The symmetric standard elliptic integrals with two or three large parameters

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ABSTRACT

Symmetric standard elliptic integrals are considered when two or three of their variables are large without any prescribed asymptotic relation between them. The analytic continuation method is used for deriving seven expansions of these integrals in inverse powers of the asymptotic parameters. These expansions are uniformly convergent when the asymptotic parameters are greater than the remaining ones. All of the expansions are accompanied by an error bound at any order of the approximation.

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1. Introduction

Elliptic integrals (EI) are integrals of the type $\int R(x,y)dx$, where $R(x,y)$ is a rational function of x and y, with y^2 a polynomial of the third or fourth degree in x. Legendre showed that all EI can be expressed in terms of three standard EI (Legendre's normal EI) [5].

A survey of properties of the standard EI can be found, for example, in [[10], chap. 12]. However, as it has been shown by Carlson and Gustafson [1], [4], for numerical computations it is more convenient to use symmetric standard EI instead of Legendre's normal EI. (They are connected by means of simple formulas [[10], eq. 12.33].) They are defined as follows:

$$R_F(x,y,z)=\frac{1}{2}\int_0^\infty\frac{dt}{\sqrt{(t+x)(t+y)(t+z)}},$$

$$R_D(x,y,z)=\frac{3}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)^3}},$$

$$R_J(x,y,z,p)=\frac{3}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}(t+p)},$$

where we assume that the parameters x, y, z are nonnegative, distinct and we will consider $p>0$ and $p\neq x$, y, z.

Asymptotic expansions of EI have been investigated by Carlson, Gustafson, Wong and López [1], [4], [6], [7], [[12], chap.6], using either Mellin transform techniques or the distributional approach [[12], chaps. 3,6]. In particular, [6] and [7] contain a very complete information about the 12 different possible expansions (with error bounds) of R_F, R_D and R_J considering one, two or three large parameters. But in these expansions, when two or three parameters are large, they go to infinity at the same speed: for example, [7] considers asymptotic expansions of $R_F(x,ax,z)$ for large x and fixed a and z. In this paper we try to solve the problem of finding asymptotic expansions of those integrals when two or three parameters go to infinity at arbitrary speeds. For example, we consider asymptotic expansions of $R_F(x,y,z)$ for large x and y and fixed z without restricting ourselves to the case $y=ax$ with fixed a. We face the challenge of obtaining easy algorithms for computing the coefficients of these expansions and simple expressions for the error bounds at any order of the approximation.

To this end we use the method of analytic continuation introduced in [11] and developed in [2], [3] and [8]. As well as we did in [6] or [7], we use the error test for finding error bounds but, in this paper, the analysis of the error bound is based on a different representation of the error term which offers a higher precision.

2. The method of analytic continuation

In this section, we resume some definitions and theorems proved in [2], [8]. We denote $h_v(t)\equiv h(t,vt)$ and $h_{v,w}(t)\equiv h(t,vt,wt)$.

Definition 1. *We denote by $\mathcal{H}_{v,a,\beta}$ the set of functions $h_v\in L^1_{\rm Loc}(0,\infty)$ uniformly for $v\in[0,1]$ verifying:*

(i) h_v has an asymptotic expansion at $t=0$ uniformly valid for $v\in[0,1]$:

$$h_v(t)=\sum_{k=0}^{n-1}A_k(v)t^{k-a}+h_n(t,vt),\quad n=1,2,3,...,\quad a\in\mathbb{R},$$

where $\{A_k(v)\}$ is a sequence of complex numbers uniformly bounded for $v\in[0,1]$ and $h_n(t,vt)=\mathcal{O}(t^{n-a})$ when $t\to 0^+$ uniformly in $v\in[0,1]$.

(ii) $h_v(t)=\mathcal{O}(t^{-\beta})$ when $t\to\infty$ for some $\beta\in\mathbb{R}$ uniformly in $v\in[0,1]$.

Definition 2. *We denote by $\mathcal{H}_{v,w,a,\beta}$ the set of functions $h_{v,w}\in L^1_{\rm Loc}(0,\infty)$ uniformly for $0\le w\le v\le 1$ verifying:*

(i) $h_{v,w}$ *has an asymptotic expansion at* $t=0$ *uniformly valid for* $0 \le w \le v \le 1$*:*

$$h_{v,w}(t) = \sum_{k=0}^{n-1} A_k(v,w) t^{k-a} + h_n(t, vt, wt), \quad n = 1,2,3,..., \quad a \in \mathbb{R},$$

where $\{A_k(v,w)\}$ *is a sequence of complex numbers uniformly bounded for* $0 \le w \le v \le 1$ *and* $h_n(t,vt,wt) = \mathcal{O}(t^{n-a})$ *when* $t \to 0^+$ *uniformly in* $0 \le w \le v \le 1$*.*

(ii) $h_{v,w}(t) = \mathcal{O}(t^{-\beta})$ *when* $t \to \infty$ *for some* $\beta \in \mathbb{R}$ *uniformly in* $0 \le w \le v \le 1$*.*

Definition 3. *We denote by* $\mathcal{F}_{b,\alpha}$ *the set of functions* $f \in L^1_{\mathrm{Loc}}(0,\infty)$ *verifying:*

(i) f *has an asymptotic expansion at infinity:*

$$f(t) = \sum_{k=0}^{n-1} B_k t^{-k-b} + f_n(t), \quad n = 1,2,3,..., \quad b \in \mathbb{R},$$

where $\{B_k\}$ *is a sequence of complex numbers and* $f_n(t) = \mathcal{O}(t^{-n-b})$ *when* $t \to \infty$*.*

(ii) $f(t) = \mathcal{O}(t^{-\alpha})$ *when* $t \to 0^+$ *for some* $\alpha \in \mathbb{R}$*.*

We require for the parameters a, b, α and β to satisfy the following conditions [8]:
CONDITION I: $a + \alpha < 1 < b + \beta$ CONDITION II: $\alpha < b$ and $a < \beta$.

Let $f \in \mathcal{F}_{b,\alpha}$, $h_v \in \mathcal{H}_{v,a,\beta}$, $f \in \mathcal{F}_{b,\alpha}$, $h_{v,w} \in \mathcal{H}_{v,w,a,\beta}$ and $0 \le w \le v \le u$. Then, for any $n = 1,2,3,...$, the following theorems hold:

Theorem 1. *For* $a + b \notin Z$ *and* $m = n + \lfloor a+b \rfloor$*,*

$$\begin{aligned}\int_0^\infty h(ut,vt)f(t)dt = &\sum_{k=0}^{n-1} B_k M[h_{v/u}; 1-k-b]u^{k+b-1} + \\ &\sum_{k=0}^{m-1} A_k(v/u) M[f; k+1-a]u^{k-a} + \int_0^\infty f_n(t) h_m(ut,vt)dt,\end{aligned} \tag{1}$$

with $\int_0^\infty f_n(t)h_m(ut,vt)dt = \mathcal{O}(u^{n+b-1})$ *when* $u \to 0^+$*.*

Theorem 2. *For* $a + b \in \mathbb{N}$ *and* $m = n + a + b - 1$*,*

$$\begin{aligned}\int_0^\infty h(ut,vt)f(t)dt = &\sum_{k=0}^{a+b-2} A_k(v/u) M[f; k+1-a]u^{k-a} + \\ &\sum_{k=0}^{n-1} u^{k+b-1}\big\{-B_k A_{k+a+b-1}(v/u)\log u + \lim_{\epsilon\to 0}\big[B_k M[h_{v/u}; \epsilon+1-k-b] + \\ &A_{k+a+b-1}(v/u) M[f; \epsilon+k+b]\big]\big\} + \int_0^\infty f_n(t)h_m(ut,vt)dt,\end{aligned} \tag{2}$$

with $\int_0^\infty f_n(t)h_m(ut,vt)dt = \mathcal{O}(u^{n+b-1}\log u)$ *when* $u \to 0^+$*.*

Theorem 3. *For $a+b\notin\mathbb{Z}$ and $m=n+\lfloor a+b\rfloor$,*

$$\int_0^\infty h(ut,vt,wt)f(t)dt=\sum_{k=0}^{n-1}B_kM[h_{v/u,w/u};1-k-b]u^{k+b-1}+\sum_{k=0}^{m-1}A_k(v/u,w/u)M[f;k+1-a]u^{k-a}+\int_0^\infty f_n(t)h_m(ut,vt,wt)dt, \tag{3}$$

with $\int_0^\infty f_n(t)h_m(ut,vt,wt)dt=\mathcal{O}(u^{n+b-1})$ when $u\to 0^+$.

Theorem 4. *For $a+b\in\mathbb{N}$ and $m=n+a+b-1$,*

$$\int_0^\infty h(ut,vt,wt)f(t)dt=\sum_{k=0}^{a+b-2}A_k(v/u,w/u)M[f;k+1-a]u^{k-a}+\sum_{k=0}^{n-1}u^{k+b-1}\{-B_kA_{k+a+b-1}(v/u,w/u)\log u+\lim_{\epsilon\to 0}\big[B_kM[h_{v/u,w/u};\epsilon+1-k-b]+A_{k+a+b-1}(v/u,w/u)M[f;\epsilon+k+b]\big]\}+\int_0^\infty f_n(t)h_m(ut,vt,wt)dt, \tag{4}$$

with $\int_0^\infty f_n(t)h_m(ut,vt,wt)dt=\mathcal{O}(u^{n+b-1}\log u)$ when $u\to 0^+$.

3. Uniform and convergent expansions of the symmetric standard EI

Corollary 1. *For $0\le z<x\le y$ and $n=1,2,3,...,$*

$$R_F(x,y,z)=\frac{1}{2}\sqrt{\frac{\pi}{y}}\sum_{k=0}^{n-1}\left[\frac{k!A_k(x/y)z^{k+1/2}}{\Gamma(k+3/2)x^{k+1/2}}+\frac{\Gamma(k+1/2)z^k}{k!x^k}F\left(\begin{matrix}-k+1/2,1/2\\1\end{matrix}\middle|1-\frac{x}{y}\right)\right]+R_n(x,y,z), \tag{5}$$

where

$$A_k(x/y)=-\frac{x^k}{k!y^k\sqrt{\pi}}\Gamma\left(k+\frac{1}{2}\right)F\left(\begin{matrix}1/2,-k\\1/2-k\end{matrix}\middle|\frac{y}{x}\right). \tag{6}$$

The remainder term $R_n(x,y,z)$ satisfies

$$|R_n(x,y,z)|\le\frac{(1/2)_nz^n}{2(n!)^2}\sum_{k=0}^{n}\frac{\Gamma(k+1/2)\Gamma(n-k+1/2)}{x^ky^{n-k+1/2}}F\left(\begin{matrix}1/2,n-k+1/2\\n+1\end{matrix}\middle|1-\frac{x}{y}\right). \tag{7}$$

Proof. After the change of variable $t \to zt$, the integral $2\sqrt{xy/z}R_F(x,y,z)$ has the form considered in Theorem 1 with $u = z/x$, $v = z/y$ and

$$\begin{aligned} h_v(t) &= \frac{1}{\sqrt{(1+t)(1+vt)}} = \sum_{k=0}^{n-1}(-1)^{k-1}A_k(v)t^k + h_n(t,vt), \\ f(t) &= \frac{1}{\sqrt{1+t}} = \sum_{k=0}^{n-1}\binom{-1/2}{k}t^{-k-1/2} + f_n(t), \end{aligned} \tag{8}$$

with $A_k(v)$ given in (6). Then $a = 0$, $b = 1/2$ and $m = n$ and the asymptotic expansion of $2\sqrt{xy/z}R_F(x,y,z)$ for large x and y (small u and v) follows form eq. (1) with the coefficients $A_k(v/u)$ and B_k given by formula (8). On the other hand, the Mellin transform of f is a beta function, whereas the Mellin transform of h_v can be obtained from [[9], p.303, eq. 24]. Introducing these Mellin transforms in (1) we obtain (5).

Integrating by parts in the integral defining the remainder term in (1) we obtain

$$2\sqrt{\frac{xy}{z}}\,|R_n(x,y,z)| \le \int_0^\infty \left|f_{n,n}(t){h_n}^{(n)}(ut,vt)\right| dt,$$

where [[12], chap. 6, eq. (2.10)] $f_{n,n}(t) = ((-1)^n/(n-1)!)\int_t^\infty (u-t)^{n-1}f_n(u)du$. Using that f verifies the error test [6]: $|f_n(u)| \le \left|\binom{-1/2}{n}\right| u^{-n-1/2}$, the Leibniz rule for the derivation of a product of functions and [[9], p.303, eq. 24] we obtain (7). □

Corollary 2. *For $0 \le x < y \le z$ and $n = 1, 2, 3, ...,$*

$$\begin{aligned} R_D(x,y,z) =& \frac{3}{2}\sqrt{\frac{\pi}{z^3}}\sum_{k=0}^{n-1}\left[\frac{k!A_k(y/z)}{\Gamma(k+3/2)}\frac{x^{k+1/2}}{y^{k+1/2}} + \right. \\ & \left.\frac{\Gamma(k+3/2)x^k}{k!y^k}F\left(\left.\begin{matrix}-k+1/2, 3/2\\ 2\end{matrix}\right| 1-\frac{y}{z}\right)\right] + R_n(x,y,z), \end{aligned} \tag{9}$$

where

$$A_k(y/z) = -\frac{2y^k}{z^k k!\sqrt{\pi}}\Gamma\left(k+\frac{3}{2}\right)F\left(\left.\begin{matrix}1/2, -k\\ -1/2-k\end{matrix}\right|\frac{z}{y}\right). \tag{10}$$

For $n = 1, 2, 3, ...,$ the remainder term $R_n(x,y,z)$ satisfies

$$|R_n(x,y,z)| \le \frac{3(1/2)_n x^n}{(n!)^2}\sum_{k=0}^{n}\frac{\Gamma(k+1/2)\Gamma(n-k+3/2)}{y^k z^{n-k+3/2}}F\left(\left.\begin{matrix}1/2, n-k+1/2\\ n+1\end{matrix}\right| 1-\frac{y}{z}\right). \tag{11}$$

Proof. After the change of variable $t \to xt$, the integral $2\sqrt{yz^3/x}R_D(x,y,z)/3$ has the form considered in Theorem 1 with $u = x/y$, $v = x/z$, the function f given in (8) and $h_v(t) = 1/\sqrt{(1+t)(1+vt)^3}$. Following the same arguments than in Corollary 1, we obtain (9) and (11). □

Corollary 3. *For $0 < z < x \le y$ and $n = 1, 2, 3, ...,$*

$$R_D(x,y,z) = -\frac{3}{2}\sqrt{\frac{\pi}{y}}\left[\sum_{k=0}^{n}\frac{2k!A_k(x/y)z^{k-1/2}}{\Gamma(k+1/2)x^{k+1/2}}+\right.$$
$$\left.\sum_{k=0}^{n-1}\frac{(3/2)_k\sqrt{\pi}z^k}{k!x^{k+1}}F\left(\begin{matrix}-k-\frac{1}{2},\frac{1}{2}\\1\end{matrix}\middle|1-\frac{x}{y}\right)\right]+R_n(x,y,z), \tag{12}$$

where, for $k = 0, 1, 2, ...,$ $A_k(x/y)$ are given in (6). For $n = 1, 2, 3, ...,$ the remainder term $R_n(x, y, z)$ satisfies

$$|R_n(x,y,z)| \le \frac{3(3/2)_n z^n}{2(n+1)!n!}\sum_{k=0}^{n+1}\frac{\Gamma(k+1/2)\Gamma(n-k+3/2)}{x^k y^{n-k+3/2}}F\left(\begin{matrix}1/2, n-k+3/2\\n+2\end{matrix}\middle|1-\frac{x}{y}\right). \tag{13}$$

Proof. After the change of variable $t \to zt$, the integral $2\sqrt{xyz}R_D(x,y,z)/3$ has the form considered in Theorem 1 with $u = z/x$, $v = z/y$, the function h_v given in (8) and $f(t) = 1/\sqrt{(1+t)^3}$. Following the same arguments from Corollary 1 we obtain (9).

Integrating by parts in the integral defining the remainder term in (1) we obtain

$$\frac{3}{2\sqrt{xyz}}|R_n(x,y,z)| \le \int_0^\infty \left|f_{n,n+1}(t)h_{n+1}{}^{(n+1)}(ut,vt)\right|dt, \tag{14}$$

where [[12], chap. 6, eq. 15] $f_{n,n+1}(t) = \left((-1)^{n+1}/n!\right)\int_t^\infty (u-t)^n f_n(u)du$. Using the error test for f and the Leibniz rule for the derivation of a product we obtain (13). □

Corollary 4. *For $0 \le x < y \le z$, $0 < p$ and $n = 1, 2, 3, ...,$*

$$R_J(x,y,z,p) = \frac{3}{2}\sqrt{\frac{\pi}{z}}\left[\sum_{k=0}^{n-1}B_k(p/x)\frac{\sqrt{\pi}x^k}{y^{k+1}}F\left(\begin{matrix}-k-1/2,1/2\\1\end{matrix}\middle|1-\frac{y}{z}\right)\right.$$
$$\left.-\sum_{k=0}^{n}A_k(y/z)\frac{2k!x^{k+1/2}}{p\Gamma(k+1/2)y^{k+1/2}}F\left(\begin{matrix}k+1,1\\3/2\end{matrix}\middle|1-\frac{x}{p}\right)\right]+R_n(x,y,z,p), \tag{15}$$

where $A_k(y/z)$ are given in (6) and

$$B_k\left(\frac{p}{x}\right) = -\frac{p^{k+1/2}}{x^k\sqrt{p-x}}+\frac{x}{p(k+1)!\sqrt{\pi}}\Gamma\left(k+\frac{3}{2}\right)F\left(\begin{matrix}1,3/2+k\\2+k\end{matrix}\middle|\frac{x}{p}\right). \tag{16}$$

For $n = 1, 2, 3, ...,$ the remainder term $R_n(x, y, z, p)$ satisfies

$$|R_n(x,y,z,p)| \le \frac{3|B_n(p/x)|x^n}{2(n+1)!}\sum_{k=0}^{n+1}\frac{\Gamma(k+1/2)\Gamma(n-k+3/2)}{y^k z^{n-k+3/2}}\times$$
$$\times F\left(\begin{matrix}1/2, n-k+3/2\\n+2\end{matrix}\middle|1-\frac{y}{z}\right). \tag{17}$$

Proof. After the change of variable $t \to xt$, the integral $2\sqrt{yz/x}R_J(x,y,z,p)/3$ has the form considered in Theorem 1 with $u = x/y$, $v = x/z$, the function h_v given in (8) and $f(t) = 1/((xt+p)\sqrt{1+t})$. The same arguments than before provide (15) and the error bound (17). □

Corollary 5. *For $0 < x < p \le z$, $0 \le y$ and $n = 1, 2, 3, \ldots$,*

$$R_J(x,y,z,p) = \frac{3}{2\sqrt{z}} \sum_{k=0}^{n-1} \frac{x^k}{p^{k+1}} \left\{ A_k(p/z)B_k(y/x)\log\left(\frac{x}{p}\right) - \sqrt{\frac{x}{y}}A_k(p/z)\times \right.$$
$$F'\left(\begin{array}{c} k+1, 1/2 \\ 1 \end{array}\middle| 1-\frac{x}{y}\right) - \frac{2B_k(y/x)\Gamma(k+3/2)}{\sqrt{\pi}k!}\left[F'\left(\begin{array}{c} -k, 1/2 \\ 3/2 \end{array}\middle| 1-\frac{p}{z}\right)\right.$$
$$\left.\left. + \ (\psi(k+1) - \psi(k+3/2))\, F\left(\begin{array}{c} -k, 1/2 \\ 3/2 \end{array}\middle| 1-\frac{p}{z}\right)\right]\right\} + R_n(x,y,z,p). \tag{18}$$

B_k are the coefficients A_k of (6) and $A_k(p/z) = \dfrac{p^k}{z^k k!\sqrt{\pi}}\Gamma\left(k+\dfrac{1}{2}\right) F\left(\begin{array}{c} 1, -k \\ 1/2-k \end{array}\middle| \dfrac{z}{p}\right)$. *$F'(a,b,c,z)$ represents the derivative of the Gauss hypergeometric function with respect the variable a. For $n = 1, 2, 3, \ldots$, the remainder term $R_n(x,y,z,p)$ satisfies*

$$|R_n(x,y,z,p)| \le 3\left(\left|A_{n-1}\left(\frac{p}{z}\right)B_n\left(\frac{y}{x}\right)\right| + \left|A_n\left(\frac{p}{z}\right)B_{n-1}\left(\frac{y}{x}\right)\right| + \right.$$
$$\left.\frac{1}{2}\left|A_n\left(\frac{p}{z}\right)B_n\left(\frac{y}{x}\right)\log\left(\frac{x}{p}\right)\right|\right)\frac{x^n}{\sqrt{z}p^{n+1}}. \tag{19}$$

Proof. After the change of variable $t \to xt$, the integral $2p\sqrt{z/x}R_J(x,y,z,p)/3$ has the form considered in Theorem 2 with $u = x/p$, $v = x/z$, $h_v(t) = 1/(t+1)\sqrt{vt+1}$ and $f(t) = 1/\sqrt{(1+t)(xt+y)}$. Following the same argument than before we obtain (18).

Consider now the integral expression of the remainder given in Theorem 2 for $m = n$. Write

$$\int_0^\infty f_n(t)h_n(ut,vt)dt = \int_0^1 f_n(t)h_n(ut,vt)dt + \int_1^{1/u} f_n(t)h_n(ut,vt)dt +$$
$$\int_{1/u}^\infty f_n(t)h_n(ut,vt)dt.$$

Perform the change of variable $t \to t/u$ in the last integral, introduce the decomposition $f_n(t) = f_{n-1}(t) - B_{n-1}t^{-n}$ into the first integral in the right hand side and the decomposition $h_n(t,vt) = h_{n-1}(t,vt) - A_{n-1}(v)t^{n-1}$ into the last integral. Bound (19) follows using the error test in both functions: $|f_n(t)| \le |B_n|t^{-n-1}$ and $|h_n(t,vt)| \le |A_n(v)|t^n$.□

Corollary 6. *For $0 \le x < y \le p \le z$ and $n = 1, 2, 3, \ldots$,*

$$R_J(x,y,z,p) = \frac{3\sqrt{\pi}}{2p\sqrt{z}} \sum_{k=0}^{n-1} \frac{x^{k+1/2}}{y^{k+1/2}} \left[\frac{k!A_k(y/p,y/z)}{\Gamma(k+3/2)} + \right.$$
$$\left.\frac{(-1)^k\Gamma(k+1/2)p\sqrt{z}}{k!\pi\sqrt{xy}}B_k\left(\frac{y}{p},\frac{y}{z}\right)\right] + R_n(x,y,z,p), \tag{20}$$

where $A_k(y/p,y/z)=-\sum_{j=0}^{k}\frac{(1/2)_{k-j}(1/2)_j}{(k-j)!j!}\left(\frac{y}{z}\right)^j F\left(\begin{array}{c}1,-j\\1/2-j\end{array}\bigg|\frac{z}{p}\right)$, *and the coefficients* $B_k(y/p,y/z)$ *verify, for* $k=1,2,3,\ldots$, *the recurrence*

$$B_k(y/p,y/z)=\frac{y}{p}\left[\frac{(-1)^k\pi\sqrt{y}}{\sqrt{z}}F\left(\begin{array}{c}-k+1/2,1/2\\1\end{array}\bigg|1-\frac{y}{z}\right)-B_{k-1}(y/p,y/z)\right], \qquad (21)$$

where $B_0(y/p,y/z)=\frac{2}{3}R_J(1,z/y,0,p/y)$. *For* $n=1,2,3,\ldots$, *the remainder term* $R_n(x,y,z,p)$ *satisfies*

$$|R_n(x,y,z,p)|\le\frac{3(1/2)_n}{n!}\left(\frac{|A_n|}{n+1}\epsilon^{n+1}+4|A_{n-1}|\epsilon^{-1/2}\right)\frac{x^n}{p\sqrt{z}y^n}, \qquad (22)$$

where $\epsilon=|2A_{n-1}/A_n|^{1/(n+3/2)}$.

Proof. After the change of variable $t\to xt$, the integral $2p\sqrt{yz/x}R_J(x,y,z,p)/3$ has the form considered in Theorem 4 with $u=x/y$ $v=x/p$, $w=x/z$, the function f given in (8) and $h_{v,w}(t)=1/((vt+1)\sqrt{(t+1)(wt+1)})$. Following the same arguments and, for the Mellin transform, it is straightforward to see that, for $k=0,1,2,\ldots$, the following recurrence holds: $M[h_{v,w};-k-1/2]=M[\sqrt{(t+1)^{-1}(wt+1)^{-1}};-k-1/2]-vM[h_{v,w};1/2-k]$. Then, using formula [[9], p.303, eq. 24] and defining $B_k(y/p,y/z)=\frac{y}{p}\sqrt{\frac{y}{z}}M[h_{y/p,y/z};1/2-k]$, we obtain the recurrence (21) and therefore expansion (20).

Consider now the integral expression of the remainder given in Theorem 4 for $m=n$. For $\epsilon>0$ write

$$\int_0^\infty f_n(t)h_n(ut,vt,wt)dt=\frac{1}{u}\int_0^\epsilon f_n(\frac{t}{u})h_n(t,\frac{v}{u}t,\frac{w}{u}t)dt+\frac{1}{u}\int_\epsilon^\infty f_n(\frac{t}{u})h_n(t,\frac{v}{u}t,\frac{w}{u}t)dt.$$

Introduce the decomposition $h_n=h_{n-1}-A_{n-1}t^{n-1}$ into the last integral in the right side above. Bound (22) follows after using the bounds provided by the error test for the functions $h_{v,w}$ and f and taking the optimum ϵ. □

Corollary 7. *For* $0<p<x\le y\le z$ *and* $n=1,2,3,\ldots$,

$$\begin{aligned}R_J(x,y,z,p)=&\frac{3}{2\sqrt{yz}}\sum_{k=0}^{n-1}\frac{p^k}{x^{k+1/2}}\left[A_k(x/y,x/z)\log\left(\frac{p}{x}\right)+\frac{2\sqrt{yz}}{k!\sqrt{\pi}x}\Gamma\left(k+\frac{3}{2}\right)\times\right.\\&\left[\left(\psi(k+1)-\psi\left(k+\frac{3}{2}\right)\right)F_1\left(k+\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2};1-\frac{y}{x},1-\frac{z}{x}\right)-\right.\\&\left.\left.F_1'\left(k+\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2};1-\frac{y}{x},1-\frac{z}{x}\right)\right]\right]+R_n(x,y,z,p),\end{aligned} \qquad (23)$$

where $F_1(a,b,c,d;x,y)$ *represents an Appell's function and* $F_1'(a,b,c,d;x,y)$ *its derivative with respect to the variable* a. *For* $k=0,1,2,\ldots$, *the coefficients* A_k *are given by*

$A_k(x/y,x/z) = -\sum_{j=0}^{k} \frac{(1/2)_{k-j}\Gamma(j+1/2)}{(k-j)!j!\sqrt{\pi}} \frac{x^j}{y^j} F\left(\begin{matrix} 1/2,-j \\ 1/2-j \end{matrix} \middle| \frac{y}{z} \right)$. *For* $n = 1,2,3,...,$ *the remainder term* $R_n(x,y,z,p)$ *satisfies*

$$|R_n(x,y,z,p)| \le 3\left(|A_{n-1}| + |A_n|\left(1+\frac{1}{2}\left|\log\left(\frac{x}{p}\right)\right|\right)\right)\frac{p^n}{\sqrt{yz}x^{n+1/2}}. \tag{24}$$

Proof. After the change of variable $t \to pt$, the integral $2\sqrt{xyz}R_J(x,y,z,p)/3$ has the form considered in Theorem 4 with $u = p/x$, $v = p/y$, $w = p/z$, $h_{v,w}(t) = 1/\sqrt{(t+1)(wt+1)(vt+1)}$ and $f(t) = 1/(1+t)$. Expansion (20) follows with the same argument using (4) and taking into account that the Mellin transform of $h_{v,w}$ is the first Appell hypergeometric function of two variables F_1. Bound (24) follows using the same argument as in Corollary 5 and the error test in both functions. □

4. Numerical examples

x	$R_F(x,x\log x,1)$	1st OrAp	ReEr	ReErBo	*2nd* OrAp	ReEr	ReErBo
10	0.344184	0.333002	0.032	0.04	0.343572	0.0018	0.002
50	0.144162	0.14337	0.0055	0.006	0.144154	5.6e-5	5.7e-5
100	0.099046	0.0987863	0.0026	0.0028	0.0990447	1.316e-5	1.319e-5
y	$R_D(1,y,y^2)$						
10	0.00411923	0.00396237	0.038	0.05	0.00411154	0.0019	0.002
50	5.37946e-5	5.35581e-5	0.004	0.005	5.37923e-5	4.2e-5	4.6e-5
100	7.83482e-6	7.82013e-6	0.0019	0.002	7.83475e-6	9.1e-6	9.7e-6
x	$R_D(x,2x\log x,1)$						
10	0.100142	0.104209	0.04	0.049	0.100539	0.004	0.0046
50	0.0184569	0.0185161	0.0032	0.0034	0.018458	6.3e-5	6.7e-5
100	0.00888594	0.00889579	0.0011	0.0012	0.00888604	1.09e-5	1.13e-5
y	$R_J(1,y,y^2,2)$						
10	0.0509229	0.05478	0.07	0.1	0.051573	0.013	0.017
50	0.00561821	0.00565251	0.006	0.007	0.00561938	0.0002	0.00025
100	0.00208589	0.00209034	0.002	0.0024	0.00208596	3.6e-5	4.e-5
p	$R_J(1,2,p^3,p)$						
10	0.0105996	0.0090172	0.15	0.4	0.0103535	0.02	0.06
50	0.000309006	0.000299864	0.03	0.06	0.00030872	0.001	0.002
100	6.43773e-5	6.34236e-5	0.01	0.02	6.43622e-5	0.0002	0.0004
y	$R_J(1,y,y^2,2y)$						
10	0.0134692	0.0124728	0.07	0.1	0.0133997	0.005	0.006
50	0.00065188	0.000643092	0.01	0.02	0.00065176	0.00018	0.0002
100	0.000170126	0.000168996	0.007	0.01	0.000170118	4.5e-5	4.9e-5
x	$R_J(x,x\log x,x^2,1)$						
10	0.0266916	0.0245712	0.08	0.1	0.02653	0.006	0.01
50	0.0013541	0.00133727	0.012	0.018	0.00135386	2.e-4	3.e-4
100	0.000364351	0.000362194	0.005	0.008	0.000364336	4.e-5	7.e-4

Second, third and sixth columns represent the EI function, approximation of corollaries 1-7 for $n = 1$ (1^{st} OrAp) and $n = 2$ (2^{nd} OrAp) respectively ($n = 2$ and $n = 3$ in the last example). Fourth and seventh columns represent the respective relative errors (ReEr) $|R_n/R_F|$. Fifth and last columns represent the respective relative error bounds (ReErBo) given by the corresponding equation in corollaries 1-7.

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