a,b;c;z)$ followed by contour deformations and series expansions.

Several authors have focused their attention on the asymptotic behaviour of

$$F(a+e_1\lambda, b+e_2\lambda; c+e_3\lambda; z), \quad e_j=0,\pm 1, \quad \lambda\to\infty. \tag{2}$$

In [13], Watson obtained an asymptotic expansion of $F(a+\lambda,b+\lambda;c+2\lambda;z)$, $F(a+\lambda,b-\lambda;c;z)$ and $F(a,b;c+\lambda;z)$ in terms of inverse powers of λ via contour integrals and the steepest descent method, see also [7, Chapter 5, Section 9]. However, these expansions are only valid in small regions of z. In [4], Jones obtains a uniform asymptotic expansion of $F(a+\lambda,b-\lambda;c;1/2-1/2z)$ when $\lambda\to\infty$ with $|\arg z|<\pi$ in terms of Bessel functions. Jones uses for his analysis Olver's method [7], which is based on the linear second order differential equation satisfied by $F(a,b;c;z)$. More recently, Olde Daalhuis has obtained an asymptotic expansion of $F(a,b-\lambda;c+\lambda;-z)$ in terms of parabolic cylinder functions and of $F(a+\lambda,b+2\lambda;c;-z)$ in terms of Airy functions [6]. These expansions hold for fixed values of a, b and c, and are uniformly valid for z with $|\arg z|<\pi$. Olde Daalhuis uses Bleinstein's method applied on a contour integral representation of $F(a,b;c;z)$ in which a saddle point and a branch point coalesce.

In [9], Temme has shown that the set of 26 possible cases in (2) can be reduced to only four cases:

(A) $e_1=e_2=0, e_3=1$, (B) $e_1=1, e_2=-1, e_3=0$,
(C) $e_1=0, e_2=-1, e_3=1$, (D) $e_1=1, e_2=2, e_3=0$.

For case (A), Temme obtains the uniform asymptotic expansion

$$F(a,b;c+\lambda;z)\sim\frac{\Gamma(c+\lambda)\zeta^{b-a}}{\Gamma(c+\lambda-b)}\sum_{s=0}^{\infty}g_s(z)(b)_s\zeta^s U(b+s,b-a+1+s,\zeta\lambda), \tag{3}$$

where U is the confluent hypergeometric function, $\zeta=\ln[(z-1)/z]$ and g_s are the coefficients of the Taylor expansion of $g(t)\equiv(t+\zeta)^a[(e^t-1)/t]^{b-1}e^{(1-c)t}(1-z+ze^{-t})^{-a}$ at $t=0$: $g(t)=\sum_{s=0}^{\infty}g_s(z)t^s$. Formula (3) is an asymptotic expansion when $\lambda\to\infty$, uniformly with respect to bounded values of ζ (z bounded away from the origin).

In this paper we are concerned with a generalization of cases (A) and (C). We study asymptotic expansions of $F(a,b;c;z)$ for large values of c uniformly in b with bounded b/c. In [2] we used a modification of the steepest descent method (see [3]) to derive uniform asymptotic expansions of the incomplete gamma functions $\Gamma(a,z)$ and $\gamma(a,z)$ for large values of a and z in terms of elementary functions. We apply here the same idea to derive a uniform asymptotic expansion of $F(a,b;c;z)$ for large b and c using the integral representation (4) of $F(a,b;c;z)$ given below. The approach consists of: (i) a factorization of the integrand in that integral in an exponential factor times another factor and (ii) an expansion of this second factor at the asymptotically relevant point of the exponential factor. The main benefit of this procedure is the derivation of easy asymptotic expansions (in terms of elementary functions) with easily computable coefficients.

In Section 2 we derive a convergent expansion of $F(a,b+1;c+2;z)$ valid under the restrictions $|b||z|<|c-bz|$ and $|c-b||z|<|c-bz|$ which has also an asymptotic character for large c uniformly in b and z with bounded b/c. This expansion is not new, it was already obtained by Nørlund in [5, Eq. (1.21)], although with more restrictive conditions for the convergence and without mention to its asymptotic properties. In Sections 3 and 4 we show that the expansion obtained in the previous section keeps its asymptotic character for large c (uniformly in b and z with bounded b/c) even if the restrictions $|b||z|<|c-bz|$ and $|c-b||z|<|c-bz|$ do not hold. In the remaining of the paper we consider $a,b,c\in\mathbb{C}$, $c\neq -2,-3,-4,\ldots$ and $|\arg(1-z)|<\pi$.

2. The expansion

The Gauss hypergeometric function may be written in the form [8, p. 110, Eq. (5.4)]

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt, \quad \Re c>\Re b>0.$$

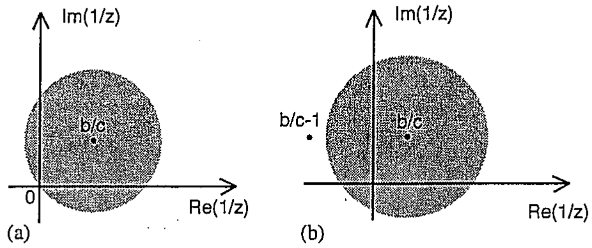

Fig. 1. Eqs. (7) are equivalent to $|z^{-1}-b/c|>|b/c|$ and $|z^{-1}-b/c|>|b/c-1|$ (last inequality in formula (6) for $t=0$ and $t=1$). Then, z^{-1} must be outside of a disk of center b/c and radius $\mathrm{Max}\{|b/c|,|b/c-1|\}$. When $\Re(b/c)\geqslant\frac{1}{2}$, the origin is on the boundary of the disk. When $\Re(b/c)<\frac{1}{2}$, it is inside the disk. (a) $\Re(b/c)\geqslant 1/2$ (b) $\Re(b/c)<1/2$.

For convenience, we consider a shift in the parameters b and c and write the hypergeometric function in the form

$$F(a,b+1;c+2;z)=\frac{\Gamma(c+2)}{\Gamma(b+1)\Gamma(c-b+1)}\int_0^1 e^{c\,f(t)}g(t)\,\mathrm{d}t, \tag{4}$$

with

$$f(t)\equiv\frac{b}{c}\log t+\left(1-\frac{b}{c}\right)\log(1-t),\quad g(t)\equiv(1-tz)^{-a},\quad \Re c+1>\Re b>-1. \tag{5}$$

The unique saddle point of $f(t)$ is located at $t=b/c$. We replace the function $g(t)$ in (5) by its Taylor expansion at $t=b/c$ with convergence radius $|1/z-b/c|$ [3]

$$g(t)=\sum_{k=0}^{\infty}\frac{(a)_k z^k}{k!(1-(b/c)z)^{k+a}}\left(t-\frac{b}{c}\right)^k,\quad \left|t-\frac{b}{c}\right|<\left|\frac{1}{z}-\frac{b}{c}\right|. \tag{6}$$

This expansion converges uniformly with respect to $t\in[0,1]$ when the following conditions hold:

$$|b||z|<|c-bz| \quad\text{and}\quad |c-b||z|<|c-bz|. \tag{7}$$

Several possible z-regions are illustrated in Fig. 1.

For the values of z, b and c verifying (7), we can introduce (6) in (4) to obtain, after interchanging summation and integration,

$$F(a,b+1;c+2;z)=\sum_{k=0}^{\infty}\frac{(a)_k z^k}{k!(1-(b/c)z)^{k+a}}\Phi_k(b,c), \tag{8}$$

where the functions $\Phi_k(b,c)$ are defined by

$$\Phi_k(b,c)\equiv\frac{\Gamma(c+2)}{\Gamma(b+1)\Gamma(c-b+1)}\int_0^1 e^{c\,f(t)}\left(t-\frac{b}{c}\right)^k \mathrm{d}t. \tag{9}$$

Using again the integral representation (4) we see that $\Phi_k(b,c)$ is a very simple hypergeometric function which is also a rational function of b and c:

$$\Phi_k(b,c)=\left(-\frac{b}{c}\right)^k F\left(-k,b+1;c+2;\frac{c}{b}\right)=\sum_{j=0}^{k}\binom{k}{j}\left(-\frac{b}{c}\right)^{k-j}\frac{(b+1)_j}{(c+2)_j}. \tag{10}$$

The first few functions $\Phi_k(b,c)$ are detailed in Table 1.

We have derived the above expansion under the restrictions $\Re c+1>\Re b>-1$, $|b||z|<|c-bz|$ and $|c-b||z|<|c-bz|$. But the restriction $\Re c+1>\Re b>-1$ is superfluous: for large values of k we have that [10]

$$F(-k,b;c;z)=\frac{\Gamma(c)}{\Gamma(c-b)}(kz)^{-b}[1+\mathrm{o}(1)]+\frac{\Gamma(c)}{\Gamma(b)}\mathrm{e}^{\pm\pi\mathrm{i}(b-c)}(1-z)^{c-b+k}(kz)^{b-c}[1+\mathrm{o}(1)]$$

Table 1
First few functions $\Phi_k(b,c)$ defined in (9) and used in (8)

k	$\Phi_k(b,c)$
0	1
1	$\dfrac{c-2b}{c(c+2)}$
2	$\dfrac{(2+b)c^2-b(6+b)c+6b^2}{c^2(c+2)(c+3)}$
3	$\dfrac{(6+5b)c^3-3b(8+5b)c^2+2b^2(18+5b)c-24b^3}{c^3(c+2)(c+3)(c+4)}$

when $k\to\infty$ with b and c fixed complex numbers, $c\neq 0,-1,-2,\ldots$, $z\neq 0$ and $|\arg(1-z)|<\pi$. Therefore,

$$\Phi_k(b,c)=\mathcal{O}(\gamma(k)\alpha^k)\quad\text{when } k\to\infty$$

with $\gamma(k)\equiv\mathrm{Max}\{k^{-b-1},k^{b-c-1}\}$ and $\alpha\equiv\mathrm{Max}\{|(b/c)|,|1-(b/c)|\}$.

Then, the terms of the series (8) verify

$$\frac{(a)_k z^k}{k!(1-(b/c)z)^{k+a}}\Phi_k(b,c)=\mathcal{O}(k^{a-1}\gamma(k)\beta^k)\quad\text{when } k\to\infty,$$

with

$$\beta\equiv\mathrm{Max}\left\{\left|\frac{bz}{c-bz}\right|,\left|\frac{(c-b)z}{c-bz}\right|\right\}<1.$$

Therefore, expansion (8) has almost a power rate of convergence under the restrictions $|b||z|<|c-bz|$ and $|c-b||z|<|c-bz|$ (and the restrictions $\Re c+1>\Re b>-1$ are not necessary).

On the other hand, from [1, Eq. (15.2.10)] we find the recurrence

$$G_k(b,c)=\frac{1}{c+k+1}\left[k\left(1-2\frac{b}{c}\right)+(k-1)\frac{b}{c}\left(1-\frac{b}{c}\right)G_{k-1}^{-1}(b,c)\right],\quad k\geqslant 2. \tag{11}$$

where

$$G_k(b,c)\equiv\frac{\Phi_k(b,c)}{\Phi_{k-1}(b,c)},\quad k=1,2,3,\ldots.$$

From the explicit values of Φ_0 and Φ_1 given in Table 1 we see that $G_1(b,c)=\mathcal{O}(c^{-1})$ and $G_2(b,c)=\mathcal{O}(b/c)$ when $|b|+|c|\to\infty$ with bounded b/c and $b\neq 0$.

From this behaviour of G_1 and G_2 and the above recurrence, it may be shown by induction that, for $b\neq 0$,

$$G_k(b,c)=\mathcal{O}(b/c)\quad\text{when } |b|+|c|\to\infty \text{ with bounded } b/c \text{ and even } k.$$

$$G_k(b,c)=\mathcal{O}(c^{-1})\quad\text{when } |b|+|c|\to\infty \text{ with bounded } b/c \text{ and odd } k.$$

Therefore, for $b\neq 0$,

$$\Phi_k(b,c)=\mathcal{O}\left(\frac{b^{1-k\,\mathrm{Mod}\,2}}{c}\Phi_{k-1}(b,c)\right)\quad\text{when } |b|+|c|\to\infty \text{ with bounded } b/c. \tag{12}$$

The asymptotic properties of the expansion (8) improve when the saddle point $t=b/c$ of (4) coalesces with an end point of the contour of integration $t=0$ or $t=1$, that is, when $b=0$ or $b=c$. In these cases, from the recurrence (11) we have that

$$\Phi_k(0,c)=(-1)^k\Phi_k(c,c)=\frac{k!}{(c+2)_k}. \tag{13}$$

Table 2
A numerical experiment about the relative error in the approximation of $F(-\mathrm{i}, b+1; c+2; -5-3\mathrm{i})$ for several values of b and c by using (8) with n terms

$b+1$	$c+2$	$n=1$	$n=3$	$n=5$	$n=7$	$n=9$
$10e^{i\pi/4}$	$20e^{i\pi/6}$	0.015019	0.001762	0.000234	0.000039	7.814 (−6)
$50e^{-i\pi/3}$	$100e^{-i\pi/4}$	0.006754	0.000072	1.682 (−6)	6.055 (−8)	2.886 (−9)
$100-30i$	$200+2i$	0.003617	0.000019	2.255 (−7)	3.987 (−9)	9.308 (−11)
$200e^{i\pi/18}$	$400e^{-i\pi/18}$	0.001036	2.672 (−6)	1.359 (−8)	1.065 (−10)	5.651 (−12)
$10-500i$	$40-300i$	0.003759	0.000010	5.701 (−8)	4.285 (−10)	2.930 (−12)

Then, (8) is an asymptotic expansion for large c with fixed a uniformly in b and z with bounded b/c. Table 2 contains some numerical experiments which show the accuracy achieved by expansion (8).

The expansion (8) was already obtained by Nørlund in [5, Eq. (1.21)], although without any mention to the asymptotic properties of the expansion. Also, the conditions for the convergence of (8) given there are more restrictive: $(|b|+|c|)|z|<|c-bz|$.

3. Asymptotic properties of the expansion (8) for real b/c

In the previous section we have shown that expansion (8) is convergent and asymptotic for large c (uniformly in b and z with bounded b/c) if b, c and z satisfy (7). In this section we will show that the expansion (8) keeps that asymptotic character if $0<b/c<1$ (even if conditions (7) do not hold). In the remaining of this section we consider $0<b/c<1$ and $-1<\Re b<\Re c+1$.

Expansion (6) is not uniformly convergent for $t\in[0,1]$ if conditions (7) do not hold. Nevertheless, we can approximate the integral (4) by replacing the function $g(t)$ by its Taylor expansion at the point $t=b/c$:

$$g(t)=\sum_{k=0}^{n-1}\frac{(a)_k z^k}{k!(1-(b/c)z)^{k+a}}\left(t-\frac{b}{c}\right)^k+g_n(t) \tag{14}$$

with $g_n(t)=\mathcal{O}((t-b/c)^n)$ when $t\to b/c$. Introducing (14) in (4) and interchanging summation and integration we obtain

$$F(a,b+1;c+2;z)=\sum_{k=0}^{n-1}\frac{(a)_k z^k}{k!(1-(b/c)z)^{k+a}}\Phi_k(b,c)+R_n(a,b;c;z), \tag{15}$$

where the functions $\Phi_k(b,c)$ are given in (9) or (10) and

$$R_n(a,b;c;z)\equiv\frac{\Gamma(c+2)}{\Gamma(b+1)\Gamma(c-b+1)}\int_0^1 e^{c f(t)}g_n(t)\,dt. \tag{16}$$

The key point here is to use the idea given in [3]: the critical point $b/c\in(0,1)$. Then, the Laplace method can be applied to the integrals (9) to obtain their asymptotic behaviour for large c (or large b and c with bounded b/c) [3]:

$$\Phi_k(b,c)=\mathcal{O}(c^{-(k+1)/2})\quad\text{when } |b|+|c|\to\infty \text{ with bounded } b/c. \tag{17}$$

On the other hand, we can also apply the Laplace's method to the remainder $R_n(a,b;c;z)$ in (16) to obtain [3]:

$$R_n(a,b;c;z)=\mathcal{O}(c^{-(n+1)/2})\quad\text{when } |b|+|c|\to\infty \text{ with bounded } b/c. \tag{18}$$

Thus, from (12) or (13) and (18), we see that (15) is an asymptotic expansion of $F(a,b+1;c+2;z)$ for large c (uniformly in b with bounded b/c). Moreover, from the Lagrange form for the Taylor remainder we have

$$g_n(t)=\frac{(a)_n(t-b/c)^n}{n!z^a(1/z-\zeta)^{n+a}},\qquad \zeta\in(t,b/c)\subset[0,1].$$

$b+1$	$c+2$	$n=1$	$n=3$	$n=5$	$n=7$	$n=9$
20	50	0.1675	0.025198	0.003424	0.000256	0.000084
50	100	0.07295	0.005519	0.000468	0.000044	4.769 (−6)
100	200	0.036612	0.001398	0.000060	2.972 (−6)	1.640 (−7)
250	500	0.014674	0.000225	3.947 (−6)	7.886 (−8)	1.780 (−9)
500	1000	0.007342	0.000056	4.965 (−7)	4.988 (−9)	5.673 (−11)

Conditions (7) do not hold for these values of b, c and z.

Then,

$$|g_n(t)|\leqslant\frac{1}{n!}\left|\frac{(a)_n}{z^a}\right|A(z,a,n)e^{\pi|\Im a|}\left|t-\frac{b}{c}\right|^n,$$

with

$$A(z,a,n)\equiv\begin{cases}|\Im(z^{-1})|^{-n-\Re a} & \text{if } 0<\Re z^{-1}<1 \text{ and } \Re a+n>0\\ \text{Max}\left\{|z|^{n+\Re a},\left|\dfrac{z}{1-z}\right|^{n+\Re a}\right\} & \text{in the remaining cases.}\end{cases}$$

Then, for real b and c and even n we have

$$|R_n(a,b,c,z)|\leqslant\frac{1}{n!}\left|\frac{(a)_n}{z^a}\right|A(z,a,n)e^{\pi|\Im a|}\Phi_n(b,c).$$

We remark that the asymptotic properties of the sequence $\{\Phi_k(b,c)\}_k$ obtained in (12) or (13) making use of (11) are slightly better than those derived from the Laplace's method in (17).

Table 3 shows a numerical experiment which illustrates the approximation supplied by (8) for large positive real values of b and c with $c>b>0$ when (7) does not hold.

4. Asymptotic properties of the expansion (8) for complex b and $c>0$

Write $t=x+\mathrm{i}y$ and $b/c=u+\mathrm{i}v$, with $x,y,u,v\in\mathbb{R}$. Consider a contour Γ defined as (see Fig. 2): $\Gamma\equiv\Gamma_1\cup\Gamma_2\cup\Gamma_3$, with

$$\Gamma_1\equiv\left\{\left(-\sqrt{v^2/4-(y-v/2)^2},y\right);0<y<v\right\},$$
$$\Gamma_2\equiv\{(x,v);0<x<1\},$$
$$\Gamma_3\equiv\left\{\left(1+\sqrt{v^2/4-(y-v/2)^2},y\right);0<y<v\right\}.$$

Consider the domain Ω bounded by $\Gamma\cup[0,1]$ and defined by

$$\Omega\equiv\left\{t\in\mathbb{C},-\sqrt{(\Im t)\left(\Im\frac{b}{c}-\Im t\right)}\leqslant\Re t\leqslant 1+\sqrt{(\Im t)\left(\Im\frac{b}{c}-\Im t\right)}\right\}. \quad (19)$$

In this section we extend the results of the previous section to the case $b\in\mathbb{C}$ with $0<\Re b<c$ and $z^{-1}\in\mathbb{C}\backslash\Omega$. In the remaining of this section we consider $0<\Re b<c$ and $z^{-1}\in\mathbb{C}\backslash\Omega$ and use the ideas of the modified saddle point method introduced in [3].

The integrand in (4) is an analytic function of $t\in\mathbb{C}$ with branch cuts at $(-\infty,0]$, $[1,\infty)$ and, if $a\notin\mathbb{Z}$, also at $[1/z,\infty)$. Then, if $z^{-1}\notin\Omega$, the integrand in (4) is an analytic function of t in the interior of Ω (see Fig. 2).

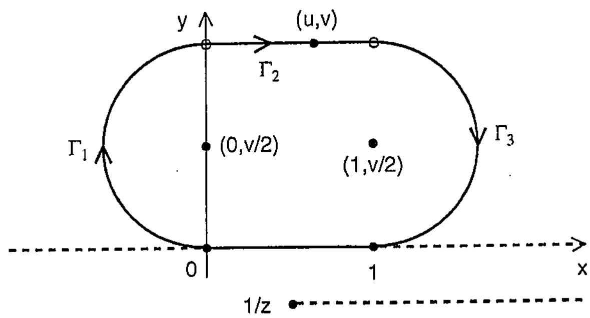

Fig. 2. The contour Γ is the union of the arc Γ_1, the segment Γ_2 and the arc Γ_3. The arc Γ_1 is a half of the circle $x^2+(y-v/2)^2=v^2/4$ of center $(0, v/2)$ and radius $v/2$. The segment Γ_2 is the segment $y=v$, $0<x<1$. The arc Γ_3 is a half of the circle $(x-1)^2+(y-v/2)^2=v^2/4$ of center $(1, v/2)$ and radius $v/2$. The functions f and g are analytic in Ω if $z^{-1}\notin\Omega$.

Using the Cauchy's Residue Theorem, we deform the integration contour [0, 1] in (4) to the contour Γ:

$$F(a,b+1;c+2;z)=\frac{\Gamma(c+2)}{\Gamma(b+1)\Gamma(c-b+1)}\int_\Gamma e^{c\,f(t)}g(t)\,\mathrm{d}t. \tag{20}$$

The real part of the function $f(t)$ in the exponent of the integrand in (20) reads

$$\Re(f(t))\equiv h(x,y)=u\log\sqrt{x^2+y^2}+(1-u)\log\sqrt{(1-x)^2+y^2} + v\arctan[y/(x-1)]-v\arctan(y/x) \tag{21}$$

and verifies the following properties:

(i) For $x\in[0,1]$, the function $h(x,v)$ has an absolute maximum at $x=u$. It is a strictly increasing function of x for $x\in[0,u)$ and strictly decreasing for $x\in(u,1]$. That is, it has an absolute maximum at $x=u$ over Γ_2.

(ii) The function $h(-\sqrt{v^2/4-(y-v/2)^2},y)$ is an strictly increasing function of y for $y\in[0,v]$. That is, it is strictly increasing over Γ_1.

(iii) The function $h(1+\sqrt{v^2/4-(y-v/2)^2},y)$ is an strictly increasing function of y for $y\in(0,v)$. That is, it is strictly decreasing over Γ_3.

Taking into account (i)–(iii) we conclude that, over the path Γ, $\Re(f(t))$ has an absolute maximum at $t=b/c$.

We divide the path Γ in two pieces: $\Gamma=\Gamma_S\cup\Gamma_T$, where Γ_S is that part of Γ contained inside a circle of center b/c and radius $r\equiv|1/z-b/c|$, and $\Gamma_T=\Gamma\backslash\Gamma_S$ (see Fig. 3).

Then,

$$F(a,b+1;c+2;z)=F_S(a,b;c;z)+F_T(a,b;c;z), \tag{22}$$

with

$$F_S(a,b;c;z)\equiv\frac{\Gamma(c+2)}{\Gamma(b+1)\Gamma(c-b+1)}\int_{\Gamma_S}e^{c\,f(t)}g(t)\,\mathrm{d}t$$

and

$$F_T(a,b;c;z)\equiv\frac{\Gamma(c+2)}{\Gamma(b+1)\Gamma(c-b+1)}\int_{\Gamma_T}e^{c\,f(t)}g(t)\,\mathrm{d}t.$$

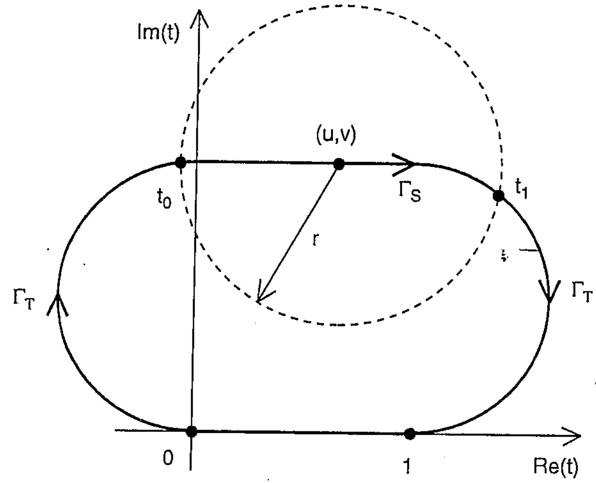

Fig. 3. Γ_S is the piece of the path Γ inside the circle of center (u, v) and radius $r \equiv |1/z - b/c|$. Expansion (6) is uniformly convergent for $t \in \Gamma_S$ (for those points t of Γ_S located between t_0 and t_1).

On the one hand, $\Re(f(t))$ has an absolute maximum at $t = b/c$ and increases from $t = 0$ up to $t = b/c$ and decreases from $t = b/c$ up to $t = 1$ following the path Γ. On the other hand, $g(t)$ is bounded on Γ. Then

$$\int_{\Gamma_T} e^{c\,f(t)} g(t)\,dt = \mathcal{O}(e^{c\,f(t_0)} + e^{c\,f(t_1)}) \quad \text{when } c \to \infty, \tag{23}$$

where t_0 and t_1 are the points of the path Γ located at a distance r from b/c (see Fig. 2).

On the other hand, because of the expansion (6) is uniformly convergent for t inside the circle of radius r and center b/c, we can repeat the reasoning of Section 2 to conclude that

$$F_S(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{k!(1 - (b/c)z)^{k+a}} \Phi_k^{(S)}(b, c), \tag{24}$$

where the functions $\Phi_k^{(S)}(b, c)$ are defined by

$$\Phi_k^{(S)}(b, c) \equiv \frac{\Gamma(c+2)}{\Gamma(c+1)\Gamma(c-b+1)} \int_{\Gamma_S} e^{c\,f(t)} \left(t - \frac{b}{c}\right)^k dt.$$

Using again that $\Re(f(t))$ has an absolute maximum over Γ at $t = b/c$ we have that

$$\Phi_k^{(S)}(b, c) = \frac{\Gamma(c+2)}{\Gamma(c+1)\Gamma(c-b+1)} \left\{ \int_{\Gamma} e^{c\,f(t)} \left(t - \frac{b}{c}\right)^k dt + \mathcal{O}(e^{c\,f(t_0)} + e^{c\,f(t_1)}) \right\} \tag{25}$$

when $c \to \infty$. Using that $e^{c\,f(t)}(t - (b/c))^k$ is an analytic function of $t \in \mathbb{C}$ with branch cuts at $(-\infty, 0]$ and $[1, \infty)$ we deform the integration contour Γ above back to $[0, 1]$: $\Gamma \to [0, 1]$. Then, using the results of Section 2 we have:

$$\Phi_k^{(S)}(b, c) = \Phi_k(b, c) + \frac{\Gamma(c+2)}{\Gamma(c+1)\Gamma(c-b+1)} \mathcal{O}(e^{c\,f(t_0)} + e^{c\,f(t_1)}) \quad \text{when } c \to \infty, \tag{26}$$

where $\Phi_k(b, c)$ are defined in (9), calculated in (10) and verify the recurrence (11).

Therefore, joining (22)–(24) and (26) we have that, even if the right-hand side of (8) is not convergent, it is an asymptotic expansion of $F(a, b+1; c+2; z)$ for large c and fixed a uniformly in b and z (with bounded b/c, $0 < \Re b < c$ and $z^{-1} \notin \Omega$). Tables 4 and 5 show numerical experiments which illustrates the approximation supplied by (24) for large values of b and c with b complex and $c > 0$ when (7) does not hold.

Table 4
A numerical experiment about the relative error in the approximation of $F(6-5i, b+1; c+2; -4+3i)$ for several values of b and c by using (24) with n terms

$b+1$	$c+2$	$n=1$	$n=3$	$n=5$	$n=7$	$n=9$
$50+27i$	160	0.138447	0.006438	0.000817	0.000281	0.000055
$115+16i$	290	0.074796	0.003316	0.000104	2.185 (−6)	2.910 (−7)
$155+2i$	375	0.059992	0.002348	0.000079	2.632 (−6)	5.259 (−8)

Conditions (7) do not hold for these values of b, c and z.

Table 5
A numerical experiment about the relative error in the approximation of $F(-4+7i, b+1; c+2; -7-3i)$ for several values of b and c by using (24) with n terms

$b+1$	$c+2$	$n=1$	$n=3$	$n=5$	$n=7$	$n=9$
$60-2i$	140	0.177146	0.013372	0.000620	0.000023	9.267 (−7)
$130-30i$	240	0.134019	0.005836	0.000034	8.676 (−6)	6.385 (−7)
$150-7i$	345	0.068925	0.001922	0.000031	4.468 (−7)	7.242 (−9)

Conditions (7) do not hold for these values of b, c and z.

5. Conclusions

We can resume the analysis of the previous sections in the following theorems.

Theorem 1. *For $a, b, c \in \mathbb{C}$, $c \neq -2, -3, -4, \ldots$, $|\arg(1-z)| < \pi$, $|b||z| < |c-bz|$ and $|c-b||z| < |c-bz|$,*

$$F(a, b+1; c+2; z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{k!(1-(b/c)z)^{k+a}} \Phi_k(b, c),$$

where

$$\Phi_k(b, c) \equiv \left(-\frac{b}{c}\right)^k F\left(-k, b+1; c+2; \frac{c}{b}\right), \tag{27}$$

$\Phi_k(b, c) = \mathcal{O}((b^{1-k \,\mathrm{Mod}\, 2}/c)\Phi_{k-1}(b, c))$ when $|b| + |c| \to \infty$ uniformly in $b(\neq 0)$ with bounded b/c and verify the recurrence

$$\Phi_k(b, c) = \frac{1}{c+k+1}\left[k\left(1-2\frac{b}{c}\right)\Phi_{k-1}(b, c) + (k-1)\frac{b}{c}\left(1-\frac{b}{c}\right)\Phi_{k-2}(b, c)\right], \quad k \geqslant 2.$$

For the particular cases $b=0$ or $b=c$ we have

$$\Phi_k(0, c) = (-1)^k \Phi_k(c, c) = \frac{k!}{(c+2)_k}.$$

Theorem 2. *For fixed $a \in \mathbb{C}$, $|\arg(1-z)| < \pi$, $-1 < \Re b < \Re c + 1$ and*

(i) *$0 < b/c < 1$ or*
(ii) *$0 < \Re b < c$, bounded b/c and $z^{-1} \in \mathbb{C} \backslash \Omega$,*

$$F(a, b+1; c+2; z) \sim \sum_{k=0}^{\infty} \frac{(a)_k z^k}{k!(1-(b/c)z)^{k+a}} \Phi_k(b, c) \quad \text{when } c \to \infty$$

uniformly in b and z. The functions $\Phi_k(b, c)$ are given in (27) and Ω is defined in (19).

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