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Symmetric standard elliptic integrals with two or three large parameters

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Symmetric standard elliptic integrals are considered when two or three of their variables are large without any prescribed asymptotic relation between them. The analytic continuation method is used for deriving seven expansions of these integrals in inverse powers of the asymptotic parameters. These expansions are uniformly convergent when the asymptotic parameters are greater than the remaining ones. All of the expansions are accompanied by an error bound at any order of the approximation.

Keywords: Elliptic integrals; Uniform asymptotic expansions; Analytic continuation approach

2000 Mathematics Subject Classification: 41A60; 33E05

1. Introduction

Elliptic integrals (EI) are integrals of the type $\int R(x, y)\,\mathrm{d}x$, where $R(x, y)$ is a rational function of x and y, with y^2 a polynomial of the third or fourth degree in x. Legendre showed that all EI can be expressed in terms of three standard EI (Legendre's normal EI) [1].

A survey of properties of the standard EI can be found, for example, in ref. [2, Chapter 12]. However, as it has been shown by Carlson and Gustafson [3, 4], for numerical computations, it is more convenient to use symmetric standard EI instead of Legendre's normal EI. They are connected by means of simple formulas [2, equation (12.33)]. They are defined as follows:

$$R_F(x, y, z) = \frac{1}{2} \int_0^\infty \frac{\mathrm{d}t}{\sqrt{(t + x)(t + y)(t + z)}},$$

$$R_D(x, y, z) = \frac{3}{2} \int_0^\infty \frac{\mathrm{d}t}{\sqrt{(t + x)(t + y)(t + z)^3}},$$

$$R_J(x, y, z, p) = \frac{3}{2} \int_0^\infty \frac{\mathrm{d}t}{\sqrt{(t + x)(t + y)(t + z)}(t + p)},$$

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where we assume that the parameters x, y, z are nonnegative and distinct, $p > 0$ and $p \neq x, y, z$.

Asymptotic expansions of EI have been investigated by Carlson, Gustafson, Wong and López [3–6; 7, Chapter 6], using either Mellin transform techniques or the distributional approach [7, Chapter 3, 6]. In particular, [5, 6] contain very complete information about the 12 different possible expansions (with error bounds) of R_F, R_D and R_J considering one, two or three large parameters. But in these expansions, when two or three parameters are large, they go to infinity at the same speed: for example, ref. [6] considers asymptotic expansions of $R_F(x, ax, z)$ for large x and fixed a and z. In this article, we try to solve the problem of finding asymptotic expansions of those integrals when two or three parameters go to infinity at arbitrary speeds. For example, we consider $R_F(x, y, z)$ for large x and y and fixed z with $y = x^2$, $y = e^x$, and so on. We face the challenge of obtaining easy algorithms for computing the coefficients of these expansions and simple expressions for the error bounds at any order of the approximation.

To this end, we use the method of analytic continuation introduced in ref. [8] and developed in refs. [9–11]. As in refs. [5, 6] we use the error test for finding error bounds but, in this article, the analysis of the error bound is based on a different representation of the error term which offers a higher precision.

2. The method of analytic continuation

In this section, we resume some definitions and theorems proved in refs. [9, 11]. We denote $h_v(t) \equiv h(t, vt)$ and $h_{v,w}(t) \equiv h(t, vt, wt)$.

DEFINITION 1 *We denote by $\mathcal{H}_{v,a,\beta}$ the set of functions $h_v \in L^1_{\text{Loc}}(0, \infty)$ uniformly for $v \in [0, 1]$ verifying:*

(i) *h_v has an asymptotic expansion at $t = 0$ uniformly valid for $v \in [0, 1]$:*

$$h_v(t) = \sum_{k=0}^{n-1} A_k(v)t^{k-a} + h_n(t, vt), \quad n = 1, 2, 3, \ldots, \quad a \in \mathbb{R},$$

where $\{A_k(v)\}$ is a sequence of complex numbers uniformly bounded for $v \in [0, 1]$ and $h_n(t, vt) = \mathcal{O}(t^{n-a})$ when $t \to 0^+$ uniformly in $v \in [0, 1]$.
(ii) *$h_v(t) = \mathcal{O}(t^{-\beta})$ when $t \to \infty$ for some $\beta \in \mathbb{R}$ uniformly in $v \in [0, 1]$.*

DEFINITION 2 *We denote by $\mathcal{H}_{v,w,a,\beta}$ the set of functions $h_{v,w} \in L^1_{\text{Loc}}(0, \infty)$ uniformly for $0 \leq w \leq v \leq 1$ verifying:*

(i) *$h_{v,w}$ has an asymptotic expansion at $t = 0$ uniformly valid for $0 \leq w \leq v \leq 1$:*

$$h_{v,w}(t) = \sum_{k=0}^{n-1} A_k(v, w)t^{k-a} + h_n(t, vt, wt), \quad n = 1, 2, 3, \ldots, \quad a \in \mathbb{R},$$

where $\{A_k(v, w)\}$ is a sequence of complex numbers uniformly bounded for $0 \leq w \leq v \leq 1$ and $h_n(t, vt, wt) = \mathcal{O}(t^{n-a})$ when $t \to 0^+$ uniformly in $0 \leq w \leq v \leq 1$.
(ii) *$h_{v,w}(t) = \mathcal{O}(t^{-\beta})$ when $t \to \infty$ for some $\beta \in \mathbb{R}$ uniformly in $0 \leq w \leq v \leq 1$.*

DEFINITION 3 *We denote by $\mathcal{F}_{b,\alpha}$ the set of functions $f \in L^1_{\text{Loc}}(0, \infty)$ verifying:*

(i) *f has an asymptotic expansion at infinity:*

$$f(t) = \sum_{k=0}^{n-1} B_k t^{-k-b} + f_n(t), \quad n = 1, 2, 3, \ldots, \quad b \in \mathbb{R},$$

where $\{B_k\}$ is a sequence of complex numbers and $f_n(t) = \mathcal{O}(t^{-n-b})$ when $t \to \infty$.

(ii) *$f(t) = \mathcal{O}(t^{-\alpha})$ when $t \to 0^+$ for some $\alpha \in \mathbb{R}$.*

We require for the parameters a, b, α and β to satisfy the following conditions [11]:

Condition I: $a + \alpha < 1 < b + \beta$
Condition II: $\alpha < b$ and $a < \beta$.

Let $f \in \mathcal{F}_{b,\alpha}$, $h_v \in \mathcal{H}_{v,a,\beta}$, $h_{v,w} \in \mathcal{H}_{v,w,a,\beta}$ and $0 \leq w \leq v \leq u$. Then, for any $n = 1$, $2, 3, \ldots$, the following theorems hold.

THEOREM 1 *For $a + b \notin \mathbb{Z}$ and $m = n + \lfloor a + b \rfloor$,*

$$\int_0^\infty h(ut, vt)f(t)\,dt = \sum_{k=0}^{n-1} B_k M[h_{v/u}; 1 - k - b]u^{k+b-1} + \sum_{k=0}^{m-1} A_k \left(\frac{v}{u}\right) M[f; k + 1 - a]u^{k-a}$$

$$+ \int_0^\infty f_n(t)h_m(ut, vt)\,dt, \tag{1}$$

with $\int_0^\infty f_n(t)h_m(ut, vt)\,dt = \mathcal{O}(u^{n+b-1})$ when $u \to 0^+$.

THEOREM 2 *For $a + b \in \mathbb{N}$ and $m = n + a + b - 1$,*

$$\int_0^\infty h(ut, vt)f(t)\,dt = \sum_{k=0}^{a+b-2} A_k \left(\frac{v}{u}\right) M[f; k + 1 - a]u^{k-a}$$

$$+ \sum_{k=0}^{n-1} u^{k+b-1} \left\{ - B_k A_{k+a+b-1} \left(\frac{v}{u}\right) \log u \right.$$

$$+ \lim_{\epsilon \to 0} \left[B_k M[h_{v/u}; \epsilon + 1 - k - b] \right.$$

$$\left. \left. + A_{k+a+b-1} \left(\frac{v}{u}\right) M[f; \epsilon + k + b] \right] \right\}$$

$$+ \int_0^\infty f_n(t)h_m(ut, vt)\,dt, \tag{2}$$

with $\int_0^\infty f_n(t)h_m(ut, vt)\,dt = \mathcal{O}(u^{n+b-1} \log u)$ when $u \to 0^+$.

THEOREM 3 *For $a + b \notin \mathbb{Z}$ and $m = n + \lfloor a + b \rfloor$,*

$$\int_0^\infty h(ut, vt, wt)f(t)\,dt = \sum_{k=0}^{n-1} B_k M[h_{v/u,w/u}; 1 - k - b]u^{k+b-1} + \sum_{k=0}^{m-1} A_k \left(\frac{v}{u}, \frac{w}{u}\right)$$

$$\times M[f; k + 1 - a]u^{k-a} + \int_0^\infty f_n(t)h_m(ut, vt, wt)\,dt, \tag{3}$$

with $\int_0^\infty f_n(t)h_m(ut, vt, wt)\,dt = \mathcal{O}(u^{n+b-1})$ when $u \to 0^+$.

THEOREM 4   *For $a + b \in \mathbb{N}$ and $m = n + a + b - 1$,*

$$\int_0^\infty h(ut, vt, wt) f(t)\, dt$$

$$= \sum_{k=0}^{a+b-2} A_k \left(\frac{v}{u}, \frac{w}{u}\right) M[f; k+1-a] u^{k-a} + \sum_{k=0}^{n-1} u^{k+b-1} \left\{ -B_k A_{k+a+b-1} \left(\frac{v}{u}, \frac{w}{u}\right) \log u \right.$$

$$+ \lim_{\epsilon \to 0} \left[B_k M[h_{v/u, w/u}; \epsilon + 1 - k - b] + A_{k+a+b-1} \left(\frac{v}{u}, \frac{w}{u}\right) M[f; \epsilon + k + b] \right] \right\}$$

$$+ \int_0^\infty f_n(t) h_m(ut, vt, wt)\, dt, \tag{4}$$

with $\int_0^\infty f_n(t) h_m(ut, vt, wt)\, dt = \mathcal{O}(u^{n+b-1} \log u)$ when $u \to 0^+$.

3.   Uniform and convergent expansions of the symmetric standard EI

COROLLARY 1   *For $0 \le z < x \le y$ and $n = 1, 2, 3, \ldots$,*

$$R_F(x, y, z) = \frac{1}{2} \sqrt{\frac{\pi}{y}} \sum_{k=0}^{n-1} \left[\frac{k! A_k(x/y) z^{k+1/2}}{\Gamma(k+3/2) x^{k+1/2}} + \frac{\Gamma(k+1/2) z^k}{k! x^k} \right.$$

$$\left. \times F\left(\begin{matrix} -k+\frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| 1 - \frac{x}{y} \right) \right] + R_n(x, y, z), \tag{5}$$

where

$$A_k \left(\frac{x}{y}\right) = -\frac{x^k}{k! y^k \sqrt{\pi}} \Gamma\left(k + \frac{1}{2}\right) F\left(\begin{matrix} \frac{1}{2}, -k \\ \frac{1}{2} - k \end{matrix} \middle| \frac{y}{x} \right). \tag{6}$$

The remainder term $R_n(x, y, z)$ satisfies

$$|R_n(x, y, z)| \le \frac{(1/2)_n z^n}{2(n!)^2} \sum_{k=0}^{n} \frac{\Gamma(k+1/2)\Gamma(n-k+1/2)}{x^k y^{n-k+1/2}} F\left(\begin{matrix} \frac{1}{2}, n-k+\frac{1}{2} \\ n+1 \end{matrix} \middle| 1 - \frac{x}{y} \right). \tag{7}$$

Proof   After the change of variable $t \to zt$, the integral $2\sqrt{xy/z} R_F(x, y, z)$ has the form considered in Theorem 1 with $u = z/x$, $v = z/y$ and

$$h_v(t) = \frac{1}{\sqrt{(1+t)(1+vt)}} = \sum_{k=0}^{n-1} (-1)^{k-1} A_k(v) t^k + h_n(t, vt),$$

$$f(t) = \frac{1}{\sqrt{1+t}} = \sum_{k=0}^{n-1} \binom{-\frac{1}{2}}{k} t^{-k-1/2} + f_n(t), \tag{8}$$

with $A_k(v)$ given in equation (6). Then $a = 0$, $b = 1/2$ and $m = n$, and the asymptotic expansion of $2\sqrt{xy/z} R_F(x, y, z)$ for large x and y (small u and v) follows from equation (1) with

the coefficients $A_k(v/u)$ and B_k given by formula (8). On the other hand, the Mellin transform of f is a beta function, whereas the Mellin transform of h_v can be obtained from ref. [2, p. 303, equation (24)]. Introducing these Mellin transforms in equation (1), we obtain equation (5).

Integrating by parts in the integral defining the remainder term in equation (1), we obtain

$$2\sqrt{\frac{xy}{z}}|R_n(x,y,z)| \le \int_0^\infty |f_{n,n}(t)h_n^{(n)}(ut,vt)|\,dt,$$

where [7, chapter 6, equation (2.10)] $f_{n,n}(t) = ((-1)^n/(n-1)!)\int_t^\infty (u-t)^{n-1}f_n(u)\,du$. Using that f verifies the error test [5]: $|f_n(u)| \le \left|\binom{-1/2}{n}\right| u^{-n-1/2}$, the Leibniz rule for the derivation of a product of functions and [12, p. 303, equation (24)] we obtain equation (7). ∎

COROLLARY 2 *For* $0 \le x < y \le z$ *and* $n = 1, 2, 3, \ldots$,

$$R_D(x,y,z) = \frac{3}{2}\sqrt{\frac{\pi}{z^3}}\sum_{k=0}^{n-1}\left[\frac{k!A_k(y/z)}{\Gamma(k+3/2)}\frac{x^{k+1/2}}{y^{k+1/2}} + \frac{\Gamma(k+3/2)x^k}{k!y^k}\right]$$

$$\times F\left(\begin{matrix}-k+\frac{1}{2},\frac{3}{2}\\2\end{matrix}\middle| 1-\frac{y}{z}\right)\right] + R_n(x,y,z),\tag{9}$$

where

$$A_k\left(\frac{y}{z}\right) = -\frac{2y^k}{z^k k!\sqrt{\pi}}\Gamma\left(k+\frac{3}{2}\right)F\left(\begin{matrix}\frac{1}{2},-k\\-\frac{1}{2}-k\end{matrix}\middle| \frac{z}{y}\right).\tag{10}$$

For $n = 1, 2, 3, \ldots$, *the remainder term* $R_n(x,y,z)$ *satisfies*

$$|R_n(x,y,z)| \le \frac{3(1/2)_n x^n}{(n!)^2}\sum_{k=0}^n \frac{\Gamma(k+1/2)\Gamma(n-k+3/2)}{y^k z^{n-k+3/2}}F\left(\begin{matrix}\frac{1}{2},n-k+\frac{1}{2}\\n+1\end{matrix}\middle| 1-\frac{y}{z}\right).\tag{11}$$

Proof After the change of variable $t \to xt$, the integral $2\sqrt{yz^3/x}R_D(x,y,z)/3$ has the form considered in Theorem 1 with $u = x/y$, $v = x/z$, the function f given in equation (8) and $h_v(t) = 1/\sqrt{(1+t)(1+vt)^3}$. Following the same arguments as Corollary 1, we obtain equations (9) and (11). ∎

COROLLARY 3 *For* $0 < z < x \le y$ *and* $n = 1, 2, 3, \ldots$,

$$R_D(x,y,z) = -\frac{3}{2}\sqrt{\frac{\pi}{y}}\left[\sum_{k=0}^n \frac{2k!A_k(x/y)z^{k-1/2}}{\Gamma(k+1/2)x^{k+1/2}} + \sum_{k=0}^{n-1}\frac{(3/2)_k\sqrt{\pi}z^k}{k!x^{k+1}}\right]$$

$$\times F\left(\begin{matrix}-k-\frac{1}{2},\frac{1}{2}\\1\end{matrix}\middle| 1-\frac{x}{y}\right)\right] + R_n(x,y,z),\tag{12}$$

where, for $k = 0, 1, 2, \ldots,$ $A_k(x/y)$ *are given in equation* (6). *For* $n = 1, 2, 3, \ldots,$ *the remainder term* $R_n(x, y, z)$ *satisfies*

$$|R_n(x, y, z)| \le \frac{3(3/2)_n z^n}{2(n+1)!n!} \sum_{k=0}^{n+1} \frac{\Gamma(k+1/2)\Gamma(n-k+3/2)}{x^k y^{n-k+3/2}} F\left(\begin{array}{c} \frac{1}{2}, n-k+\frac{3}{2} \\ n+2 \end{array} \middle| 1 - \frac{x}{y}\right).$$

(13)

Proof After the change of variable $t \to zt$, the integral $2\sqrt{xyz}R_D(x, y, z)/3$ has the form considered in Theorem 1 with $u = z/x$, $v = z/y$, the function h_v given in equation (8) and $f(t) = 1/\sqrt{(1+t)^3}$. Following the same arguments from Corollary 1, we obtain equation (9). Integrating by parts in the integral defining the remainder term in equation (1), we obtain

$$\frac{3}{2\sqrt{xyz}}|R_n(x, y, z)| \le \int_0^\infty \left| f_{n,n+1}(t) h_{n+1}^{(n+1)}(ut, vt) \right| dt,$$

(14)

where ref. [7, chapter 6, equation (15)] $f_{n,n+1}(t) = ((-1)^{n+1}/n!) \int_t^\infty (u-t)^n f_n(u) \, du$. Using the error test for f and the Leibniz rule for the derivation of a product, we obtain equation (13). ∎

COROLLARY 4   *For* $0 \le x < y \le z,$ $0 < p$ *and* $n = 1, 2, 3, \ldots,$

$$R_J(x, y, z, p) = \frac{3}{2}\sqrt{\frac{\pi}{z}} \left[\sum_{k=0}^{n-1} B_k\left(\frac{p}{x}\right) \frac{\sqrt{\pi}x^k}{y^{k+1}} F\left(\begin{array}{c} -k - \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \middle| 1 - \frac{y}{z}\right) - \sum_{k=0}^n A_k\left(\frac{y}{z}\right) \right.$$

$$\left. \times \frac{2k!x^{k+1/2}}{p\Gamma(k+1/2)y^{k+1/2}} F\left(\begin{array}{c} k+1, 1 \\ \frac{3}{2} \end{array} \middle| 1 - \frac{x}{p}\right) \right] + R_n(x, y, z, p), \quad (15)$$

where $A_k(y/z)$ *are given in equation* (6) *and*

$$B_k\left(\frac{p}{x}\right) = -\frac{p^{k+1/2}}{x^k\sqrt{p-x}} + \frac{x}{p(k+1)!\sqrt{\pi}} \Gamma\left(k+\frac{3}{2}\right) F\left(\begin{array}{c} 1, \frac{3}{2}+k \\ 2+k \end{array} \middle| \frac{x}{p}\right).$$

(16)

For $n = 1, 2, 3, \ldots,$ *the remainder term* $R_n(x, y, z, p)$ *satisfies*

$$|R_n(x, y, z, p)| \le \frac{3|B_n(p/x)|x^n}{2(n+1)!} \sum_{k=0}^{n+1} \frac{\Gamma(k+1/2)\Gamma(n-k+3/2)}{y^k z^{n-k+3/2}}$$

$$\times F\left(\begin{array}{c} \frac{1}{2}, n-k+\frac{3}{2} \\ n+2 \end{array} \middle| 1 - \frac{y}{z}\right).$$

(17)

Proof After the change of variable $t \to xt$, the integral $2\sqrt{yz/x}R_J(x, y, z, p)/3$ has the form considered in Theorem 1 with $u = x/y$, $v = x/z$, the function h_v given in equation (8) and $f(t) = 1/((xt + p)\sqrt{1+t})$. The same arguments as before provide equation (15) and the error bound equation (17). ∎

COROLLARY 5 *For* $0 < x < p \le z,\ 0 \le y$ *and* $n = 1, 2, 3, \ldots,$

$$
R_J(x, y, z, p) = \frac{3}{2\sqrt{z}} \sum_{k=0}^{n-1} \frac{x^k}{p^{k+1}} \Bigg\{ A_k\left(\frac{p}{z}\right) B_k\left(\frac{y}{x}\right) \log\left(\frac{x}{p}\right) - \sqrt{\frac{x}{y}} A_k\left(\frac{p}{z}\right)
$$

$$
\times F'\left(\begin{matrix} k+1, \frac{1}{2} \\ 1 \end{matrix} \middle| 1 - \frac{x}{y}\right) - \frac{2B_k(y/x)\Gamma(k+3/2)}{\sqrt{\pi}k!} \Bigg[F'\left(\begin{matrix} -k, \frac{1}{2} \\ \frac{3}{2} \end{matrix} \middle| 1 - \frac{p}{z}\right)
$$

$$
+ \left(\psi(k+1) - \psi\left(k+\frac{3}{2}\right)\right) F\left(\begin{matrix} -k, \frac{1}{2} \\ \frac{3}{2} \end{matrix} \middle| 1 - \frac{p}{z}\right) \Bigg] \Bigg\} + R_n(x, y, z, p).
$$

$$(18)$$

B_k *are the coefficients* A_k *of equation* (6) *and*

$$
A_k\left(\frac{p}{z}\right) = \frac{p^k}{z^k k! \sqrt{\pi}} \Gamma\left(k + \frac{1}{2}\right) F\left(\begin{matrix} 1, -k \\ \frac{1}{2} - k \end{matrix} \middle| \frac{z}{p}\right).
$$

$F'(a, b, c, z)$ *represents the derivative of the Gauss hypergeometric function with respect to the variable a. For* $n = 1, 2, 3, \ldots,$ *the remainder term* $R_n(x, y, z, p)$ *satisfies*

$$
|R_n(x, y, z, p)| \le 3 \left(\left| A_{n-1}\left(\frac{p}{z}\right) B_n\left(\frac{y}{x}\right) \right| + \left| A_n\left(\frac{p}{z}\right) B_{n-1}\left(\frac{y}{x}\right) \right| \right.
$$

$$
\left. + \frac{1}{2} \left| A_n\left(\frac{p}{z}\right) B_n\left(\frac{y}{x}\right) \log\left(\frac{x}{p}\right) \right| \right) \frac{x^n}{\sqrt{z}p^{n+1}}. \tag{19}
$$

Proof After the change of variable $t \to xt$, the integral $2p\sqrt{z/x}R_J(x, y, z, p)/3$ has the form considered in Theorem 2 with $u = x/p,\ v = x/z,\ h_v(t) = 1/(t + 1)\sqrt{vt + 1}$ and $f(t) = 1/\sqrt{(1 + t)(xt + y)}$. After a straightforward computation, we obtain equation (18).

Consider now the integral expression of the remainder given in Theorem 2 for $m = n$. It can be written as:

$$
\int_0^\infty f_n(t)h_n(ut, vt)\,dt = \int_0^1 f_n(t)h_n(ut, vt)\,dt + \int_1^{1/u} f_n(t)h_n(ut, vt)\,dt
$$

$$
+ \int_{1/u}^\infty f_n(t)h_n(ut, vt)\,dt.
$$

Perform the change of variable $t \to t/u$ in the last integral, introduce the decomposition $f_n(t) = f_{n-1}(t) - B_{n-1}t^{-n}$ into the first integral in the right hand side and the decomposition $h_n(t, vt) = h_{n-1}(t, vt) - A_{n-1}(v)t^{n-1}$ into the last integral. Bound (19) follows using the error test in both functions: $|f_n(t)| \le |B_n|t^{-n-1}$ and $|h_n(t, vt)| \le |A_n(v)|t^n$. ∎

COROLLARY 6   *For $0 \le x < y \le p \le z$ and $n = 1, 2, 3, \ldots,$*

$$R_J(x, y, z, p) = \frac{3\sqrt{\pi}}{2p\sqrt{z}} \sum_{k=0}^{n-1} \frac{x^{k+1/2}}{y^{k+1/2}} \left[\frac{k! A_k(y/p, y/z)}{\Gamma(k+3/2)} \right.$$
$$\left. + \frac{(-1)^k \Gamma(k+1/2) p\sqrt{z}}{k! \pi \sqrt{xy}} B_k \left(\frac{y}{p}, \frac{y}{z} \right) \right] + R_n(x, y, z, p), \qquad (20)$$

where $A_k(y/p, y/z) = -\sum_{j=0}^{k} \frac{(1/2)_{k-j}(1/2)_j}{(k-j)!j!} (\frac{y}{z})^j F\left(\begin{array}{c} 1, -j \\ \frac{1}{2} - j \end{array} \middle| \frac{z}{p} \right)$,   and   the   coefficients $B_k(y/p, y/z)$ verify, for $k = 1, 2, 3, \ldots,$ the recurrence

$$B_k(y/p, y/z) = \frac{y}{p} \left[\frac{(-1)^k \pi \sqrt{y}}{\sqrt{z}} F \left(\begin{array}{c} -k + \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \middle| 1 - \frac{y}{z} \right) - B_{k-1} \left(\frac{y}{p}, \frac{y}{z} \right) \right], \qquad (21)$$

where $B_0(y/p, y/z) = (2/3) R_J(1, z/y, 0, p/y)$. For $n = 1, 2, 3, \ldots,$ the remainder term $R_n(x, y, z, p)$ satisfies

$$|R_n(x, y, z, p)| \le \frac{3(1/2)_n}{n!} \left(\frac{|A_n|}{n+1} \epsilon^{n+1} + 4|A_{n-1}| \epsilon^{-1/2} \right) \frac{x^n}{p\sqrt{z} y^n}, \qquad (22)$$

where $\epsilon = |2 A_{n-1}/A_n|^{1/(n+3/2)}$.

Proof   After the change of variable $t \to xt$, the integral $2p\sqrt{yz/x} R_J(x, y, z, p)/3$ has the form considered in Theorem 4 with $u = x/y$, $v = x/p$, $w = x/z$, the function f given in equation (8) and $h_{v,w}(t) = 1/((vt+1)\sqrt{(t+1)(wt+1)})$. It is straightforward to see that, for $k = 0, 1, 2, \ldots,$ the following recurrence holds: $M[h_{v,w}; -k - 1/2] = M[\sqrt{(t+1)^{-1}(wt+1)^{-1}}; -k - 1/2] - v M[h_{v,w}; 1/2 - k]$. Then, using formula [12, p. 303, equation (24)] and defining $B_k(y/p, y/z) = (y/p)\sqrt{y/z} M[h_{y/p,y/z}; 1/2 - k]$, we obtain the recurrence (21) and therefore expansion (20).

Consider now the integral expression of the remainder given in Theorem 4 for $m = n$. For $\epsilon > 0$, write

$$\int_0^\infty f_n(t) h_n(ut, vt, wt) \, dt = \frac{1}{u} \int_0^\epsilon f_n \left(\frac{t}{u} \right) h_n \left(t, \frac{v}{u} t, \frac{w}{u} t \right) dt$$
$$+ \frac{1}{u} \int_\epsilon^\infty f_n \left(\frac{t}{u} \right) h_n \left(t, \frac{v}{u} t, \frac{w}{u} t \right) dt.$$

Introduce the decomposition $h_n = h_{n-1} - A_{n-1} t^{n-1}$ into the last integral in the right side above. Bound (22) follows after using the bounds provided by the error test for the functions $h_{v,w}$ and f and taking the optimum ϵ.   ∎

COROLLARY 7   *For $0 < p < x \le y \le z$ and $n = 1, 2, 3, \ldots,$*

$$R_J(x, y, z, p) = \frac{3}{2\sqrt{yz}} \sum_{k=0}^{n-1} \frac{p^k}{x^{k+1/2}} \left[A_k \left(\frac{x}{y}, \frac{x}{z} \right) \log \left(\frac{p}{x} \right) + \frac{2\sqrt{yz}}{k! \sqrt{\pi} x} \Gamma \left(k + \frac{3}{2} \right) \right.$$
$$\times \left[\left(\psi(k+1) - \psi \left(k + \frac{3}{2} \right) \right) F_1 \left(k + \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1 - \frac{y}{x}, 1 - \frac{z}{x} \right) \right.$$
$$\left. \left. - F_1' \left(k + \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1 - \frac{y}{x}, 1 - \frac{z}{x} \right) \right] \right] + R_n(x, y, z, p), \qquad (23)$$

where $F_1(a, b, c, d; x, y)$ *represents an Appell's function and* $F_1'(a, b, c, d; x, y)$ *its derivative with respect to the variable a. For* $k = 0, 1, 2, \ldots$, *the coefficients* A_k *are given by*

$$A_k\left(\frac{x}{y}, \frac{x}{z}\right) = -\sum_{j=0}^{k} \frac{(1/2)_{k-j}\Gamma(j + 1/2)}{(k - j)!j!\sqrt{\pi}} \frac{x^j}{y^j} F\left(\begin{array}{cc} \frac{1}{2}, & -j \\ & \frac{1}{2} - j \end{array}\middle| \begin{array}{c} \frac{y}{z} \end{array}\right).$$

For $n = 1, 2, 3, \ldots$, *the remainder term* $R_n(x, y, z, p)$ *satisfies*

$$|R_n(x, y, z, p)| \le 3\left(|A_{n-1}| + |A_n|\left(1 + \frac{1}{2}\left|\log\left(\frac{x}{p}\right)\right|\right)\right)\frac{p^n}{\sqrt{yz}x^{n+1/2}}. \tag{24}$$

Proof After the change of variable $t \to pt$, the integral $2\sqrt{xyz}R_J(x, y, z, p)/3$ has the form considered in Theorem 4 with $u = p/x$, $v = p/y$, $w = p/z$, $h_{v,w}(t) = 1/\sqrt{(t + 1)(wt + 1)(vt + 1)}$ and $f(t) = 1/(1 + t)$. Expansion (20) follows from equation (4) and taking into account that the Mellin transform of $h_{v,w}$ is the first Appell hypergeometric function of two variables F_1. Bound (24) follows using the same argument as in Corollary 5 and the error test in both functions. ∎

4. Numerical examples

		First OrAp	ReEr	ReErBo	Second OrAp	ReEr	ReErBo
x	$R_F(x, x\log x, 1)$						
10	0.344184	0.333002	0.032	0.04	0.343572	0.0018	0.002
50	0.144162	0.14337	0.0055	0.006	0.144154	5.6E–5	5.7E–5
100	0.099046	0.0987863	0.0026	0.0028	0.0990447	1.316E–5	1.319E–5
y	$R_D(1, y, y^2)$						
10	0.00411923	0.00396237	0.038	0.05	0.00411154	0.0019	0.002
50	5.37946E–5	5.35581E–5	0.004	0.005	5.37923E–5	4.2E–5	4.6E–5
100	7.83482E–6	7.82013E–6	0.0019	0.002	7.83475E–6	9.1E–6	9.7E–6
x	$R_D(x, 2x\log x, 1)$						
10	0.100142	0.104209	0.04	0.049	0.100539	0.004	0.0046
50	0.0184569	0.0185161	0.0032	0.0034	0.018458	6.3E–5	6.7E–5
100	0.00888594	0.00889579	0.0011	0.0012	0.00888604	1.09E–5	1.13E–5
y	$R_J(1, y, y^2, 2)$						
10	0.05478	0.0509229	0.07	0.1	0.051573	0.013	0.017
50	0.00561821	0.00565251	0.006	0.007	0.00561938	0.0002	0.00025
100	0.00208589	0.00209034	0.002	0.0024	0.00208596	3.6E–5	4E–5
p	$R_J(1, 2, p^3, p)$						
10	0.0105996	0.0090172	0.15	0.4	0.0103535	0.02	0.06
50	0.000309006	0.000299864	0.03	0.06	0.00030872	0.001	0.002
100	6.43773E–5	6.34236E–5	0.01	0.02	6.43622E–5	0.0002	0.0004
y	$R_J(1, y, y^2, 2y)$						
10	0.0134692	0.0124728	0.07	0.1	0.0133997	0.005	0.006
50	0.00065188	0.000643092	0.01	0.02	0.00065176	0.00018	0.0002
100	0.000170126	0.000168996	0.007	0.01	0.000170118	4.5E–5	4.9E–5
x	$R_J(x, x\log x, x^2, 1)$						
10	0.0266916	0.0245712	0.08	0.1	0.02653	0.006	0.01
50	0.0013541	0.00133727	0.012	0.018	0.00135386	2E–4	3E–4
100	0.000364351	0.000362194	0.005	0.008	0.000364336	4E–5	7E–4

The second, third and sixth columns represent the EI function, approximation of Corollaries 1–7 for $n = 1$ (first OrAp) and $n = 2$ (second OrAp), respectively ($n = 2$ and

$n = 3$ in the last example). The fourth and seventh columns represent the respective relative errors (ReEr) $|R_n/R_F|$. The fifth and last columns represent the respective relative error bounds (ReErBo) given by the corresponding equation in Corollaries 1–7.

Acknowledgements

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