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Asymptotic expansions of the generalized Epstein–Hubbell integral

JOSÉ L. LÓPEZ†
Departamento de Matemática e Informática, Universidad Pública de Navarra, Spain

AND

CHELO FERREIRA‡
Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Zaragoza, Spain

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The generalized Epstein–Hubbell integral recently introduced by Kalla & Tuan (*Comput. Math. Applic.* **32**,1996) is considered for values of the variable k close to its upper limit $k = 1$. Distributional approach is used for deriving two convergent expansions of this integral in increasing powers of $1 - k^2$. For certain values of the parameters, one of these expansions involves also a logarithmic term in the asymptotic variable $1 - k^2$. Coefficients of these expansions are given in terms of the Appell function and its derivative. All the expansions are accompanied by an error bound at any order of the approximation. Numerical experiments show that this bound is considerably accurate.

Keywords: Epstein–Hubbell integral; asymptotic expansions; distributional approach; generalized Stieltjes transforms.

1. Introduction

The Epstein–Hubbell elliptic-type integral is the two parameter integral defined by (Epstein & Hubbell, 1963)

$$\Omega_j(k) \equiv \int_0^{\pi} (1 - k^2 \cos\theta)^{-j-\frac{1}{2}} \, d\theta \quad 0 \leqslant k < 1, \; j = 0, 1, 2, \ldots. \tag{1}$$

This integral appears in the application of a Legendre polynomial expansion method to computations involved in certain radiation problems (Berger & Lamkin, 1958). In particular, to the computation of the radiation field off axis from a uniform circular disc radiating according to an arbitrary distribution law (Hubbell *et al.*, 1961).

On the other hand, the Epstein–Hubbell elliptic-type integral is a generalization of the complete elliptic integrals of the first and second kind (Al-Zamel & Kalla, 1996):

$$\Omega_0(k) = \frac{2}{\sqrt{1+k^2}} K\left(\sqrt{\frac{2k^2}{1+k^2}}\right), \qquad \Omega_1(k) = \frac{2}{(1-k^2)\sqrt{1+k^2}} E\left(\sqrt{\frac{2k^2}{1+k^2}}\right),$$

†Email: jl.lopez@unavarra.es
‡Email: cferrei@posta.unizar.es

where K and E are the complete elliptic integrals of the first and second kind respectively (Abramowitz & Stegun, 1970; Byrd & Friedman, 1971).

The literature contains several generalizations of the Epstein–Hubbell integral (1). The most general one was proposed recently by Kalla & Tuan (1996):

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k) \equiv \int_0^{\pi} \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2)}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}\left[1-\rho\sin^2(\theta/2)\right]^{\lambda}\left[1+\delta\cos^2(\theta/2)\right]^{\gamma}}\,\mathrm{d}\theta, \tag{2}$$

where $\alpha, \beta, \lambda, \gamma, \mu, \rho, \delta \in \mathbb{C}$, $\Re\alpha, \Re\beta > 0$, $\rho - 1 \notin \mathbb{R}^+ \cup \{0\}$ if $\Re\lambda \geqslant 1$, $\delta + 1 \notin \mathbb{R}^- \cup \{0\}$ if $\Re\gamma \geqslant 1$ and $0 \leqslant k < 1$.

This integral and some other particular cases (including the Epstein–Hubbell integral (1)) have been investigated by several authors. Some important results are the following: limit relationships for the generalized Epstein–Hubbell integral $S_\mu(k,\lambda) \equiv 2^{2\lambda}\Lambda^{(\lambda+\frac{1}{2},\lambda+\frac{1}{2})}_{(0,0,\mu)}(0,0;k)$ are given in Pinter & Srivastava (1998). A numerical evaluation of $\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k)$ is obtained in Khajah (1999) by using the tau method approximation with a Chebyshev polynomial basis. A survey of properties and evaluation techniques for the integral (1) can be found in Al-Zamel & Kalla (1996), as well as important properties of several generalizations of this integral.

Complete power series expansions at $k = 0$ of the integral (1) or its generalizations may be obtained by means of a series expansion of the integrand at $k = 0$. In particular, a basic representation of (1) by means of the Gauss hypergeometric function may be found in Saxena & Kalla (2000) and Todorov & Hubbell (1994), whereas Kalla *et al.* (1986) contains a series expansion at $k = 0$ of $R_\mu(k,\alpha,\gamma) \equiv \Lambda^{(\alpha,\gamma-\alpha)}_{(0,0,\mu)}(0,0;k)$. On the other hand, an asymptotic formula for $\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k)$ in the neighbourhood of $k = 1$ has been derived by Kalla & Tuan (1996). Although derived in a clever way, the expansion is given by means of triple series in which the explicit calculation of its coefficients is not straightforward. Therefore, complete asymptotic expansions at $k = 1$ of these integrals have not been fully investigated.

We consider in this paper the problem of finding complete asymptotic expansions of $\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k)$ in the neighbourhood of $k = 1$. We face the challenge of obtaining easy algorithms for computing the coefficients of these expansions as well as error bounds at any order of the approximation. The asymptotic method used for obtaining the expansions will be the distributional approach applied on generalized Stieltjes transforms (Wong, 1980, 1989, Chapter 6; López, 2000, 2001). Then, the generalized Epstein–Hubbell integral $\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k)$ should be written as a Stieltjes transform. For that purpose we perform in (2) the change of variable $t^{-\frac{1}{2}} = \tan(\theta/2)$, obtaining

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k) = (1+\delta)^{-\gamma}(1-k^2)^{-\mu-\frac{1}{2}}\int_0^{\infty}\frac{F(t)}{(t+\bar{k})^{\mu+\frac{1}{2}}}\,\mathrm{d}t, \tag{3}$$

where

$$F(t) \equiv t^{\alpha-1}\frac{(1+t)^{\lambda+\gamma+\mu-\beta-\alpha+\frac{1}{2}}}{(t+\bar{\rho})^{\lambda}(t+\bar{\delta})^{\gamma}},$$
$$\bar{\rho} \equiv 1-\rho, \qquad \bar{\delta} \equiv \frac{1}{1+\delta}, \qquad \bar{k} \equiv \frac{1+k^2}{1-k^2}. \tag{4}$$

Then, up to a factor, the Epstein–Hubbell integral is the generalized Stieltjes transform of $F(t)$. For $\Re(\alpha) > 0$, $\bar{\rho} \notin \mathbb{R}^- \cup \{0\}$ if $\Re\lambda \geqslant 1$ and $\bar{\delta} \notin \mathbb{R}^- \cup \{0\}$ if $\Re\gamma \geqslant 1$, $F(t)$ is a locally integrable function on $[0, \infty)$ and satisfies

$$F(t) = \sum_{k=0}^{n-1} A_k t^{-k+\mu-\beta-\frac{1}{2}} + F_n(t), \tag{5}$$

where

$$A_k \equiv \sum_{l=0}^{k}\sum_{j=0}^{l}\binom{\lambda+\gamma+\mu-\beta-\alpha+\frac{1}{2}}{j}\binom{-\lambda}{l-j}\binom{-\gamma}{k-l}\bar{\rho}^{l-j}\bar{\delta}^{k-l} \tag{6}$$

and $F_n(t) = O(t^{-n+\mu-\beta-\frac{1}{2}})$ when $t \to \infty$. Therefore, the asymptotic methods developed in Wong (1980, 1989, chapter 6) and López (2000, 2001) apply to this integral, although only for real values of the parameters: asymptotic theorems there consider only real values for the parameters of the integrals. Therefore, in order to apply these theorems to the integral (3), they must be generalized to the case of complex parameters. In Section 2, the extension to the complex case of distributional asymptotic methods for generalized Stieltjes transforms is performed, including theorems about error bounds. In Section 3 we apply these methods to the generalized Epstein–Hubbell integral (3), obtaining asymptotic expansions with error bounds. Several numerical examples are shown as illustrations. A brief summary and a few comments are postponed to Section 4.

2. Distributional approach

The purpose of this section is to obtain asymptotic expansions with error bounds of generalized Stieltjes transforms

$$S_f(w; z) \equiv \int_0^{\infty} \frac{f(t)}{(t+z)^w}\,\mathrm{d}t \tag{7}$$

for large z. The parameters w and z are complex and $f(t)$ is a locally integrable function on $[0, \infty)$ which satisfies

$$f(t) = \sum_{k=K}^{n-1} a_k t^{-k-s} + f_n(t), \tag{8}$$

where $K \in \mathbb{Z}$, $0 < \Re s \leqslant 1$, $\{a_k, k = K, K+1, K+2, \dots\}$ is a sequence of complex numbers and $f_n(t) = O(t^{-n-s})$ when $t \to \infty$.

2.1 *Asymptotic expansion of $S_f(w; z)$ for large z*

In the following, we use the notation introduced in Wong (1989). In particular, we denote by $\mathcal{S}$ the space of rapidly decreasing functions and by $\langle \Lambda, \varphi \rangle$ the image of a tempered distribution Λ acting over a function $\varphi \in \mathcal{S}$. Recall that we can associate to any locally integrable function $g(t)$ on $[0, \infty)$ a tempered distribution $\Lambda_{\mathbf{g}}$ defined by

$$\langle \Lambda_{\mathbf{g}}, \varphi \rangle \equiv \int_0^\infty g(t)\varphi(t)\,\mathrm{d}t.$$

Since $f(t)$ in (7) is a locally integrable function on $[0, \infty)$, it defines a distribution

$$\langle \mathbf{f}, \varphi \rangle \equiv \int_0^\infty f(t)\varphi(t)\,\mathrm{d}t.$$

The distributions associated with t^{-k-s}, $k = 0, 1, 2, \ldots, n-1$ (Wong, 1989, Chapter 5) are given by,

$$\langle \mathbf{t^{-k-s}}, \varphi \rangle \equiv \frac{1}{(s)_k} \int_0^\infty t^{-s}\varphi^{(k)}(t)\,\mathrm{d}t \qquad \text{if} \qquad 0 < \Re s < 1,$$

where $(s)_k$ denotes Pochhammer's symbol,

$$\langle \mathbf{t^{-k-s}}, \varphi \rangle \equiv \frac{1}{(\mathrm{i}\Im s)_{k+1}} \int_0^\infty t^{-\mathrm{i}\Im s}\varphi^{(k+1)}(t)\,\mathrm{d}t \qquad \text{if } 1 \neq s = 1 + \mathrm{i}\Im s$$

and

$$\langle \mathbf{t^{-k-1}}, \varphi \rangle \equiv -\frac{1}{k!} \int_0^\infty \log(t)\varphi^{(k+1)}(t)\,\mathrm{d}t.$$

To assign a distribution to the function $f_n(t)$ introduced in (8), we first define recursively the k−esim integral $f_{n,k}(t)$ of $f_n(t)$ by $f_{n,0}(t) \equiv f_n(t)$ and

$$f_{n,k+1}(t) \equiv -\int_t^\infty f_{n,k}(u)\,\mathrm{d}u = \frac{(-1)^{k+1}}{k!} \int_t^\infty (u-t)^k f_n(u)\,\mathrm{d}u. \tag{9}$$

For $s \neq 1$, it is trivial to show that $f_{n,n}(t)$ is bounded on $[0, T]$ for any $T > 0$ and is $O(t^{-s})$ as $t \to \infty$. For $s = 1$ we have $f_{n,n}(t) = O(t^{-1})$ as $t \to \infty$ and $f_{n,n}(t) = O(\log(t))$ as $t \to 0^+$. Therefore, for $0 < \Re s \leqslant 1$ we can define the distribution associated to $f_n(t)$ by

$$\langle \mathbf{f_n}, \varphi \rangle \equiv (-1)^n \langle \mathbf{f_{n,n}}, \varphi^{(n)} \rangle \equiv (-1)^n \int_0^\infty f_{n,n}(t)\varphi^{(n)}(t)\,\mathrm{d}t.$$

Once we have assigned a distribution to each function involved in the identity (8), we are interested in finding a relation (if any) between these distributions. In fact, this relation is established in the following two lemmas.

LEMMA 1 For $0 < \Re s < 1$, $n \geqslant K + 1$, and $n \in \mathbb{N}$, the identity

$$\mathbf{f} = \sum_{k=K}^{n-1} a_k \mathbf{t^{-k-s}} + \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} M[f; k+1]\delta^{(\mathbf{k})} + \mathbf{f_n}$$

holds for any rapidly decreasing function $\varphi \in \mathcal{S}$, where δ is the delta distribution in the origin and $M[f; k+1]$ denotes the Mellin transform of $f(t)$: $\int_0^\infty t^k f(t)\,\mathrm{d}t$, or its analytic continuation.

Proof. It is a trivial generalization of Wong (1989, Chapter 6, Lemma 1) from real to complex values of s. □

LEMMA 2 For $\Re s = 1$, $n \geqslant K + 1$ and $n \in \mathbb{N}$, the identity

$$\mathbf{f} = \sum_{k=K}^{n-1} a_k \mathbf{t^{-k-s}} + \sum_{k=0}^{n-1} b_{k+1}\delta^{(\mathbf{k})} + \mathbf{f_n}$$

holds for any rapidly decreasing function $\varphi \in \mathcal{S}$, where, for $n = 0, 1, 2, \ldots$,

$$b_{n+1} \equiv \frac{(-1)^n}{n!}\left[\int_0^1 t^n f_n(t)\,\mathrm{d}t + \int_1^\infty t^n f_{n+1}(t)\,\mathrm{d}t + a_n \sum_{k=1}^{n+1} \frac{(n-k+2)_{k-1}}{(n+s-k)_k}\right] \tag{10}$$

$$= \frac{(-1)^n}{n!}\left\{M[f; n+1] + \frac{a_n}{1-s} + a_n \sum_{k=1}^{n+1} \frac{(n-k+2)_{k-1}}{(n+s-k)_k}\right\} \tag{11}$$

if $\Im s \neq 0$ or

$$b_{n+1} \equiv \frac{(-1)^n}{n!}\left[\int_0^1 t^n f_n(t)\,\mathrm{d}t + \int_1^\infty t^n f_{n+1}(t)\,\mathrm{d}t + a_n \sum_{k=1}^{n} \frac{1}{k}\right] \tag{12}$$

$$= \frac{(-1)^n}{n!}\left\{\lim_{z\to n}\left[M[f; z+1] + \frac{a_n}{z-n}\right] + a_n \sum_{k=1}^{n} \frac{1}{k}\right\}, \tag{13}$$

if $\Im s = 0$.

Proof. Let $f_0(t) \equiv f(t) - \sum_{k=K}^{-1} a_k t^{-k-s}$. Then, for $n = 0, 1, 2, \ldots$,

$$f_{n+1}(t) = f_n(t) - \frac{a_n}{t^{n+s}}$$

and

$$f_{n+1,n}(t) = f_{n,n}(t) - (-1)^n \frac{a_n}{(s)_n} \frac{1}{t^s}.$$

From this, by integration, it follows that

$$\int_0^t f_{n,n}(u)\,\mathrm{d}u = f_{n+1,n+1}(t) + (-1)^n a_n g_n(s,t) + b_{n+1},$$

where

$$g_n(s,t) \equiv \begin{cases} \log(t)/n! & \text{if } \Im s = 0 \\ -t^{-i\Im s}/(i\Im s)_{n+1} & \text{if } \Im s \neq 0, \end{cases}$$

and where we have defined the integration constant

$$b_{n+1} \equiv -\lim_{t\to 0}\left[f_{n+1,n+1}(t) + (-1)^n a_n g_n(s,t)\right].$$

From here, the proof is the same as the proofs of Wong (1989, Chapter 6, Lemma 2 and Theorem 2) from formulae (2.21) and (2.35) respectively: just replace $\log t$ by $n! g_n(s,t)$ and d_{n+1} by b_{n+1} in those proofs. □

To apply Lemmas 1 and 2 to the integral (7) we choose a specific function in $\mathcal{S}$:

$$\varphi_\eta(t) \equiv \frac{e^{-\eta t}}{(t+z)^w} \in \mathcal{S},$$

where $\eta > 0$ and $z \notin \mathbb{R}^- \cup \{0\}$. We will need also the following lemma.

LEMMA 3 Let $f(t)$ satisfy (8). Then, for $0 < \Re s \leqslant 1$, $k = 0, 1, 2, \ldots$ and $n = 1, 2, 3, \ldots$, the following identities hold;

$$\lim_{\eta\to 0}\langle \mathbf{f}, \varphi_\eta\rangle = \int_0^\infty \frac{f(t)}{(t+z)^w}\,\mathrm{d}t \quad \text{for } \Re(s+w)+K>1,$$

$$\lim_{\eta\to 0}\langle \delta, \varphi_\eta^{(k)}\rangle = \frac{(-1)^k(w)_k}{z^{k+w}},$$

$$\lim_{\eta\to 0}\langle \mathbf{t^{-s}}, \varphi_\eta^{(k)}\rangle = \frac{(-1)^k\Gamma(k+w+s-1)\Gamma(1-s)}{\Gamma(w)z^{k+w+s-1}} \quad \text{for } \Re(s+w)+k>1, \quad s\neq 1,$$

$$\lim_{\eta\to 0}\langle \mathbf{log(t)}, \varphi_\eta^{(k+1)}\rangle = \frac{(-1)^{k+1}}{z^{k+w}}(w)_k\big(\log(z)-\gamma-\psi(k+w)\big) \quad \text{for } \Re(s+w)>0,$$

where γ is the Euler constant and ψ the digamma function and

$$\lim_{\eta\to 0}\langle \mathbf{f_{n,n}}, \varphi_\eta^{(n)}\rangle = (-1)^n(w)_n\int_0^\infty \frac{f_{n,n}(t)}{(t+z)^{n+w}}\,\mathrm{d}t \quad \text{for } \Re(s+w)+n>1.$$

Proof. This is a trivial generalization of the proofs of López (2000, lemma 2), López (2001, lemma 3), from real to complex values of s and w. □

With these preparations, we are able now to obtain asymptotic expansions of the integral (7) for large z. This is achieved in the following theorems.

THEOREM 1 Let $f(t)$ be a locally integrable function on $[0,\infty)$ which satisfies (8) with $0 < \Re s \leqslant 1$, $s \neq 1$. Then, for $z \in \mathbb{C}\setminus\mathbb{R}^-$, $z \neq 0$, $\Re(s+w)+K>1$ and $n = 1, 2, 3, \ldots$,

$$\int_0^\infty \frac{f(t)}{(t+z)^w}\,\mathrm{d}t = \sum_{k=K}^{n-1} \frac{(-1)^k\pi a_k\Gamma(w+s+k-1)}{\Gamma(s+k)\Gamma(w)\sin(\pi s)z^{w+s+k-1}} + \sum_{k=0}^{n-1}\frac{(-1)^k(w)_kM_k}{z^{k+w}} + R_n(w;z), \tag{14}$$

where

$$M_k \equiv \begin{cases} M[f;k+1]/k! & \text{if } \Re s \neq 1, \\ (-1)^k b_{k+1} & \text{if } \Re s = 1 \end{cases} \tag{15}$$

and, for $k = 0, 1, 2, \ldots$, the coefficients b_{k+1} are given by (10), (11) or

$$b_{n+1} = \frac{(-1)^n}{n!}\left\{\lim_{T\to\infty}\left[\int_0^T t^n f(t)\,\mathrm{d}t + \sum_{k=K}^{n}\frac{a_kT^{n-k}}{k-n+s-1}\right] + \frac{a_n}{1-s} + a_n\sum_{k=1}^{n+1}\frac{(n-k+2)_{k-1}}{(n+s-k)_k}\right\}. \tag{16}$$

The remainder term satisfies

$$R_n(w;z) \equiv (w)_n \int_0^\infty \frac{f_{n,n}(t)\,\mathrm{d}t}{(t+z)^{n+w}}; \tag{17}$$

empty sums must be understood as zero and $f_{n,n}(t)$ is defined in (9).

Proof. For $\Re s \neq 1$ this follows from Lemmas 1 and 3 using the reflection formula of the gamma function. For $\Re s = 1$, from Lemmas 2 and 3 we obtain immediately formulae (14) and (15), but with b_{k+1} as given in formulae (10) or (11). Introducing

$$f_n(t) = f(t) - \sum_{k=K}^{n-1} \frac{a_k}{t^{k+s}} \tag{18}$$

in the integrands on the right-hand side of (10) and using simple manipulations we obtain (16). □

THEOREM 2 Let $f(t)$ be a locally integrable function on $[0,\infty)$ which satisfies (8) with $s = 1$. Then, for $z \in \mathbb{C} \setminus \mathbb{R}^-$, $z \neq 0$, $\Re w + K > 0$ and $n = 1, 2, 3, \ldots,$

$$\begin{aligned}\int_0^\infty \frac{f(t)}{(t+z)^w}\,\mathrm{d}t = {} & \sum_{k=K}^{-1} a_k \frac{\Gamma(w+k)\Gamma(-k)}{\Gamma(w) z^{w+k}} \\ & + \sum_{k=0}^{n-1} \left[a_k \frac{(-1)^k (w)_k}{k! z^{k+w}} \big(\log(z) - \gamma - \psi(k+w)\big) + b_{k+1} \frac{(w)_k}{z^{k+w}} \right] \\ & + R_n(w;z),\end{aligned} \tag{19}$$

where, for $k = 0, 1, 2, \ldots,$ the coefficients b_{k+1} are given by (12), (13) or

$$b_{n+1} = \frac{(-1)^n}{n!} \left\{ \lim_{T\to\infty} \left[\int_0^T t^n f(t)\,\mathrm{d}t + \sum_{k=K}^{n-1} \frac{a_k T^{n-k}}{k-n} - a_n \log(T) \right] + a_n \sum_{k=1}^{n} \frac{1}{k} \right\}, \tag{20}$$

empty sums being understood as zero. The remainder term $R_n(w;z)$ is given in (17).

Proof. From Lemmas 2 and 3 we obtain immediately formulae (17) and (19), but with b_{k+1} as given in formulae (12) and(13). Introducing (18) (with $s = 1$) in the integrands on the right-hand side of (12) and using simple manipulations we obtain (20). □

2.2 *Error bounds*

In the following theorem we show that the expansions (14) and (19) given in Theorems 1, 2 respectively are in fact asymptotic expansions for large z.

THEOREM 3 In the region of validity of the expansions (14) and (19), the remainder term $R_n(w;z)$ in these expansions satisfies

$$|R_n(w;z)| \leqslant \frac{C_n}{|z|^{n+\Re s+\Re w-1}} \tag{21}$$

if $0 < \Re s < 1$ and

$$|R_n(w;z)| \leqslant \frac{C_n \log|z|}{|z|^{n+\Re w}} \tag{22}$$

if $\Re s = 1$, where the constants C_n are independent of $|z|$ (but may depend on the remaining parameters of the problem).

Proof. On the one hand, $f_n(t) = O(t^{-n-s})$ for $t \to \infty$ (with $0 < \Re s \leqslant 1$); then there is a certain $t_0 \in (0,\infty)$ and a constant $C_{1,n}$ such that $|f_n(t)| \leqslant C_{1,n} t^{-n-\Re s}$ for all $t \in [t_0,\infty)$. Then, introducing this bound in the definition (9) of $f_{n,n}(t)$ we obtain the bound $|f_{n,n}(t)| \leqslant C_{2,n} t^{-\Re s}$ for all $t \in [t_0,\infty)$, where $C_{2,n}$ is a certain positive constant. On the other hand, $f_{n,n}(t)$ is bounded on any compact interval in $[0,\infty)$ for $s \neq 1$ and $f_{n,n}(t)$ is bounded on any compact interval in $(0,\infty)$ and $O(\log t)$ as $t \to 0^+$ for $s = 1$. Then, for all $t \in [0,t_0]$ we have $|f_{n,n}(t)| \leqslant C_{3,n} t^{-\Re s}$ for $0 < \Re s < 1$ and $|f_{n,n}(t)| \leqslant C_{3,n}(|\log t| + 1)$ for $\Re s = 1$, where $C_{3,n}$ is a certain positive constant.

If we divide the integration interval $[0,\infty)$ in the definition (17) of $R_n(w;z)$ at the point t_0 and introduce these bounds in each one of the intervals $[0,t_0]$ and $[t_0,\infty)$, we obtain the bounds (21) and (22). □

The bounds (21) and (22) are not useful for numerical computations unless we are able to calculate the constants C_n in terms of the data of the problem (w, $\operatorname{Arg}(z)$ and $f(t)$). The following two propositions show that, if the bound $|f_n(t)| \leqslant C_{1,n} t^{-n-\Re s}$ holds for all $t \in [0,\infty)$ and not only for $t \in [t_0,\infty)$ then the constants C_n may be calculated in terms of $C_{1,n}$.

PROPOSITION 1 If, for $0 < \Re s < 1$, the remainder $f_n(t)$ in the expansion (8) of the function $f(t)$ satisfies the bound $|f_n(t)| \leqslant c_n t^{-n-\Re s}$ for all $t \in [0,\infty)$ for some positive constant c_n, then the remainder $R_n(w;z)$ in the expansions (14) and (19) satisfies

$$|R_n(w;z)| \leqslant \frac{c_n \pi (|w|)_n \Gamma(n+\Re w+\Re s-1) h(z,w)}{\Gamma(n+\Re s)\Gamma(n+\Re w)|\sin(\pi \Re s)||z|^{n+\Re w+\Re s-1}} \times F\left(\left.\begin{matrix} 1-\Re s,\ n+\Re s+\Re w-1 \\ (n+\Re w+1)/2 \end{matrix}\right| \sin^2\left(\frac{\operatorname{Arg}(z)}{2}\right)\right),$$

where

$$h(z,w) \equiv \begin{cases} 1 & \text{if } \operatorname{Arg}(z)\Im w \geqslant 0, \\ e^{|\operatorname{Arg}(z)\Im w|} & \text{if } \operatorname{Arg}(z)\Im w < 0. \end{cases} \tag{23}$$

Proof. Introducing the bound $|f_n(t)| \leqslant c_n t^{-n-\Re s}$ in the definition (9) of $f_{n,n}(t)$ we obtain

$$|f_{n,n}(t)| \leqslant \frac{c_n \Gamma(\Re s)}{\Gamma(n+\Re s) t^{\Re s}} \qquad \forall t \in [0,\infty).$$

Introducing this bound in the definition (17) of $R_n(w;z)$ and using the duplication formula of the gamma function and (Prudnikov *et al.*, 1990, p. 309, equation 7) we obtain the desired result. □

PROPOSITION 2 If, for $\Re s = 1$, each remainder $f_n(t)$ in the expansion (8) of the function $f(t)$ satisfies the bound $|f_n(t)| \leqslant c_n t^{-n-1}$ for all $t \in [0, \infty)$ for some positive constant c_n, then the remainder $R_n(w; z)$ in the expansion (19) satisfies

$$|R_n(w;z)| \leqslant \frac{\bar{c}_n \pi (|w|)_n \Gamma(n+\Re w-\frac{1}{2}) h(z,w)}{\Gamma(n+\frac{1}{2})\Gamma(n+\Re w)|z|^{n+\Re w-\frac{1}{2}}} \times F\left(\left.\begin{matrix}\frac{1}{2}, n+\Re w-\frac{1}{2}\\ (n+\Re w+1)/2\end{matrix}\right| \sin^2\left(\frac{\operatorname{Arg}(z)}{2}\right)\right), \tag{24}$$

where $h(z, w)$ is defined in (23) and $\bar{c}_n \equiv \mathrm{Max}\{c_n, c_{n-1} + |a_{n-1}|\}$, and

$$\begin{aligned}|R_n(w;z)| \leqslant \frac{(|w|)_n}{|z|^{n+\Re w}} &\left\{ \frac{\epsilon(c_{n-1}+|a_{n-1}|)+c_n}{(n-1)!\,\Theta_\epsilon(z)^{n+\Re w}} + \frac{c_n}{n!}\left|1+\frac{\epsilon}{z}\right|^{-n-\Re w} \left[\log|z|\right.\right.\\ &+ \frac{(n+\Re w)[(2\epsilon+\Re z+|\Re z|)(|z|^{-1}-1)+(|\Re z|-\Re z)\log|z|]}{2(n+\Re w+1)|z+\epsilon|}F_1\\ &+\left.\left.\frac{4\epsilon+\Re z+|\Re z|-2\epsilon|z|}{2\epsilon(n+\Re w+1)|z|}F_0+\frac{2|\epsilon+z|F_{-1}}{\epsilon((n+\Re w)^2-1)|z|}\right]\right\} h(z,w),\end{aligned} \tag{25}$$

where ϵ is an arbitrary positive number,

$$F_k \equiv F\left(\left.\begin{matrix}2-k, n+\Re w+k\\ (n+\Re w+3)/2\end{matrix}\right| \sin^2\left(\frac{\operatorname{Arg}(z+\epsilon)}{2}\right)\right) \tag{26}$$

and

$$\Theta_\epsilon(z) \equiv \begin{cases} 1 & \text{if } \Re z \geqslant 0,\\ |\sin(\operatorname{Arg}(z)| & \text{if } \epsilon \geqslant -\Re z > 0,\\ |1+\epsilon/z| & \text{if } -\Re z > \epsilon > 0.\end{cases} \tag{27}$$

For large z and fixed n, the optimum value for ϵ is given approximately by

$$\epsilon^2 = \frac{c_n}{n(c_{n-1}+|a_{n-1}|)}\left[\frac{2F_{-1}}{(n+\Re w)^2-1}+\frac{(\Re z+|\Re z|)F_0}{2(n+\Re w+1)|z|}\right]. \tag{28}$$

Proof. From $|f_{n-1}(t)| \leqslant c_{n-1}t^{-n}$ for all $t \in [0, \infty)$ and $f_n(t) = f_{n-1}(t) - a_{n-1}t^{-n}$ we obtain $|f_n(t)| \leqslant (c_{n-1} + |a_{n-1}|)t^{-n}$ for all $t \in [0, \infty)$. To obtain the bound (25) we divide the integral defining $f_{n,n}(t)$ in (9) by a fixed point $u = \epsilon \geqslant t$ and use the bound $|f_n(t)| \leqslant (c_{n-1} + |a_{n-1}|)t^{-n}$ in the integral over $[t, \epsilon]$ and the bound $|f_n(t)| \leqslant c_n t^{-n-1}$ in the integral over $[\epsilon, \infty)$. Using $u - t \leqslant u$ in the integral over $[t, \epsilon]$ we obtain

$$|f_{n,n}(t)| \leqslant \frac{1}{(n-1)!}\left[(c_{n-1}+|a_{n-1}|)\log\left(\frac{\epsilon}{t}\right)+\frac{c_n}{\epsilon}\right] \qquad \forall t \in [0,\epsilon], \epsilon > 0. \tag{29}$$

On the other hand, for all $t \in [0, \infty)$ we introduce the bound $|f_n(t)| \leqslant c_n t^{-n-1}$ in the integral definition of $f_{n,n}(t)$ and perform the change of variable $u = tv$. We obtain

$$|f_{n,n}(t)| \leqslant \frac{c_n}{n!}\frac{1}{t} \qquad \forall t \in [0,\infty). \tag{30}$$

We divide the integral in the right-hand side of (17) at the point $t = \epsilon$ and use the bound (30) in the integral over $[\epsilon, \infty)$ and the bound (29) in the integral over $[0, \epsilon]$. We obtain

$$|R_n(w; z)| \leqslant \frac{(|w|)_n}{n!}\left[c_n \int_1^\infty \frac{\mathrm{d}t}{t|\epsilon t + z|^{n+\Re w}} + nc_n \int_0^1 \frac{\mathrm{d}t}{|\epsilon t + z|^{n+\Re w}} \right.$$
$$\left. + n\epsilon(c_{n-1} + |a_{n-1}|) \int_0^1 \frac{\log(t^{-1})\,\mathrm{d}t}{|\epsilon t + z|^{n+\Re w}}\right] h(z, w). \qquad (31)$$

Removing a factor $|z|^{n+\Re w}$ from the denominator in the integrand of the two last integrals in the right-hand side of (31) and using the bound $|\epsilon t/z + 1| \geqslant \Theta_\epsilon(z)$ we easily obtain that those two integrals are bounded by $(|z|\Theta_\epsilon(z))^{-n-\Re w}$. On the other hand, we perform the change of variable $t \to |z|t$ in the first integral in the right-hand side of (31) and integrate by parts obtaining

$$|z|^{n+\Re w} \int_1^\infty \frac{\mathrm{d}t}{t|\epsilon t + z|^{n+\Re w}} = \frac{\log|z|}{|1 + \epsilon/z|^{n+\Re w}} + \epsilon(n + \Re w)$$
$$\times \int_{|z|^{-1}}^\infty \frac{(\epsilon t + \cos(\mathrm{Arg}(z))) \log t\,\mathrm{d}t}{[(\epsilon t + \cos(\mathrm{Arg}(z)))^2 + \sin^2(\mathrm{Arg}(z))]^{(n+\Re w)/2+1}}.$$

Now, with the change of variable $t \to t/\epsilon + |z|^{-1}$ and using $-\log|z| \leqslant \log(t/\epsilon + |z|^{-1}) \leqslant t/\epsilon + |z|^{-1} - 1$ for all $t \in [0, \infty)$ and (Prudnikov *et al.*, 1990, p. 309, equation 7) we obtain (25).

To obtain (24) we use $|f_n(t)| \leqslant c_n t^{-n-1}$ and $|f_n(t)| \leqslant (c_{n-1} + |a_{n-1}|)t^{-n}$. Then we have $f_n(t) \leqslant c_n t^{-n-\frac{1}{2}}$ if $t \geqslant 1$ and $f_n(t) \leqslant (c_{n-1} + |a_{n-1}|)t^{-n-\frac{1}{2}}$ if $t \leqslant 1$. Therefore, $f_n(t) \leqslant \bar{c}_n t^{-n-\frac{1}{2}}$ for all $t \in [0, \infty)$. Then, $f_n(t)$ satisfies the bound required in proposition 1 with $\Re s = \frac{1}{2}$ and c_n replaced by $\bar{c}_n$. Now applying Proposition 1 we obtain (24).

If we get rid of irrelevant terms for large z, the right-hand side of (25), as a function of ϵ, has a minimum for ϵ given in (28). □

The following two lemmas introduce two families of functions $f(t)$ which satisfy the bound $|f_n(t)| \leqslant c_n t^{-n-\Re s}$ for all $t \in [0, \infty)$. Moreover, for these functions $f(t)$, the constants c_n can be easily obtained from $f(t)$.

LEMMA 4 Suppose $f(t)$ satisfies (8) with $\Re s > 0$ and $K = 0$ and consider the function $g(u) \equiv u^{-s} f(u^{-1})$. If $g(w)$ is a bounded analytic function in the region W of the complex w-plane comprised by points situated at a distance less than σ from the positive real axis (see Fig. 1), then

$$|f_n(t)| \leqslant Cr^{-n} t^{-n-\Re s},$$

where C is a bound of $|g(w)|$ in W and $0 < r < \sigma$.

Proof. From the asymptotic expansion (8) and the Lagrange formula for the remainder in the Taylor expansion of $g(u)$ at $u = 0$, we have

$$g(u) = \sum_{k=0}^{n-1} a_k u^k + R_n(u),$$

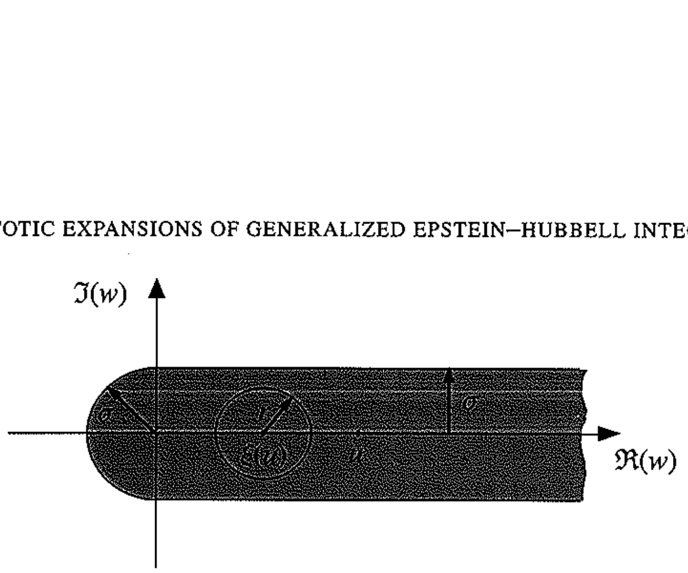

FIG. 1. Analyticity region W for the function $g(u)$ considered in Lemma 5. The integration variable u in (9) is real and unbounded and, therefore, the analyticity region for $g(u)$ must contain the positive real axis. The circle of radius r centred at $\xi(u)$, with $0 < \xi(u) < u$, used in the proof of Lemma 5 must be contained in this region and, therefore, $r < \sigma$.

where

$$R_n(u) = \frac{1}{n!} \left.\frac{\mathrm{d}^n g(u)}{\mathrm{d}u^n}\right|_{u=\xi} u^n, \quad \xi \in (0, u).$$

Using the Cauchy formula for the derivative of an analytic function,

$$\frac{\mathrm{d}^n g(u)}{\mathrm{d}u^n} = \frac{n!}{2\pi \mathrm{i}} \int_C \frac{g(w)}{(w-\xi)^{n+1}} \mathrm{d}w,$$

where C is a circle of radius r around ξ contained inside the region W. Then, for fixed ξ and r, performing the change of variable $w = \xi + re^{i\theta}$, and using $|g(\xi + re^{i\theta})| \leqslant C$ for $\theta \in [0, 2\pi)$ with C independent of θ, r and ξ, we obtain the desired result. □

LEMMA 5 If the expansion (8) satisfies the error test, then

$$|f_n(t)| \leqslant |a_n| t^{-n-\Re s} \quad \text{and} \quad |f_n(t)| \leqslant |a_{n-1}| t^{-(n-1)-\Re s}.$$

Proof. A proof of the first inequality can be found in Olver (1974, p. 68). The second inequality follows from the first, from $\mathrm{sign}(f_n(t)) \neq \mathrm{sign}(f_{n-1}(t))$ and

$$f_n(t) = f_{n-1}(t) - \frac{a_{n-1}}{t^{n-1+s}}.$$

□

COROLLARY 1 If $f(t)$ satisfies the hypotheses of Lemma 4, then $R_n(w; z)$ satisfies the bounds given in Propositions 1 and 2 with $c_n = Cr^{-n}$. Moreover, the expansions given in Theorems 1 and 2 are convergent when the parameter $|z|$ is longer than the inverse of the width of the region considered in Lemma 4 (see Fig. 1); more precisely, when $r|z| \geqslant 1$ if $\Re w < 1$ or $r|z| > 1$ if $\Re w \geqslant 1$.

For $\Re s = 1$, the convergence of these expansions requires $\lim_{n\to\infty} n^{w-1} a_n z^{-n} = 0$ also.

COROLLARY 2 If the expansion (8) of $f(t)$ satisfies the error test, then $R_n(w; z)$ satisfies the bounds given in Propositions 1 and 2 replacing c_n by $|a_n|$ and c_{n-1} by 0. Moreover, the expansions given in Theorems 1 and 2 are convergent when the coefficients a_n in the asymptotic expansion (8) satisfy $\lim_{n\to\infty} n^{w-1} a_n z^{-n} = 0$.

3. Asymptotic expansions of the Epstein–Hubbell integral

In order to obtain asymptotic expansions of the generalized Epstein–Hubbell integral (2) for $k \to 1$ we just apply Theorems 1 and 2 to the integral (3). Error bounds for the remainders are obtained from Corollaries 1 and 2.

COROLLARY 3 For $\Re\alpha, \Re\beta > 0$, $\rho - 1 \notin \mathbb{R}^+ \cup \{0\}$ if $\Re\lambda \geqslant 1$, $\delta + 1 \notin \mathbb{R}^- \cup \{0\}$ if $\Re\gamma \geqslant 1$, $\beta - \mu + \frac{1}{2} \notin \mathbb{Z}$ and $|\mathrm{Arg}(\bar{k})| < \pi$,

$$(1+\delta)^\gamma (1-k^2)^{\mu+\frac{1}{2}} \Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho,\delta;k) = \sum_{m=0}^{n-1} \frac{\Gamma(m+\mu+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})} \frac{(-1)^m B_m}{\bar{k}^{m+\mu+\frac{1}{2}}} + \sum_{m=0}^{n-M-1} \frac{(-1)^m \pi \Gamma(m+\beta)}{\Gamma(\mu+\frac{1}{2})\Gamma(m+\beta-\mu+\frac{1}{2})\sin(\pi\kappa)} \frac{A_m}{\bar{k}^{m+\beta}} + R_n(\bar{k}), \quad (32)$$

where $\bar{k}$ is defined in (4), $M \equiv \lfloor \Re(\beta - \mu + \frac{1}{2}) \rfloor$ and the coefficients A_m are defined in (6). Coefficients B_m are given by

$$B_m \equiv \begin{cases} \dfrac{M[F;m+1]}{m!} & \text{if } \Re\left(\beta-\mu+\frac{1}{2}\right) \notin \mathbb{Z}, \\ \dfrac{M[F;m+1]}{m!} + \dfrac{A_{m-M}}{m!}\left[\dfrac{1}{1-\kappa} + \displaystyle\sum_{j=1}^{m+1} \dfrac{(m-j+2)_{j-1}}{(m+\kappa-j)_j}\right] & \text{if } \Re\left(\beta-\mu+\frac{1}{2}\right) \in \mathbb{Z}, \end{cases}$$

where $\kappa \equiv \mathrm{Fr}\left(\Re\left(\beta-\mu+\frac{1}{2}\right)\right) + i\Im\left(\beta-\mu+\frac{1}{2}\right)$ and

$$M[F;m+1] = \frac{B(\beta-m-\mu-\frac{1}{2}, m+\alpha)}{\bar{\rho}^\lambda \bar{\delta}^\gamma} F_1 \left(\begin{matrix} \alpha+m, \lambda, \gamma \\ \alpha+\beta-\mu-\frac{1}{2} \end{matrix} \middle| \frac{\bar{\rho}-1}{\bar{\rho}}, \frac{\bar{\delta}-1}{\bar{\delta}} \right). \quad (33)$$

Here, $F_1\left(\begin{matrix} a,b,c \\ d \end{matrix} \middle| x,y\right)$ is the Appell hypergeometric function of two variables (Prudnikov *et al.*, 1990, p. 789) and the parameters $\bar{\rho}$ and $\bar{\delta}$ are given in (4).

If $\Re(\beta - \mu + \frac{1}{2}) \notin \mathbb{Z}$ and $n \geqslant M$, a bound for the remainder is given by

$$|R_n(\bar{k})| \leqslant \frac{c_n \pi (|\mu+\frac{1}{2}|)_n \Gamma(n+\Re(\mu+\kappa)-\frac{1}{2}) h(\bar{k},\mu)}{\Gamma(n+\Re\kappa)\Gamma(n+\Re\mu+\frac{1}{2})|\sin(\pi\Re\kappa)||\bar{k}|^{n-M+\Re\beta}} \times F\left(\begin{matrix} 1-\Re\kappa, n+\Re\kappa+\Re\mu-\frac{1}{2} \\ (n+\Re\mu+3/2)/2 \end{matrix} \middle| \sin^2\left(\frac{\mathrm{Arg}(\bar{k})}{2}\right)\right), \quad (34)$$

where we can take $c_n = |A_{n-M}|$ if the following conditions over the parameters hold:

$$\alpha, \beta, \mu \in \mathbb{R}, \quad \beta+\alpha \geqslant \lambda+\gamma+\mu+\tfrac{1}{2}, \quad \lambda \geqslant 0, \quad \gamma \geqslant 0, \quad \rho < 1 \text{ and } \delta > -1. \quad (35)$$

In any case, we can take $c_n = Cr^{-n}$, where

$$C \geqslant \mathrm{Sup}_{u\in W} \left| (1+u)^{\lambda+\gamma+\mu-\alpha-\beta+\frac{1}{2}} (1+\bar{\rho}u)^{-\lambda} (1+\bar{\delta}u)^{-\gamma} \right|, \quad (36)$$

W is the region considered in Lemma 5 for $g(u) = u^{\mu-\beta-\frac{1}{2}}F(u^{-1})$, and

$$0 < r < \mathrm{Min}\left\{1, |\bar{\rho}|^{-1}\xi(\lambda), |\bar{\delta}|^{-1}\xi(\gamma)\right\}, \qquad \xi(z) \equiv \begin{cases} 1 & \text{if } z \notin \mathbb{Z}^- \cup \{0\}, \\ +\infty & \text{if } z \in \mathbb{Z}^- \cup \{0\}, \end{cases} \tag{37}$$

where $h(\bar{k}, \mu)$ is given in (23).

On the other hand, if $\Re(\beta - \mu + \frac{1}{2}) \in \mathbb{Z}$ and $n \geqslant M$, $n \in \mathbb{N}$, two bounds for the remainder are given by

$$|R_n(\bar{k})| \leqslant \frac{\bar{c}_n \pi (|\mu + \frac{1}{2}|)_n \Gamma(n + \Re\mu) h(\bar{k}, \mu)}{\Gamma(n + \frac{1}{2})\Gamma(n + \Re\mu + \frac{1}{2})|\bar{k}|^{n+\Re\mu}} \times F\left(\left.\begin{matrix} \frac{1}{2}, n + \Re\mu \\ (n + \Re\mu + \frac{3}{2})/2 \end{matrix}\right| \sin^2\left(\frac{\mathrm{Arg}(\bar{k})}{2}\right)\right) \tag{38}$$

and

$$\begin{aligned} |R_n(\bar{k})| \leqslant \frac{(|\mu + \frac{1}{2}|)_n}{|\bar{k}|^{n+\Re\mu+\frac{1}{2}}} \Bigg\{ & \frac{\epsilon(c_{n-1} + |A_{n-M-1}|) + c_n}{(n-1)!\,\Theta_\epsilon(\bar{k})^{n+\Re\mu+\frac{1}{2}}} + \frac{c_n}{n!}\left|1 + \frac{\epsilon}{\bar{k}}\right|^{-n-\Re\mu-\frac{1}{2}} \Bigg[\log|\bar{k}| \\ & + \frac{(n + \Re\mu + \frac{1}{2})[(2\epsilon + \Re\bar{k} + |\Re\bar{k}|)(|\bar{k}|^{-1} - 1) + (|\Re\bar{k}| - \Re\bar{k})\log|\bar{k}|]}{2(n + \Re\mu + \frac{3}{2})|\bar{k} + \epsilon|} F_1 \\ & + \frac{4\epsilon + \Re\bar{k} + |\Re\bar{k}| - 2\epsilon|\bar{k}|}{2\epsilon(n + \Re\mu + \frac{3}{2})|\bar{k}|} F_0 + \frac{2|\epsilon + \bar{k}| F_{-1}}{\epsilon((n + \Re\mu + \frac{1}{2})^2 - 1)|\bar{k}|} \Bigg] \Bigg\} h(\bar{k}, \mu). \end{aligned} \tag{39}$$

In these formulae $\bar{c}_n$ =Max$\{|A_{n-M}|, |A_{n-M-1}|\}$ with $c_n = |A_{n-M}|$ and $c_{n-1} = 0$ if conditions (35) hold. In any case, we can take $\bar{c}_n$ =Max$\{c_n, c_{n-1} + |A_{n-M-1}|\}$, with $c_n = Cr^{-n}$ given above. In (39), ϵ is an arbitrary positive number, $\Theta_\epsilon(z)$ is given in (27) and F_k is given in (26) setting $w = \mu + \frac{1}{2}$ and $z = \bar{k}$. For large $\bar{k}$ and fixed n, the optimum value for ϵ is given approximately by (28) setting $z = \bar{k}$ and $w = \mu + \frac{1}{2}$. Moreover, the expansion (32) is convergent when Max$\{|\bar{\rho}|\xi(\lambda)^{-1}, |\bar{\delta}|\xi(\gamma)^{-1}, 1\} < \bar{k}$.

Proof. To obtain the expansion (32), just apply Theorem 1 to the integral (3) with $f(t) = F(t)$ given in (4), $s = \kappa$, $a_m = A_{m-M}$ given in (6), $z = \bar{k}$ and $w = \mu + \frac{1}{2}$. After the change of variable $t = u(1-u)^{-1}$, the Mellin transform of $F(t)$ reads

$$M[F; k+1] = \frac{1}{\bar{\rho}^\lambda \bar{\delta}^\gamma} \int_0^1 \frac{u^{k+\alpha-1}}{(1-u)^{k+\mu-\beta+\frac{3}{2}}} \left(1 + \frac{1-\bar{\rho}}{\bar{\rho}}u\right)^{-\lambda} \left(1 + \frac{1-\bar{\delta}}{\bar{\delta}}u\right)^{-\gamma} du.$$

Then, (33) follows from Prudnikov *et al.* (1990, p. 306, equation 5).

If (35) holds, then, by López (2000, Lemmas 3 and 4) the function $F(t)$ satisfies the error test. Therefore, by Corollary 2, the remainder in the expansion (32) satisfies the bounds given in Propositions 1 and 2 with $c_n = |A_{n-M}|$, $c_{n-1} = 0$. In any case, by Lemma 5 and Corollary 1, the remainder in the expansion (32) satisfies the bounds given in Propositions 1 and 2 with $c_n = Cr^{-n}$, C and r satisfies (36) and (37) respectively. Therefore, the bounds (34), (38) and (39) hold.

Introducing (37) and

$$|A_n| \leqslant C \left| \binom{\mu-\beta-\alpha+\frac{1}{2}}{n} \right| \left[\mathrm{Max}\left\{1, |\bar{\rho}|\xi(\lambda)^{-1}, |\bar{\delta}|\xi(\gamma)^{-1}\right\}\right]^n,$$

where C is a constant independent of n, in (34) and (38) we obtain that $\lim_{n\to\infty} R_n(\bar{k}) = 0$ if $\mathrm{Max}\{|\bar{\rho}|\xi(\lambda)^{-1}, |\bar{\delta}|\xi(\gamma)^{-1}, 1\} < \bar{k}$. □

Tables 1, 2 and 3 show numerical experiments about the approximation supplied by (32) and the accuracy of bounds (34), (38) and (39).

COROLLARY 4 For $\Re\alpha, \Re\beta > 0$, $\rho - 1 \notin \mathbb{R}^+ \cup \{0\}$ if $\Re\lambda \geqslant 1$, $\delta + 1 \notin \mathbb{R}^- \cup \{0\}$ if $\Re\gamma \geqslant 1$, $\beta - \mu + \frac{1}{2} \in \mathbb{Z}$ and $|\mathrm{Arg}(\bar{k})| < \pi$,

$$(1+\delta)^\gamma (1-k^2)^{\mu+\frac{1}{2}} \Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k) = \sum_{m=0}^{\mu-\beta-\frac{1}{2}} A_m \frac{\Gamma(m+\beta)\Gamma(\mu-\beta-m+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})\bar{k}^{m+\beta}}$$
$$+\sum_{m=0}^{n-1} \frac{(-1)^m(\mu+\frac{1}{2})_m}{m!\bar{k}^{m+\mu+\frac{1}{2}}} \left[A_{m+\mu-\beta+\frac{1}{2}}\left(\log(\bar{k}) - \gamma - \psi(m+\mu+\tfrac{1}{2})\right) + B_m\right] + R_n(\bar{k}), \tag{40}$$

where the coefficients A_m are given in (6) and the coefficients B_m are given by

$$B_m \equiv A_{m+\mu-\beta+\frac{1}{2}} \sum_{k=1}^{m} \frac{1}{k} + \frac{(-1)^{m+\mu-\beta-\frac{1}{2}}\Gamma(m+\alpha)}{\Gamma(\beta+\alpha-\mu-\frac{1}{2})(m+\mu-\beta+\frac{1}{2})!\bar{\rho}^\lambda\bar{\delta}^\gamma}$$
$$\times \left\{ [\psi(m+\alpha) - \psi(m+\mu-\beta+\tfrac{3}{2})]\, F_1\left(\begin{matrix} m+\alpha, \lambda, \gamma \\ \alpha+\beta-\mu-\frac{1}{2}\end{matrix} \middle| \frac{\bar{\rho}-1}{\bar{\rho}}, \frac{\bar{\delta}-1}{\bar{\delta}}\right) \right.$$
$$\left. + F_1'\left(\begin{matrix} m+\alpha, \lambda, \gamma \\ \alpha+\beta-\mu-\frac{1}{2}\end{matrix} \middle| \frac{\bar{\rho}-1}{\bar{\rho}}, \frac{\bar{\delta}-1}{\bar{\delta}}\right)\right\}, \tag{41}$$

where $F_1'\left(\begin{matrix} a, b, c \\ d \end{matrix} \middle| x, y\right)$ denotes the derivative of the Appell hypergeometric function with respect to the parameter a.

For $n \geqslant \beta - \mu - \frac{1}{2}$ and $n \in \mathbb{N}$, two bounds for the remainder are given by (38) and (39) in Corollary 3 replacing M by $\beta - \mu - \frac{1}{2}$ in A_{n-M}. And again, the expansion (40) is convergent if $\mathrm{Max}\{|\bar{\rho}|\xi(\lambda)^{-1}, |\bar{\delta}|\xi(\gamma)^{-1}, 1\} < \bar{k}$, where $\xi(z)$ is defined in (37).

Proof. To obtain the expansion (40), just apply Theorem 2 to the integral (3) with $f(t) = F(t)$ given in (4), $s = 1$, $a_m = A_{m+\mu+\frac{1}{2}-\beta}$, $z = \bar{k}$ and $w = \mu + \frac{1}{2}$.

On the other hand, the coefficients A_m in the expansion (5) of $F(t)$ may be written as

$$A_m = \frac{1}{m!}\frac{d^m}{dt^m}\left[t^{\mu-\beta-\frac{1}{2}}F(t^{-1})\right]_{t=0}.$$

Using the Cauchy formula for the derivative of an analytic function, we obtain

$$A_{m+\mu+\frac{1}{2}-\beta} = \frac{d^{m+\mu+\frac{1}{2}-\beta}}{dt^{m+\mu+\frac{1}{2}-\beta}}\left[\frac{t^m(1-t)^{\mu-\beta-\frac{1}{2}}}{(m+\mu+\frac{1}{2}-\beta)!}F\left(\frac{t}{1-t}\right)\right]_{t=1}. \tag{42}$$

TABLE 1 *The second, third and sixth columns represent the integral* $\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k)$, *approximation* (32) *for* $n = 2$ *and for* $n = 3$ *respectively. The fourth and seventh columns represent the respective relative errors, and the fifth and last columns are the respective relative error bounds given by* (34)

Parameter values:
$\alpha = 1{\cdot}7,\ \beta = 0{\cdot}55,\ \gamma = 0{\cdot}5,\ \lambda = 0{\cdot}25,\ \mu = 0{\cdot}73,\ \rho = 0{\cdot}4,\ \delta = 0{\cdot}001.$

k	Λ	First order approx.	Relative error	Relative error bound	Second order approx.	Relative error	Relative error bound
0·5	1·12759	−0·09105	1·08	1·5	0·32218	0·714	0·979
0·7	0·91954	0·36985	0·598	0·77	0·713	0·225	0·287
0·9	0·56255	0·464	0·175	0·202	0·551	0·02	0·023
0·95	0·40385	0·36992	0·084	0·092	0·40195	0·0047	0·0052
0·99	0·17873	0·17585	0·0161	0·0167	0·1787	1·76e−4	1·82e−4
0·999	5·22502e−2	5·21673e−2	1·59e−3	1·6e−3	5·22501e−2	1·73e−6	1·74e−6
0·9999	1·48615e−2	1·486e−2	1·581e−4	1·584e−4	1·48615e−2	1·705e−8	1·725e−8

TABLE 2 *The second, third and sixth columns represent the integral* $\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k)$, *approximation (32) for* $n=2$ *and for* $n=3$ *respectively. The fourth and seventh columns represent the respective relative errors, and the fifth and last columns are the respective relative error bounds given by (34)*

Parameter values:
$\alpha=1{\cdot}5,\ \beta=0{\cdot}7,\ \gamma=0,\ \lambda=0{\cdot}2i,\ \mu=0{\cdot}25+0{\cdot}5i,\ \rho=0{\cdot}1,\ \delta=0{\cdot}1.$

k	Λ	First order approx.	Relative error	Relative error bound	Second order approx.	Relative error	Relative error bound
0·5	0·91077 −0·07139i	0·35393 −0·20978i	0·628	9·321	0·55577 −0·18143i	0·406	5·886
0·7	0·75508 −0·13542i	0·47979 −0·16594i	0·361	4·275	0·65421 −0·15206i	0·133	1·54
0·9	0·44459 −0·18898i	0·39137 −0·17996i	0·111	0·910	0·43854 −0·18824i	0·012	0·1
0·95	0·29432 −0·17831i	0·27654 −0·17213i	0·054	0·377	0·29333 −0·17801i	0·003	0·0203
0·99	0·08844 −0·10594i	0·08731 −0·10497i	0·0107	0·059	0·08842 −0·10593i	1·15e−4	6·24e−4
0·999	6·94e−3 −0·02761i	6·929e−3 −0·02759i	1·06e−3	5·67e−3	6·94e−2 −0·02761i	1·13e−6	5·96e−6
0·9999	−4·5655e−4 −4·2913e−3i	−4·5654e−4 −4·2909e−3i	1·043e−4	7·453e−4	4·5655e−4 −4·2913e−3i	1·108e−8	7·845e−8

TABLE 3 *The second, third and sixth columns represent the integral* $\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k)$, *approximation* (32) *for* $n = 2$ *and for* $n = 3$ *respectively. The fourth and seventh columns represent the respective relative errors, and the fifth and last columns are the respective relative error bounds given by Min*{(38),(39)}

Parameter values:
$\alpha = 0{\cdot}7,\ \beta = 0{\cdot}5,\ \gamma = 0{\cdot}1 + 0{\cdot}2i,\ \lambda = 0{\cdot}2,\ \mu = i,\ \rho = 0{\cdot}01,\ \delta = 0{\cdot}01.$

k	Λ	First order approx.	Relative error	Relative error bound	Second order approx.	Relative error	Relative error bound
0·5	2·16114 −0·47149i	1·61551 −0·4415i	0·247	2·426	1·95664 −0·47183i	0·092	1·528
0·6	1·95106 −0·64402i	1·54907 −0·57396i	0·198	2·048	1·83023 −0·63062i	0·059	1·012
0·7	1·65145 −0·81032i	1·39146 −0·72221i	0·149	1·664	1·59233 −0·79503i	0·033	0·598
0·8	1·22241 −0·92728i	1·09061 −0·84789i	0·1	1·27	1·2013 −0·91728i	0·015	0·295
0·9	0·5961 −0·8679i	0·56193 −0·8247i	0·0523	0·891	0·59227 −0·8643i	0·005	0·098
0·95	0·1928 −0·62804i	0·18734 −0·61026i	0·028	0·6983	0·19247 −0·6265i	0·002	0·037

The coefficient B_n in (40) is just b_{n+1} given by (13) with $a_n = A_{n+\mu+\frac{1}{2}-\beta}$. The Mellin transform in this formula is given by (33). When $z \to n$, there are two singular terms in this limit: $A_{n+\mu+\frac{1}{2}-\beta}/(z-n)$ and $B(\beta - z - \mu - \frac{1}{2}, z+\alpha)$. Setting $z = n + \eta$, expanding these terms at $\eta = 0$ and using (42) we obtain (41).

The bounds (38) and (39) are obtained as in Corollary 3 (using only Proposition 2). □

Table 4 shows a numerical experiment about the approximation supplied by (40) and the accuracy of bounds (38) and (39).

4. Conclusions

Asymptotic expansions of generalized Stieltjes transforms of complex valued functions have been derived in Section 2, including error bounds. They extend to the complex case the known methods given in Wong (1980, Chapter 6), Wong (1989), López (2000) and López (2001) for real functions. Using these methods we have derived two expansions with error bounds of the generalized Epstein–Hubbell integral (2) in Corollaries 3 and 4. Moreover, these expansions are convergent when the asymptotic variable is greater than the remaining ones. The convergence rate increases as the difference between the asymptotic variable and the remaining ones increases. When the parameters defining the function $F(t)$ in the integrand of the Epstein–Hubbell integral (3) satisfy conditions given in (35), then $F(t)$ belongs to a special kind of functions: the remainder term in its asymptotic expansion in inverse powers of t satisfies the error test. This fundamental property allows us to use Corollary 2 to derive a more accurate error bound for the remainder in the asymptotic expansions of the Epstein–Hubbell integral given in Corollaries 3 and 4. These bounds

TABLE 4 *The second, third and sixth columns represent the integral* $\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k)$, *approximation* (40) *for* $n=2$ *and* $n=3$ *respectively. The fourth and seventh columns represent the respective relative error, and the fifth and last columns are the respective relative error bounds given by* min{(38),(39)}

Parameter values:
$\alpha=1,\ \beta=2,\ \gamma=0{\cdot}01,\ \lambda=0{\cdot}1,\ \mu=1{\cdot}5,\ \rho=0{\cdot}1,\ \delta=0{\cdot}01.$

k	Λ	First order approx.	Relative error	Relative erorr bound	Second order approx.	Relative error	Relative erorr bound
0·9	0·01878	0·01393	0·258	0·887	0·01802	0·04	0·154
0·95	0·00592	0·0052	0·121	0·468	0·00587	0·009	0·04
0·99	0·0003731	0·000365	0·022	0·127	0·00037298	2·4e−4	2·1e−3
0·999	5·9363e−6	5·9236e−6	2·14e−3	2·5e−2	5·9363e−6	3·72e−6	4·1e−5
0·9999	8·22324e−8	8·2215e−8	2·1e−4	5·7e−3	8·22325e−8	5·82e−7	9·36e−7

have been obtained from the error test and, as numerical computations show (see Tables 1 and 4), they exhibit a remarkable accuracy.

Expansions given in Corollaries 3 and 4 constitute a simple form of the asymptotic formula for $\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho,\delta;k)$ derived by Kalla & Tuan (1996). Apart from the simplicity of the expansion, the approach presented here supplies a simple algorithm for the calculation of the coefficients of these expansions and accurate error bounds at any order of approximation.

Distributional approach should succeed for deriving complete uniform asymptotic expansions of Epstein–Hubbell integrals too. This challenge is postponed for further investigations.

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