Asymptotic Expansions of the Double Zeta Function

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ABSTRACT

The double Zeta function of Barnes, $\zeta_2(v,z,w)$, is considered for large and small values of z and w with $w>0$, $|\text{Arg}(z)|<\pi$ and $v\neq 1,\,2$. Two integral representations are obtained for $\zeta_2(v,z,w)$. These integrals define the analytical continuation of the double Zeta function, primarily defined for $\Re(v)>2$ and $\Re(z)>0$, to the whole complex $z-$plane and complex $v-$plane with $|\text{Arg}(z)|<\pi$ and $v\neq 1,2$. Six asymptotic expansions for large and small w or z are derived from these integrals. The expansions are all accompanied by error bounds at any order of the approximation. Numerical experiments show that these bounds are very accurate for real values of the variables.

1991 Mathematics Subject Classification: 41A60 33E05

Keywords & Phrases: Barnes double Zeta function, asymptotic expansions.

1. Introduction

The double Zeta function $\zeta_2(v,z,w)$ is defined by the double infinite series [4]:

$$\zeta_2(v,z,w)\equiv\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(z+m+nw)^{-v},\qquad \text{Re}(z)>0,\ \ \text{Re}(w)>0,\ \ \text{Re}(v)>2. \quad (1)$$

The double Gamma function of Barnes $\Gamma_2(z,w)$ introduced in [3] is closely related to the double Zeta function,

$$\log\Gamma_2(z,w)\equiv\left.\frac{d}{dv}\zeta_2(v,z,w)\right|_{v=0}-\lim_{z\to 0}\left[\left.\frac{d}{dv}\zeta_2(v,z,w)\right|_{v=0}+\log z\right].$$

An important particular case of the function $\Gamma_2(z,w)$ is the function $G(z) = (2\pi)^{z/2}\Gamma_2^{-1}(z,1)$ (also called the double Gamma function) introduced in [2] as the infinite product

$$G(z+1) = (2\pi)^{z/2} e^{-[(1+\gamma)z^2+z]/2} \prod_{k=1}^{\infty} \left[\left(1+\frac{z}{k}\right)^k e^{-z+z^2/2k} \right],$$

where γ is the Euler-Mascheroni constant. It satisfies the recursion relation $G(z+1) = \Gamma(z)G(z)$ and, for integer positive z, it verifies $G(1) = G(2) = 1$ and $G(m) = 1!2!...(m-3)!(m-2)!$ for $m \geq 3$.

Barnes introduced these functions, studied many properties and applied them to the theory of elliptic and theta functions [2]-[4]. Some other mathematical applications of the double gamma functions may be found in [8], [9], [17] and [18]. The double gamma functions are used in [17] to prove the classical Kronecker limit formula. On the other hand, the theory of the double gamma function is used in [8], [9] and [18] to evaluate some series involving the Riemann zeta function.

A second field of applications is the study of determinants of Laplacians. In fact, multiple gamma functions evaluated at 1/2 may be expressed in terms of the functional determinant of Laplacians of the n-sphere, which have been a recent subject of research due to their relevance to superstring theory [20]. Toeplitz determinants with special rational generating functions may be evaluated in terms of the double gamma function and the Gauss hypergeometric function [5]. Several properties of the Gamma and double gamma functions may be deduced from the application of the zeta regularization to the determinants of certain operators [21]. The double gamma function plays a key role in the derivation of the determinant of the Laplacian on spinor fields on a Riemann surface in terms of the value of Selberg zeta function at the middle of the critical strip [16]. Some other applications of the double gamma function to the study of determinants of Laplacians may be found in [7],[14] and [15]. For a deep understanding of the important role that these functions, in particular the double zeta function, play on zeta-function regularization methods see [10].

Asymptotic expansions of $\zeta_2(v,z,w)$, $\Gamma_2(z,w)$ and $G(z)$ have been investigated, among other authors, by Matsumoto, Billingham & King and Ferreira & Lopez. Complete asymptotic expansions of $\zeta_2(v,z,w)$ and $\Gamma_2(z,w)$ in decreasing powers of w are obtained in [12] and [13] from an integral representation of a generalization of these functions. These expansions are valid for z, $w>0$, $3-v\notin\mathbb{N}$ and $\Re(v) > -N$ for some fixed $N\in\mathbb{N}$. The first terms of the asymptotic expansions of $\log\Gamma_2(z,w)$ for large or small z or w have been obtained in [6] by using the method of matched asymptotic expansions to solve the difference equation satisfied by this function. The first terms of uniform asymptotic expansions are also obtained there. A complete asymptotic expansion of $\log G(z)$ with error bounds has been recently obtained in [11] from an integral representation of $\log G(z)$. Several complete convergent expansions of $\log G(z)$ in powers of z are given in [8]. These expansions of $\log G(z)$ for large and small z are valid for $|\mathrm{Arg}(z)| < \pi$ and $|z| < 1$ respectively.

On the other hand, complete asymptotic expansions of $\zeta_2(v,z,w)$ for large and small z and small w are not known. The purpose of this paper is to obtain complete asymptotic expansions of $\zeta_2(v,z,w)$ in these limits with error bounds. For completeness, we also investigate asymptotic expansions of $\zeta_2(v,z,w)$ for large w. The expansions obtained here for large w are alternative to Matsumoto's expansions and are valid in a larger domain of z and v.

In section 2, we obtain two integral representations of $\zeta_2(v,z,w)$ from which, in section 3, we derive complete asymptotic expansions of this function in the limits mentioned above. We use the error test and Cauchy's integral formula to obtain error bounds at any order of the approximations. Numerical examples are shown as an illustration in section 4. A brief summary and a few comments are given in section 5.

2. Analytic continuation of the double zeta function

The starting point to derive asymptotic expansions of $\zeta_2(v,z,w)$ is a suitable integral representation. It may be obtained by substituting

$$\frac{1}{(z+m+nw)^v}=\frac{1}{\Gamma(v)}\int_0^\infty t^{v-1}e^{-(z+m+nw)t}dt$$

in definition (1) and interchanging sums and integral,

$$\zeta_2(v,z,w)=\frac{1}{\Gamma(v)}\int_0^\infty \frac{t^{v-1}e^{-zt}}{(1-e^{-t})(1-e^{-wt})}dt.$$

This integral defines an analytic function of either v, z or w for $\Re(v)>2$, $\Re(w)>0$ and $\Re(z)>0$ [[19], p. 30, theorem 2.3].

We can continue analytically $\zeta_2(v,z,w)$ in both, v or z, to larger regions in the complex plane. For this purpose we consider only positive values of w. In order to continue $\zeta_2(v,z,w)$ to a larger domain in the $z-$plane we consider an angle φ verifying $|\varphi|<\pi/2$ and $|\text{Arg}(z)+\varphi|<\pi/2$. Then, using the Cauchy residue theorem we obtain that, for fixed v with $\Re(v)>2$ and fixed $w>0$, the analytic continuation of $\zeta_2(v,z,w)$ in the z variable to $\{z\in\mathbb{C},\ |\text{Arg}(z)|<\pi\}$ is given by

$$\zeta_2(v,z,w)=\frac{1}{\Gamma(v)}\int_0^{\infty e^{i\varphi}} \frac{t^{v-1}e^{-zt}}{(1-e^{-t})(1-e^{-wt})}dt. \tag{2}$$

¿From this integral, a straightforward computation shows that, for fixed $z\in\mathbb{C}\setminus\mathbb{R}^-$, the analytical continuation of $\zeta_2(v,z,w)$ in the complex v plane to the region $v\in\{\mathbb{C},\ v\notin\mathbb{N}\}$ is given by

$$\zeta_2(v,z,w)=\frac{i\Gamma(1-v)}{2\pi}\int_{\mathcal{L}_\varphi} \frac{(-t)^{v-1}e^{-zt}}{(1-e^{-t})(1-e^{-wt})}dt, \tag{3}$$

where the contour $\mathcal{L}_\varphi$ is the Hankel's contour shown in fig. 1. This last formula may be proved by shrinking the contour $\mathcal{L}_\varphi$ around the straight $[0, \infty e^{i\varphi})$. Therefore, we have the following proposition.

Proposition 1. *For fixed $w > 0$, the analytical continuation of the double zeta function $\zeta_2(v, z, w)$ defined in (1) to $\{z \in \mathbb{C}, |\text{Arg}(z)| < \pi\}$ and $\{v \in \mathbb{C}, v \neq 1, 2\}$, is given by*

$$\zeta_2(v, z, w) = S(v) \int_{\mathcal{C}_\varphi} \frac{t^{v-1} e^{-zt}}{(1 - e^{-t})(1 - e^{-wt})} dt, \tag{4}$$

where the integration path $\mathcal{C}_\varphi$ is the straight line $[0, \infty e^{i\varphi})$ if $\Re(v) > 2$ or the Hankel's contour $\mathcal{L}_\varphi$ given in fig. 1 if $v \notin \mathbb{N}$. The parameter φ is an angle verifying $|\varphi| < \frac{\pi}{2}$ and $|\text{Arg}(z) + \varphi| < \frac{\pi}{2}$, and

$$S(v) \equiv \begin{cases} \Gamma(v)^{-1} & \text{if} \quad \text{Re}(v) > 2 \\ i(2\pi)^{-1} e^{i\pi(1-v)} \Gamma(1-v) & \text{if} \quad v \notin \mathbb{N}. \end{cases}$$

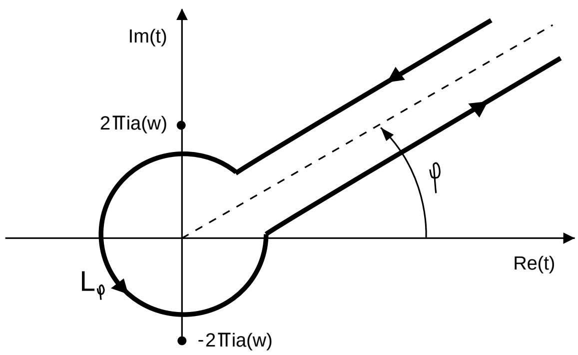

Figure 1. *Hankel's contour involved in the integral representation (3) of $\zeta_2(v, z, w)$. It surrounds the half-line $[0, \infty e^{i\varphi})$ in the counterclockwise direction and does not enclose any pole $\pm 2n\pi i$, $\pm 2n\pi i/w$, $n \in \mathbb{N}$ of the integrand in (3). Then, in this picture, $a(w) \equiv$Min$\{1, w^{-1}\}$.*

3. Asymptotic expansions of the double zeta function

The integral representations given above are the starting point to derive asymptotic expansions of $\zeta_2(v, z, w)$ for large or small z or w. These expansions are given in theorems 1-6. Empty sums must be understood as zero in the remaining of the paper.

Theorem 1. *For $w > 0$, $|\text{Arg}(z + w - 1)| < \pi$ and $v \neq 1, 2$, an asymptotic expansion of $\zeta_2(v, z, w)$ for large w and $N = 0, 1, 2, 4, 6, ...$ is given by*

$$\zeta_2(v, z, w) = \zeta(v, z) + \sum_{n=0}^{N-1} \frac{B_n}{n!} \frac{\zeta_n(v, 1 + (z-1)/w)}{w^{n+v-1}} + R_N(v, z, w), \tag{5}$$

where B_n *are the Bernoulli numbers,* $\zeta(s,a)$ *is the Hurwitz zeta function and*

$$\zeta_n(v,z) \equiv \begin{cases} \frac{(v-1)_n}{v-1}\zeta\left(n+v-1,z\right) & \text{if} \qquad n \neq 2-v \\ (-1)^n(n-2)! & \text{if} \quad n=2-v, \quad n=2,3,4,.... \end{cases} \tag{6}$$

In this formula $(v)_n$ *denotes the Pochhamer simbol of* v*. The error term verifies* $R_N(v,z,w) = \mathcal{O}(w^{-N-\Re(v)+1})$ *as* $w \to 0$*. More precisely, for* $N = 0,1,2,4,6,...$*,* $\Re(v) > 2-N$ *and* $\Re(z+w-1) > 0$*, an error bound for the remainder is given by*

$$\begin{aligned} |R_N(v,z,w)| \leq \Bigg\{ \frac{|B_N|}{N!} \Bigg[& \frac{w+\Re z-1}{w}\gamma\left(N+\Re v-2, 2\pi(w+\Re z-1)\right) + \\ & \gamma\left(N+\Re v-1, 2\pi(w+\Re z-1)\right) \Bigg] + \\ & \frac{\Gamma\left(N+\Re v-1, 2\pi(w+\Re z-1)\right)}{2\pi^{N-1}(1-e^{-2\pi w})} \Bigg\} \frac{1}{|\Gamma(v)|(w+\Re z-1)^{N+\Re v-1}}, \end{aligned} \tag{7}$$

where $\gamma(a,x)$ *and* $\Gamma(a,x)$ *are the incomplete gamma functions [[1], eqs. 6.5.2 and 6.5.3]. For* $N = 0,1,2,4,6,...$*,* $\Re(v) > 2-N$ *and* $|Arg(z+w-1)| < \pi$*, an error bound for the remainder is given by*

$$|R_N(v,z,w)| \leq \frac{\Gamma(N+\Re v-2)\left[|1+\frac{z-1}{w}|+N+\Re v-2\right]\cot\left(\frac{\pi}{4}-\frac{|\varphi|}{2}\right)}{2e^{\varphi\Im v}|\Gamma(v)|\pi^{N-1}(\cos\varphi)^N(|w+z-1|\cos\varphi)^{N+\Re v-1}}, \tag{8}$$

with $\varphi \equiv -\frac{1}{2}Arg(z+w-1)$*.*

Proof. We introduce the decomposition

$$\frac{1}{1-e^{-wt}} = 1 + \frac{e^{-wt}}{1-e^{-wt}} \tag{9}$$

in either of the integral representations given in proposition 1. Then, using [[19], p. 75, ex. 3.16] we have

$$\zeta_2(v,z,w) = \zeta(v,z) + S(v)\int_{\mathcal{C}_\varphi} \frac{t^{v-1}e^{-(z+w-1)t}}{(e^t-1)(1-e^{-wt})}dt. \tag{10}$$

We introduce now the expansion [[1], eq. 23.1.1]

$$\frac{t}{e^t-1} = \sum_{k=0}^{N-1}\frac{B_k}{k!}t^k + r_N(t), \qquad |t| < 2\pi, \qquad N = 0,1,2,4,6,..., \tag{11}$$

where $r_N(t) = \mathcal{O}(t^N)$ as $t \to 0$, into the integrand of equation (10). Interchanging sum and integral and using again [[19], p. 75, ex. 3.16] we obtain (5) with

$$R_N(v,z,w) = S(v)\int_{\mathcal{C}_\varphi} \frac{t^{v-2}e^{-(z+w-1)t}}{1-e^{-wt}}r_N(t)dt. \tag{12}$$

In [[11], sec. 3.1] it is shown that $\mathrm{sign}(r_{2N}(t)) = -(-1)^N$ for $N = 2, 3, 4, ...$ and $0 \leq t < 2\pi$. It is easy to show that also, $r_0(t) > 0$, $r_1(t) < 0$ and $r_2(t) > 0$. Therefore, two consecutive error terms $r_N(t)$ (with $N = 0, 1, 2, 4, 6, ...$) in the expansion (11) have opposite sign. Applying the error test (see [[22], p. 38]) we find,

$$|r_N(t)| \leq \frac{|B_N|}{N!} t^N, \qquad 0 \leq t < 2\pi, \qquad N = 0, 1, 2, 4, 6, ... \tag{13}$$

On the other hand, for any $t \in \mathcal{C}$ we consider the explicit expression of $r_N(t)$ given by the Lagrange form for the remainder of the Taylor expansion (11):

$$r_N(t) = \frac{1}{N!} \frac{d^N}{dx^N} \left(\frac{x}{e^x - 1} \right) \Big|_{x=\xi} t^N, \qquad N = 0, 1, 2, 4, 6, ...,$$

where $\xi \in (0, t)$. By the Cauchy's integral theorem,

$$r_N(t) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{u du}{(u - \xi)^{N+1}(e^u - 1)} t^N, \qquad N = 0, 1, 2, 4, 6, ..., \tag{14}$$

where $\mathcal{C}$ is a circle with center at the point ξ that does not enclose any singularity of $(e^u - 1)^{-1}$. In order to find bounds of $R_N(v, z, w)$, we require bounds of $r_N(t)$ valid for fixed $\mathrm{Arg}(t) = \varphi$ with $|\varphi| < \pi/2$ and $0 \leq |t| < \infty$. Therefore, we make the change of variable $u = \xi + \pi \cos\varphi e^{i\theta}$, (observe that $\pi \cos\varphi <$ the distance of the $t-$axis to the first singularities $\pm 2\pi i$ of $(e^u - 1)^{-1}$, see figure 1), obtaining

$$|r_N(t)| \leq C(\varphi) \frac{t^N}{(\pi \cos\varphi)^N}, \qquad N = 0, 1, 2, 4, 6, ..., \tag{15}$$

where $C(\varphi)$ is a bound of $|u/(e^u - 1)|$ in the shaded region depicted in figure 1. The maximum of the function $|u/(e^u - 1)|$ in that region is located on the contour of the region. When $|\varphi| < \pi/2$, a simple bound may be chosen:

$$C(\varphi) = \frac{\pi}{2} \cot\left(\frac{\pi}{4} - \frac{|\varphi|}{2} \right). \tag{16}$$

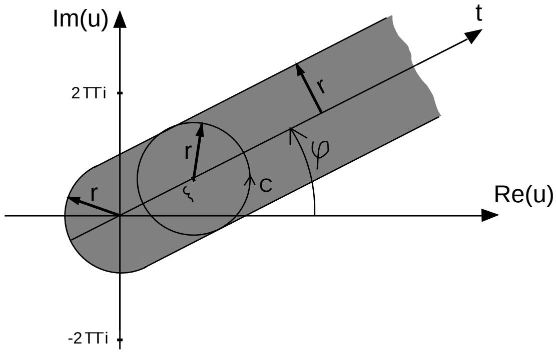

Figure 2. *The circle of radius* $r \equiv \pi\cos\varphi$ *centered at* ξ, *with* $\xi \in (0,t)$, *used in the Cauchy definition of* $r_N(t)$ *must be contained in the shaded region. This region is defined by the set* $\{u \in \mathcal{C}, \, |u-t| < \pi\cos\varphi, \, 0 \le |t| < \infty, \, Arg(t) = \varphi\}$.

When $|\text{Arg}(z+w-1)| < \pi/2$ and $\Re(v) > 2-N$ we can take $\varphi = 0$ in the integral (12) with $\mathcal{C}_0 = [0,\infty)$ (and $S(v) = \Gamma(v)^{-1}$). Then we introduce the bound (13) in that integral for $t \in [0, 2\pi)$ and the bound (15) for $t \in [2\pi, \infty)$. Taking $C(0) = \pi/2$, the bound $wt(1-e^{-wt})^{-1} \le 1 + wt$ in the first integral and $(1-e^{-wt})^{-1} \le (1-e^{-2\pi w})^{-1}$ in the second one, after trivial manipulations we obtain (7). Nevertheless, for $|\text{Arg}(z+w-1)| < \pi$ and $\Re(v) > 2-N$ we can take $\varphi = -\frac{1}{2}\text{Arg}(z+w-1)$ in the integral (12) with $\mathcal{C} = [0, \infty e^{i\varphi})$. Then introducing the bound (15) in (12) and using $|wt(1-e^{-wt})^{-1}| \le (\cos\varphi)^{-1} + |wt|$ we obtain (8).

Using again formula (14) with $\xi \in (0,t)$, $\text{Arg}(t) = \varphi$, it is easy to show that

$$|r_N(t)| \le \bar{C}_N(\varphi)|t|^N, \qquad t \in \mathcal{L}_\varphi, \qquad N = 0,1,2,4,6,..., \tag{17}$$

where $\bar{C}_N(\varphi)$ is independent of $|t|$. Introducing this bound in (12) with $\mathcal{C}_\varphi = \mathcal{L}_\varphi$ and $\varphi = -\frac{1}{2}\text{Arg}(z+w-1)$ we obtain that, for $v \notin \mathbb{N}$,

$$|R_N(v,z,w)| \le \frac{M_N(v,w,\varphi,z)}{|w+z-1|^{N+\Re v-1}},$$

where $M_N(v,w,\varphi,z)$ is bounded for large w. Therefore, (5) is an asymptotic expansion also for $\Re(v) \le 2-N$ with $v \ne 1, 2$. □

Theorem 2. *For* $w > 0$, $|\text{Arg}(z)| < \pi$, $v \ne 1,2$ *and* $N = 2,4,6,...$, *an asymptotic expansion of* $\zeta_2(v,z,w)$ *for small* w *is given by*

$$\zeta_2(v,z,w) = \sum_{n=0}^{N-1} \frac{(-1)^n B_n}{n!}\zeta_n(v,z)w^{n-1} + R_N(v,z,w), \tag{18}$$

where $\zeta_n(v,z)$ *are given in (6) and* $R_N(v,z,w) = \mathcal{O}(w^{N-1})$ *as* $w \to 0$. *More precisely, for* $N = 2,4,6,...$, $\Re(v) > 2-N$ *and* $\Re(z) > 0$, *an error bound for the remainder is given by*

$$\begin{aligned}|R_N(v,z,w)| \le \Bigg\{ \frac{|B_N|}{N!}\Bigg[\gamma\left(N+\Re v-2, \frac{2\pi\Re(z)}{w}\right)\Re(z) + \\ \gamma\left(N+\Re v-1, \frac{2\pi\Re(z)}{w}\right)\Bigg] + \frac{\Gamma\left(N+\Re v-1, \frac{2\pi\Re(z)}{w}\right)}{2\pi^{N-1}(1-e^{2\pi/w})}\Bigg\} \frac{w^{N-1}}{|\Gamma(v)|(\Re z)^{N+\Re v-1}}.\end{aligned} \tag{19}$$

For $N = 2,4,6,...$, $\Re(v) > 2-N$ *and* $|Arg(z)| < \pi$, *an error bound for the remainder is given by*

$$|R_N(v,z,w)| \le \frac{(|z|+N+\Re v-2)\Gamma(N+\Re v-2)w^{N-1}}{2e^{\varphi\Im v}|\Gamma(v)|\pi^{N-1}(\cos\varphi)^N(|z|\cos\varphi)^{N+\Re v-1}}\cot\left(\frac{\pi}{4}-\frac{|\varphi|}{2}\right), \tag{20}$$

with $\varphi \equiv -\frac{1}{2}Arg(z)$.

Proof. We introduce the expansion [[1], eq. 23.1.1]

$$\frac{wt}{1-e^{-wt}} = \sum_{k=0}^{N-1} \frac{(-1)^k B_k}{k!}(wt)^k + r_N(wt), \qquad |wt| < 2\pi, \qquad N = 0,1,2,4,6,..., \tag{21}$$

where $r_N(t) = \mathcal{O}(t^N)$ as $t \to 0$, in either of the integral representation given in proposition 1. Interchanging sum and integral and using [[19], p. 75, ex. 3.16] we obtain (18) with

$$R_N(v,z,w) = \frac{S(v)}{w}\int_{\mathcal{C}_\varphi} \frac{t^{v-2}e^{-zt}}{1-e^{-t}} r_N(wt)dt. \tag{22}$$

The terms of the expansion (11), with t replaced by wt, coincide with the terms of the expansion (21) from $n = 2$ because $B_{2n+1} = 0$ for $n = 1, 2, 3, ...$ Therefore, the remainder $r_N(wt)$ in (21) satisfies the bounds (13) and (15) for $N = 2, 4, 6, ...$ On the other hand, the bound (17) holds for every N. When $|\text{Arg}(z)| < \pi/2$ and $\Re(v) > 2-N$, we can take $\varphi = 0$ in the integral (22) with $\mathcal{C} = [0,\infty)$. Then we introduce the bound (13) of $r_N(wt)$ in that integral for $wt \in [0, 2\pi)$ and the bound (15) for $wt \in [2\pi,\infty)$. Using the bound $t(1-e^{-t})^{-1} \le 1+t$ in the first integral and $(1-e^{-t})^{-1} \le (1-e^{-2\pi/w})^{-1}$ in the second one and after trivial manipulations we obtain (19). For $|\text{Arg}(z)| < \pi$ and $\Re(v) > 2-N$ we can take $\varphi = -\frac{1}{2}\text{Arg}(z)$ in the integral (22) with $\mathcal{C} = [0,\infty e^{i\varphi})$. Using the bound $|t(1-e^{-t})^{-1}| \le (\cos\varphi)^{-1} + |t|$ for $t \in [0,\infty e^{i\varphi})$ and after trivial manipulations we obtain (20). Introducing the bound (17) in (22) with t replaced by wt for $\mathcal{C}_\varphi = \mathcal{L}_\varphi$ with $\varphi = -\frac{1}{2}\text{Arg}(z)$ we obtain that, for $v \notin \mathbb{N}$,

$$|R_N(v,z,w)| \le M_N(\varphi,z,v)w^{N-1},$$

where $M_N(\varphi,z,v)$ is independent of w. Therefore, (18) is an asymptotic expansion for small w also when $\Re(v) \le 2-N$ with $v \ne 1,\ 2$. □

The bounds given in this theorem, although accurate when z is close to the real axis, do not hold for $N = 0, 1$. The error bounds given in the following theorem are valid for every $N \in \mathbb{N}$.

Theorem 3. *For* $w > 0$, $|\text{Arg}(z-w)| < \pi$, $v \ne 1, 2$ *and* $N = 0,1,2,4,6,...$, *an asymptotic expansion of* $\zeta_2(v,z,w)$ *for small* w *is given by*

$$\zeta_2(v,z,w) = \sum_{n=0}^{N-1} \frac{B_n}{n!}\zeta_n(v,z-w)w^{n-1} + R_N(v,z,w), \tag{23}$$

where $\zeta_n(v,z)$ *are given in (6) and* $R_N(v,z,w) = \mathcal{O}(w^{N-1})$ *as* $w \to 0$. *For* $N = 0,1,2,4,6,...$, $\Re(v) > 2-N$ *and* $\Re(z-w) > 0$, *an error bound for the remainder is given by*

$$\begin{aligned} |R_N(v,z,w)| \le \Bigg\{ & \frac{|B_N|}{N!}\left[\gamma\left(N+\Re v-2, 2\pi\Re(z/w-1)\right)\Re(z-w) + \right. \\ & \left.\gamma\left(N+\Re v-1, 2\pi\Re(z/w-1)\right)\right] + \\ & \frac{\Gamma\left(N+\Re v-1, 2\pi\Re(z/w-1)\right)}{2(1-e^{-2\pi/w})\pi^{N-1}}\Bigg\} \frac{w^{N-1}}{|\Gamma(v)|(\Re(z-w))^{N+\Re v-1}}. \end{aligned} \tag{24}$$

For $N = 0, 1, 2, 4, 6, ...,$ $\Re(v) > 2 - N$ and $|Arg(z - w)| < \pi$, an error bound for the remainder is given by

$$|R_N(v,z,w)| \leq \frac{(|z-w| + N + \Re v - 2)\Gamma(N + \Re v - 2) w^{N-1}}{2e^{\varphi \Im v}|\Gamma(v)|\pi^{N-1}(\cos\varphi)^N(|z-w|\cos\varphi)^{N+\Re v - 1}} \cot\left(\frac{\pi}{4} - \frac{|\varphi|}{2}\right), \quad (25)$$

where $\varphi \equiv -\frac{1}{2}Arg(z - w)$.

Proof. We write

$$\zeta_2(v,z,w) = \frac{S(v)}{w}\int_{\mathcal{C}_\varphi} \frac{t^{v-2}e^{-(z-w)t}}{1-e^{-t}} \frac{wt}{e^{wt}-1} dt.$$

and introduce in this integral the expansion [[1], eq. 23.1.1]

$$\frac{wt}{e^{wt}-1} = \sum_{k=0}^{N-1} \frac{B_k}{k!}(wt)^k + r_N(wt), \qquad N = 0, 1, 2, 4, 6, ...,$$

where $r_N(t) = \mathcal{O}(t^N)$ as $t \to 0$. The remaining proof is similar to the proof of theorem 2, but now the bounds (13) and (15) hold from $N = 0$. □

Theorem 4. *For $w > 0$, $|\mathrm{Arg}(z)| < \pi$, $v \neq 1, 2$ and $N = 0, 1, 2, ...,$ an asymptotic expansion of $\zeta_2(v,z,w)$ for large z is given by*

$$\zeta_2(v,z,w) = \sum_{n=0}^{N-1} \frac{\tilde{B}_n(v,w)}{z^{n+v-2}} + R_N(v,z,w), \quad (26)$$

where $R_N(v,z,w) = \mathcal{O}(z^{-N-\Re(v)+2})$ as $z \to \infty$ and

$$\tilde{B}_n(v,w) \equiv \frac{(-1)^n (v-2)_n}{w(v-1)(v-2)} \sum_{k=0}^{n} \frac{w^{n-k}B_k B_{n-k}}{k!(n-k)!}. \quad (27)$$

For $N = 0, 1, 2, ...,$ $|Arg(z)| < \pi$ and $\Re(v) > 2 - N$, an error bound for the remainder is given by

$$\begin{aligned}|R_N(v,z,w)| \leq & \frac{e^{-\varphi \Im v}\Gamma(N + \Re v - 2)}{w(\pi a(w)\cos\varphi)^N|\Gamma(v)|(|z|\cos\varphi)^{N+\Re v - 2}} \left\{\pi^2\left(a(w)\cos\varphi + \right.\right.\\ & \left.\left.\frac{1}{2}\cot\left(\frac{\pi}{4} - \frac{|\varphi|}{2}\right)\right)^2 + \frac{N + \Re v - 2}{|z|\cos\varphi}\left[w\frac{N + \Re v - 1}{|z|\cos\varphi} + \right.\right.\\ & \left.\left.\pi(1+w)\left(\frac{1}{2}\cot\left(\frac{\pi}{4} - \frac{|\varphi|}{2}\right) + a(w)\cos\varphi\right)\right]\right\}, \end{aligned} \quad (28)$$

where $\varphi \equiv -\frac{1}{2}Arg(z)$ and $a(w) \equiv Min\{1, w^{-1}\}$ was introduced in fig. 1.

Proof. We introduce in the right hand side of (4) the expansion

$$\frac{t}{1-e^{-t}}\frac{wt}{1-e^{-wt}}=\sum_{k=0}^{N-1}b_k(v,w)t^k+r_N(w,t), \qquad N=0,1,2,3,..., \tag{29}$$

where $r_N(w,t)=\mathcal{O}(t^N)$ as $t\to 0$ and

$$b_k(v,w)\equiv(-1)^k\sum_{j=0}^{k}\frac{w^{k-j}B_jB_{k-j}}{j!(k-j)!}.$$

Interchanging sum and integral we obtain (26) with

$$R_N(v,z,w)=\frac{S(v)}{w}\int_{\mathcal{C}_\varphi}t^{v-3}e^{-zt}r_N(w,t)dt. \tag{30}$$

We consider now the explicit expression for $r_N(w,t)$ given by the Lagrange form for the remainder of the Taylor expansion and the Cauchy's integral theorem,

$$r_N(w,t)=\frac{w}{2\pi i}\int_{\mathcal{C}}\frac{u^2du}{(u-\xi)^{N+1}(1-e^{-u})(1-e^{-wu})}t^N, \qquad N=0,1,2,3,..., \tag{31}$$

where $\mathcal{C}$ is a circle with center at the point $\xi\in(0,t)$ that does not enclose singularities of $(1-e^{-u})^{-1}(1-e^{-wu})^{-1}$. In order to find bounds of $R_N(v,z,w)$ we require bounds of $r_N(w,t)$ valid for fixed $\text{Arg}(t)=\varphi$ with $|\varphi|<\pi/2$ and $0\le|t|<\infty$. Therefore, we proceed here as in the proof of theorem 1, but now we introduce in (31) the bound $|u(1-e^{-u})^{-1}|\le C(\varphi)+|u|$, where $C(\varphi)$ is given in (16) and u belongs to the shaded region of fig. 2. Then, it is easy to show that, for $N=0,1,2,4,6,...$,

$$|r_N(w,t)|\le[C(\varphi)+r+|t|]\,[C(\varphi)+r+w|t|]\,\frac{|t|^N}{r^N}, \qquad t\in[0,\infty e^{i\varphi}). \tag{32}$$

where $r\equiv\pi a(w)\cos\varphi$. Introducing this bound in (30) with $\mathcal{C}=[0,\infty e^{i\varphi})$ and $\Re(v)>2-N$ and after trivial manipulations we obtain (28). Using a similar argument we have that $r_N(w,t)$ satisfies a bound similar to (32) for any $t\in\mathcal{L}_\varphi$,

$$|r_N(w,t)|\le\bar{C}_N(w,\varphi)|t|^N(1+|t|^2), \qquad t\in\mathcal{L}_\varphi,$$

where $\bar{C}_N(w,\varphi)$ is independent of $|t|$. Then, introducing this bound in (30) with $\mathcal{C}_\varphi=\mathcal{L}_\varphi$ and $\varphi=-\frac{1}{2}\text{Arg}(z)$ we obtain that, for $v\notin\mathbb{N}$ and $|z|\ge z_0>0$,

$$|R_N(v,z,w)|\le\frac{M_N(v,\varphi,w)}{|z|^{N+\Re v-2}},$$

where $M_N(v,\varphi,w)$ is independent of $|z|$. Therefore, (26) is an asymptotic expansion for large z also when $\Re(v)\le 2-N$ with $v\neq 1,\,2$. □

Although the expansion given in the preceding theorem is quite simple, the error bound is not quite sharp because the coefficient in front of $(|z|\cos\varphi)^{-N-\Re(v)+2}$ is large. A more accurate error bound for large z is given in the following theorem.

Theorem 5. *For $w>0$, $|\mathrm{Arg}(z-w-1)|<\pi$, $v\neq 1,2$ and $N=0,1,2,\ldots$, an asymptotic expansion of $\zeta_2(v,z,w)$ for large z is given by*

$$\zeta_2(v,z,w)=\sum_{n=0}^{N-1}\frac{\tilde{B}_n(v,z,w)}{(z-w-1)^{2n+v-6}}+R_N(v,z,w), \tag{33}$$

where $R_N(v,z,w)=\mathcal{O}(z^{-2N-\Re(v)+6-(2+\delta_N)\Theta(2-N)})$ as $z\to\infty$,

$$\tilde{B}_n(v,z,w)\equiv\sum_{k=0}^{n}\frac{B_{2k-2+\delta_k}B_{2n-2k-2+\delta_{n-k}}}{(2k-2+\delta_k)!(2n-2k-2+\delta_{n-k})!}\times \\ \frac{(v-6+\Delta_{n,k})_{2n}}{(v-6+\Delta_{n,k})_{6-\Delta_{n,k}}}\frac{w^{2n-3-2k+\delta_{n-k}}}{(z-w-1)^{\Delta_{n,k}}} \tag{34}$$

and we have defined $\delta_k\equiv 2\delta_{k,0}+\delta_{k,1}$, $\Delta_{n,k}\equiv\delta_k+\delta_{n-k}$ and

$$\Theta(n)\equiv\begin{cases}1 & \text{if} \quad n\geq 0\\ 0 & \text{if} \quad n<0.\end{cases} \tag{35}$$

For $N=0,1,2,\ldots$, $\Re(v)>6-2N-(2+\delta_N)\Theta(2-N)$ and $\Re(z-w-1)>0$, an error bound for the remainder is given by

$$|R_N(v,z,w)|\leq\frac{C_N(v,z,w,0)}{w|\Gamma(v)|[\Re(z-w-1)]^{2N+\Re(v)-6+\delta_N}}, \tag{36}$$

where

$$\begin{aligned}C_N(v,z,w,\varphi)\equiv\Big[&C_N^{(1)}(w)\gamma(2N+\Re(v)-6+\delta_N,2\pi a(w)\Re(z-w-1))+\\ &C_N^{(2)}(w,\varphi)\Gamma(2N+\Re(v)-6+\delta_N,2\pi a(w)\Re(z-w-1))\Big]\,\Theta(N-3)+\\ &C_N^{(1)}(w)\frac{\gamma(2N+\Re(v)-4+\delta_N,2\pi a(w)\Re(z-w-1))}{[\Re(z-w-1)]^2}+\\ &C_N^{(3)}(w,\varphi)\frac{\Gamma(2N+\Re(v)-4+\delta_N,2\pi a(w)\Re(z-w-1))}{[\Re(z-w-1)]^2},\end{aligned} \tag{37}$$

$$C_N^{(1)}(w)\equiv\sum_{k=0}^{N}\frac{|B_{2k-2+\delta_k}|}{(2k-2+\delta_k)!}\frac{|B_{2N-2k-2+\delta_{n-k}}|}{(2N-2k-2+\delta_{n-k})!}w^{2N-2k-2+\delta_{n-k}}, \tag{38}$$

$$C_N^{(2)}(w,\varphi)\equiv\frac{(2\pi)^3w^{2N-2}[(\pi\cos\varphi)^2+w^2]}{(\pi\cos\varphi)^{2N}}\frac{1-\left(\frac{\cos\varphi}{2w}\right)^{2N}}{1-\left(\frac{\cos\varphi}{2w}\right)^{2}}\cot\left(\frac{\pi}{4}-\frac{|\varphi|}{2}\right), \tag{39}$$

$$C_N^{(3)}(w,\varphi)\equiv C_N^{(2)}(w,\varphi)+\frac{\pi^2}{4(\pi\cos\varphi)^{2N-2+\delta_N}}\cot^2\left(\frac{\pi}{4}-\frac{|\varphi|}{2}\right), \tag{40}$$

$a(w)$ is defined in fig. 1.

For $N=0,1,2,...$, $\Re(v)>6-2N-(2+\delta_N)\Theta(2-N)$ and $|Arg(z-w-1)|<\pi$, an error bound for the remainder is given by

$$\begin{aligned}|R_N(v,z,w)|\leq &\frac{e^{-\varphi\Im(v)}\Gamma(2N+\Re(v)-6+\delta_N)}{w|\Gamma(v)|(|z-w-1|\cos\varphi)^{2N+\Re(v)-6+\delta_N}}\Big[C_N^{(2)}(w,\varphi)\Theta(N-3)+\\ &C_N^{(3)}(w,\varphi)\frac{(2N+\Re(v)-5+\delta_N)(2N+\Re(v)-6+\delta_N)}{(|z-w-1|\cos\varphi)^2}\Big],\end{aligned} \tag{41}$$

where $\varphi\equiv-\frac{1}{2}Arg(z-w-1)$.

Proof. Write

$$\zeta_2(v,z,w)=\frac{S(v)}{w}\int_{\mathcal{C}_\varphi}t^{v-3}e^{-(z-w-1)t}\frac{t}{e^t-1}\frac{wt}{e^{wt}-1}dt \tag{42}$$

and consider the expansion

$$\frac{t}{e^t-1}=\sum_{n=0}^{N-1}\frac{B_{2n-2+\delta_n}}{(2n-2+\delta_n)!}t^{2n-2+\delta_n}+\bar r_N(t),\qquad N=0,1,2,...,$$

This expansion verifies the error test (see the proof of theorem 1) for $0\leq t<2\pi$, and therefore, from (13),

$$|\bar r_N(t)|\leq\frac{|B_{2N-2+\delta_N}|}{(2N-2+\delta_N)!}t^{2N-2+\delta_N},\qquad 0\leq t<2\pi.$$

Nevertheless, from (15),

$$|\bar r_N(t)|\leq\frac{C(\varphi)}{(\pi\cos\varphi)^{2N-2+\delta_N}}|t|^{2N-2+\delta_N},\qquad t\in[0,\infty e^{i\varphi}), \tag{43}$$

where $C(\varphi)$ is given in (16). Consider now the analogous expansion for $wt(e^{wt}-1)^{-1}$,

$$\frac{wt}{e^{wt}-1}=\sum_{n=0}^{N-1}\frac{B_{2n-2+\delta_n}}{(2n-2+\delta_n)!}(wt)^{2n-2+\delta_n}+\bar r_N(wt),\qquad N=0,1,2,...,$$

and multiply both expansions term by term. We obtain

$$\frac{t}{e^t-1}\frac{wt}{e^{wt}-1}=\sum_{n=0}^{N-1}\Phi_n(w,t)+r_N(w,t),\qquad N=0,1,2,..., \tag{44}$$

where, for $n=0,1,2,...,$

$$\Phi_n(w,t)\equiv (wt)^{2n}\sum_{k=0}^{n}\frac{B_{2k-2+\delta_k}}{(2k-2+\delta_k)!}\frac{B_{2n-2k-2+\delta_{n-k}}}{(2n-2k-2+\delta_{n-k})!}\frac{t^{\Delta_{n,k}-4}}{w^{2k+2-\delta_{n-k}}}. \tag{45}$$

The terms of this expansion are an asymptotic sequence for small t: $\Phi_n(t)=\mathcal{O}(t^{2n-2+\delta_n})$ as $t\to 0$. The remainder term $r_N(w,t)$ is given by

$$r_N(w,t)\equiv\sum_{n=0}^{N-1}\frac{B_{2n-2+\delta_n}}{(2n-2+\delta_n)!}t^{2n-2+\delta_n}\bar{r}_{N-n}(wt)+\bar{r}_N(t)\bar{r}_0(wt), \tag{46}$$

and therefore, $r_N(w,t)=\mathcal{O}(t^{2N-4+(2+\delta_N)\Theta(2-N)})$ as $t\to 0$. Introducing the expansion (44) in the right hand side of (42) and after trivial manipulations we obtain (33) with

$$R_N(v,z,w)=\frac{S(v)}{w}\int_{\mathcal{C}_\varphi}t^{v-3}e^{-(z-w-1)t}r_N(t)dt. \tag{47}$$

Using that $\text{sign}(\bar{r}_n(t))=(-1)^n$ for $0\le t<2\pi$ and $\text{sign}(B_{2n-2+\delta_n})=(-1)^n$ we have that $\text{sign}(r_N(w,t))=(-1)^N$. Therefore, using the error test and the bound

$$|t|^{\Delta_{N,k}}\le |t|^{\delta_N}[|t|^2+\Theta(N-3)] \tag{48}$$

for $k=0,1,2,...,N$, $t\in[0,\infty)$, we obtain

$$|r_N(w,t)|\le|\Phi_N(w,t)|\le C_N^{(1)}(w)|t|^{2N-4+\delta_N}\left[|t|^2+\Theta(N-3)\right],\qquad 0\le t<2\pi a(w). \tag{49}$$

Nevertheless, introducing the bound (43) in (46) for either $\bar{r}_n(t)$ or $\bar{r}_n(wt)$, using [[1], eqs. 23.1.14 and 23.1.15] and (48) we obtain

$$|r_N(w,t)|\le\left[C_N^{(3)}(w,\varphi)|t|^2+C_N^{(2)}(w,\varphi)\Theta(N-3)\right]|t|^{2N-4+\delta_N},\qquad t\in[0,\infty e^{i\varphi}). \tag{50}$$

When $|\text{Arg}(z+w-1)|<\pi/2$ and $\Re(v)>6-2N-(2+\delta_N)\Theta(2-N)$ we can take $\varphi=0$ in this integral with $\mathcal{C}=[0,\infty)$. Then we introduce the bound (49) in that integral for $t\in[0,2\pi a(w))$ and the bound (50) for $t\in[2\pi a(w),\infty)$. After trivial manipulations we obtain (36)-(40). Nevertheless, for $|\text{Arg}(z+w-1)|<\pi$ and $\Re(v)>6-2N-(2+\delta_N)\Theta(2-N)$ we can take $\varphi=-\frac{1}{2}\text{Arg}(z+w-1)$ in the integral (47) with $\mathcal{C}_\varphi=[0,\infty e^{i\varphi})$. Then introducing the bound (50) in (47) we obtain (41).

Using the Lagrange formula for the remainder $r_N(w,t)$, the Cauchy's integral formula for the derivative of an analytic function it is easy to show that, for $N=0,1,2,...$ and $\varphi=\text{Arg}(t)$,

$$|r_N(w,t)|\le\bar{C}_N(w,\varphi)|t|^{2N-4+\delta_N}(1+|t|^2),\qquad t\in\mathcal{L}_\varphi,$$

where $\bar{C}_N(w,\varphi)$ is independent of $|t|$. Introducing this bound in (47) with $\mathcal{C}_\varphi = \mathcal{L}_\varphi$ we obtain that, for $v \notin \mathbb{N}$ and $|z| \geq z_0 > 0$,

$$|R_N(v,z,w)| \leq \frac{M_N(v,\varphi,w,z)}{|z|^{2N+\Re(v)-4+\delta_N}},$$

where $M_N(v,\varphi,w,z)$ is bounded for large $|z|$. Therefore, (33) is an asymptotic expansion for large z also when $\Re(v) \leq 6-2N-(2+\delta_N)\Theta(2-N)$ with $v \neq 1, 2$. □

Remark. Observe that the expansions given in theorems 1-5 are all finite for $v = 0, -1, -2, -3, \ldots$

Theorem 6. *For $\Re(z)+w > 0$, $|z| < w$, $v \neq 1, 2$ and $N = 0, 1, 2, \ldots$, a convergent expansion of $\zeta_2(v,z,w)$ for small z and/or large w is given by*

$$\zeta_2(v,z,w) = \zeta(v,z) + \sum_{n=0}^{N-1} \frac{(-1)^n}{n!} \zeta_2(n+v,w,w) z^n + R_N(v,z,w), \tag{51}$$

where $R_N(v,z,w) = \mathcal{O}(z^N)$ as $z \to 0$. Moreover, for $\Re(v) > 2-N$ and $N = 0, 1, 2, \ldots$, the remainder term is bounded by

$$\begin{aligned} |R_N(v,z,w)| \leq & \frac{\Gamma(N+\Re(v))}{N!|\Gamma(v)|} \zeta_2\big(N+\Re(v), w+\Re(z)\Theta(-\Re(z)), w\big)|z|^N \leq \\ & \frac{\Gamma(N+\Re(v)-2)|z|^N}{N!|\Gamma(v)|[w+\Re(z)\Theta(-\Re(z))]^{N+\Re(v)-2}} \Bigg[1+ \\ & \frac{N+\Re(v)-2}{w+\Re(z)\Theta(-\Re(z))} \left(1+w+\frac{N+\Re(v)-1}{w+\Re(z)\Theta(-\Re(z))}\right)\Bigg], \end{aligned} \tag{52}$$

where $\Theta(n)$ is defined in (35).

Proof. Introduce the decomposition (9) in either of the integral representations given in proposition 1 with $\varphi = 0$. Then we obtain

$$\zeta_2(v,z,w) = \zeta(v,z) + \tilde{\zeta}_2(v,z,w),$$

where

$$\tilde{\zeta}_2(v,z,w) \equiv S(v) \int_{\mathcal{C}_0} \frac{t^{v-1}e^{-wt}}{(1-e^{-t})(1-e^{-wt})} e^{-zt} dt.$$

If we introduce now the expansion

$$e^{-zt} = \sum_{n=0}^{N-1} \frac{(-1)^n}{n!} t^n z^n + r_N(tz)$$

in this integral, we obtain (51) with

$$R_N(v,z,w) = S(v) \int_{\mathcal{C}_0} \frac{t^{v-1}e^{-wt} r_N(tz)}{(1-e^{-t})(1-e^{-wt})} dt. \tag{53}$$

Using the Lagrange form for the remainder $r_N(tz)$, we obtain

$$|r_N(tz)| \le \frac{e^{-t\Re(z)\Theta(-\Re(z))}}{N!}|tz|^N.$$

Introducing this bound in the integral (53) we find that, for $w+\Re(z)\Theta(-\Re(z))>0$, $R_N(v,z,w)=\mathcal{O}(z^N)$ as $z\to 0$, and moreover, that $\lim_{N\to\infty} R_N(v,z,w)=0$ if besides, $|z|<w$. On the other hand, for $N+\Re(v)-2>0$, after trivial manipulations in the integral (53) we obtain the first inequality in (52). The second inequality follows if we use besides the inequalities $t(1-e^{-t})\le 1+t$ and $wt(1-e^{-wt})\le 1+wt$ in that integral. □

4. Numerical experiments

Tables 1-13 show numerical experiments about the approximation supplied by theorems 1-6 and the accuracy of the error bounds.

In all these tables, the second column represents the value of $\zeta_2(v,z,w)$. The third and sixth columns represent, respectively, a first and a second approximation given by the corresponding theorem. Fourth and seventh columns represent the respective relative errors $R_N(v,z,w)/\zeta_2(v,z,w)$. Fifth and last columns represent the respective error bounds given by the corresponding theorem.

Table 1 (theorem 1) ($z=1, v=3$)

w	$\zeta_2(3,1,w)$	First ($N=1$) order approx.	Relative error	Relative er. bound	Second ($N=2$) order approx.	Relative error	Relative er. bound
10	1.20970747	1.21028157	4.74e-4	6.2e-4	1.20968054	2.22e-5	2.75e-5
20	1.20403962	1.20411307	6.1e-5	7.79e-5	1.20403794	1.39e-6	1.73e-6
50	1.20238112	1.20238589	3.96e-6	4.99e-6	1.20238108	3.43e-8	4.44e-8
100	1.20213855	1.20213915	4.98e-7	6.24e-7	1.20213855	1.83e-9	2.77e-9
200	1.20207739	1.20207746	6.25e-8	7.8e-8	1.20207739	3.67e-11	1.73e-10

Table 2 (theorem 1) ($z=1.5+1.5i$, $v=4$)

w	$\zeta_2(4,1.5+1.5i,w)$	First ($N=1$) ord. approx.	Relative error	Relative er. bound	Second ($N=2$) ord. approx.	Relative error	Relative er. bound
10	-0.014450327 - 0.020237395i	-0.0143484 - 0.020332539i	0.0056	0.0079	-0.014457534 - 0.020226785i	0.000516	0.00067
20	-0.01499289 - 0.01991197i	-0.0149878 - 0.01991369i	0.000215	0.000276	-0.014993077 - 0.019911884i	8.19e-6	9.93e-6
50	-0.01505355 - 0.019897419i	-0.01505344 - 0.01989743i	4.16e-6	5.18e-6	-0.01505354 - 0.019897418i	5.69e-8	6.83e-8
100	-0.015056684 - 0.019897134i	-0.015056678 - 0.019897134i	2.37e-7	2.94e-7	-0.015056684 - 0.019897134i	1.56e-9	1.88e-9
200	-0.0150570554 - 0.019897117i	-0.015057055 - 0.019897117i	1.42e-8	1.75e-8	-0.0150570554 - 0.019897117i	4.6e-11	5.54e-11

Table 3 (theorem 2) $(z = 1, v = 10)$

w	$\zeta_2(10,1,w)$	First $(N=2)$ order approx.	Relative error	Relative er. bound	Second $(N=4)$ order approx.	Relative error	Relative er. bound
0.1	1.69545592	1.61383995	0.048	0.054	1.69721446	0.001	0.0012
0.05	2.76864311	2.72718260	0.015	0.016	2.76886986	8.19e-5	8.97e-5
0.02	6.08387084	6.06721058	0.0027	0.003	6.08388548	2.4e-6	2.6e-6
0.01	11.64225949	11.63392387	0.00071	0.00079	11.64226132	1.57e-7	1.7e-7
0.005	22.77151895	22.76735046	0.00018	0.0002	22.77151919	1.e-8	1.1e-8

Table 4 (theorem 2) $(z = i, v = 6)$

w	$\zeta_2(6,i,w)$	First $(N=2)$ ord. approx.	Relative error	Relative er. bound	Second $(N=4)$ ord. approx.	Relative error	Relative er. bound
0.1	-0.77425775 - 1.67084705i	-0.777180201 - 1.72446627i	0.0292	0.4721	-0.7742432 - 1.67133821i	0.000267	0.0804
0.05	-1.04900585 - 3.48279081i	-1.05047254 - 3.50941527i	0.0073	0.1195	-1.04900404 - 3.48285124i	1.66e-5	0.0051
0.02	-1.86976231 - 8.85363279i	-1.87034959 - 8.864262226i	0.0012	0.0192	-1.86976219 - 8.853663665i	4.26e-7	0.00013
0.01	-3.23651764 - 17.78369395i	-3.23681133 - 17.7890072i	0.00029	0.0048	-3.23651763 - 17.783694433i	2.66e-8	8.19e-6
0.005	-5.96958795 - 35.63584073i	-5.9697348 - 35.63849719i	7.363e-5	0.0012	-5.96958795 - 35.63584079i	1.666e-9	5.12e-7

Table 5 (theorem 3) $(z = 1, v = 10)$

w	$\zeta_2(10,1,w)$	First $(N=1)$ order approx.	Relative error	Relative er. bound	Second $(N=2)$ order approx.	Relative error	Relative er. bound
0.1	1.69545592	2.87149806	0.6936	0.9304	1.43668397	0.1526	0.1707
0.05	2.76864311	3.53153438	0.2755	0.3335	2.69580351	0.0263	0.0289
0.02	6.08387084	6.67555648	0.097	0.112	6.06306638	0.0034	0.0037
0.01	11.64225949	12.18633612	0.0467	0.0527	11.63294976	0.0008	0.00088
0.005	22.77151895	22.76735046	0.0229	0.0256	22.77151919	0.00019	0.00021

Table 6 (theorem 3) $(z = 1 + i, v = 6)$

w	$\zeta_2(6,1+i,w)$	First $(N=1)$ ord. approx.	Relative error	Relative er. bound	Second $(N=2)$ ord. approx.	Relative error	Relative er. bound
0.1	-0.27425775 + 0.27867593i	-0.25874755 + 0.35432687i	0.1975	0.794	-0.27973327 + 0.27606469i	0.01551	0.1303
0.05	-0.54900585 + 0.49215054i	-54442706 + 0.56040679i	0.093	0.345	-0.55106037 0.49063914i	0.0035	0.0272
0.02	-1.36976231 + 1.13636347i	-1.37053862 + 1.1999728i	0.036	0.127	-1.37043853 + 1.13574025i	0.00052	0.0039
0.01	-2.73651764 + 2.2113056i	-2.7388938 + 2.273335243i	0.0176	0.0617	-2.73683313 + 2.21099292i	0.000126	0.000947
0.005	-5.46958795 + 4.3616592i	-5.47273082 4.422924024i	0.0088	0.031	-5.4697402 + 4.36150279i	0.0000312	0.000233

Table 7 (theorem 4) ($w = 2, v = 4$)

z	$\zeta_2(4, z, 2)$	First ($N = 1$) order approx.	Relative error	Relative er. bound	Second ($N = 2$) order approx.	Relative error	Relative er. bound
10	0.001134136	0.0008333333	0.26523	1.2103	0.0010833333	0.0448	0.25085
20	0.000242604	0.000208333	0.141	0.62	0.0002395833	0.0124	0.062
50	0.0000354082	0.0000333333	0.0856	0.25	0.0000353333	0.00212	0.00974
100	8.58796461e-6	8.3333e-6	0.02965	0.125	8.5833e-6	0.000539	0.0024
200	2.114869e-6	2.08333e-6	0.0149	0.0628	2.11458e-6	0.000135	0.0006

Table 8 (theorem 4) ($w = 0.75$, $v = 5.3$, Arg$z = \frac{\pi}{4}$)

$\|z\|$	$\zeta_2(5.3, z, 0.75)$	First ($N = 2$) ord. approx.	Relative error	Relative er. bound	Second ($N = 4$) ord. approx.	Relative error	Relative er. bound
10	-0.000110725 - 0.000075201i	-0.00007857 - 0.00007857i	0.241	3.74	-0.000107734 - 0.000078567i	0.0336	0.577
20	-0.00001173 - 9.7215791e-6i	-9.82092e-6 - 9.8209275e-6i	0.126	1.9156	-0.00001164 - 9.8209275e-6i	0.0089	0.1454
50	-6.76164481 - 6.2755743i	-6.2853936 - 6.2853936i	0.0516	0.7764	-6.75206027 - 6.2853936i	0.0015	0.0233
100	-8.151404e-8 - 7.8537101e-8i	-7.856742e-8 - 7.856742e-8i	0.026	0.39	-8.148409e-8 - 7.856742e-8i	0.00038	0.0058
200	-1.00042e-8 - 9.81998e-9i	-9.820927e-9 - 9.820927e-9i	0.013	0.195	-1.000322e-8 - 9.820927e-9i	0.000095	0.00146

Table 9 (theorem 5) ($w = 0.5, v = 5$)

z	$\zeta_2(5, z, 0.5)$	First ($N = 1$) order approx.	Relative error	Relative er. bound	Second ($N = 2$) order approx.	Relative error	Relative er. bound
10	0.0002090624	0.000271389	0.2981	0.3436	0.000199550	0.0455	0.0494
20	0.0000233252	0.00002632289	0.128	0.1372	0.0000231214	0.00873	0.009
50	1.3948e-6	1.46091e-6	0.0474	0.04859	1.393136e-6	0.0012	0.00122
100	1.704628e-7	1.743974e-7	0.023	0.0233	1.704137e-7	0.000288	0.00029
200	2.106914e-8	2.13092034e-8	0.01139	0.01146	2.10676634e-8	7.03e-5	7.06e-5

Table 10 (theorem 5) ($w = 0.05$, $v = 6$, Arg$z$=$\frac{\pi}{4}$)

$\|z\|$	$\zeta_2(3, z, 1)$	First ($N = 1$) ord. approx.	Relative error	Relative er. bound	Second ($N = 2$) ord. approx.	Relative error	Relative er. bound
10	-0.00011484 + 0.000016789i	-0.00012759 + 0.000042298i	0.2457	2.2864	-0.000116165 + 0.0000141319i	0.02557	0.34734
20	-6.7139e-6 + 4.942e-7i	-7.1635e-6 + 1.113e-6i	0.1136	0.811	-6.7222e-6 + 4.578e-7i	0.005543	0.0566
50	-1.6475e-7 + 4.8749e-9i	-1.6948e-7 + 1.023e-8i	0.0433	0.2681	-1.6476e-7 + 4.7413e-9i	0.000814	0.00715
100	-1.01485e-8 + 1.504e-10i	-1.0297e-8 + 3.0817e-10i	0.02133	0.126	-1.0148e-8 + 1.484e-10i	0.000198	0.00166
200	-6.296e-10 + 4.6706e-12i	-6.3428e-10 + 9.45e-12i	0.0106	0.0612	-6.2964e-10 + 4.6397e-12i	4.9e-5	3.99e-4

Table 11 (theorem 6) $(w=1, v=3)$

z	$\zeta_2(3,z,1)$	First $(N=1)$ order approx.	Relative error	Relative er. bound	Second $(N=2)$ order approx.	Relative error	Relative er. bound
0.1	1.52996633	1.64492907	0.07514	0.359485	1.5247234	0.003426	0.062093
0.05	1.58615778	1.64492907	0.03705	0.17337	1.58482622	0.000839	0.014973
0.02	1.62110302	1.64492907	0.0147	0.067855	1.62088793	0.000133	0.002344
0.01	1.63296244	1.64492907	0.007328	0.03368	1.6329085	3.303e-5	5.82e-4
0.005	1.63893229	1.64492907	0.00366	0.01678	1.638919	8.2416e-6	1.449e-4

Table 12 (theorem 6) $(w=1,\ v=3,\ \mathrm{Arg}z{=}\frac{\pi}{4})$

$\|z\|$	$\zeta_2(3,z,1)$	First $(N=1)$ ord. approx.	Relative error	Relative er. bound	Second $(N=2)$ ord. approx.	Relative error	Relative er. bound
0.1	1.5600488 - 0.0796089i	1.64492907	0.0745	0.352	1.5599308 - 0.08499826i	0.003387	0.0608
0.05	1.60244495 - 0.0411615i	1.64492907	0.0369	0.1716	1.6024299 - 0.04249913i	0.000835	0.01481
0.02	1.6279304 - 0.01678416i	1.64492907	0.0147	0.0676	1.627929 - 0.0169997i	0.00013	0.0023
0.01	1.63642936 - 0.00844583i	1.64492907	0.00732	0.0336	1.6364292 - 0.0084998i	3.3e-5	5.8e-4
0.005	1.640679 - 0.004236i	1.64492907	0.00366	0.0167	1.640679 - 0.0042499i	8.24e-6	1.45e-4

Table 13 (theorem 6) $(w=1,\ v=3,\ \mathrm{Arg}z{=}\frac{\pi}{2})$

$\|z\|$	$\zeta_2(3,z,1)$	First $(N=1)$ ord. approx.	Relative error	Relative er. bound	Second $(N=2)$ ord. approx.	Relative error	Relative er. bound
0.1	1.63952169- 0.12003295i	1.64492907	0.07309	0.3346	1.64492907 - 0.1202057i	0.00329	0.05779
0.05	1.643576428 - 0.0600812i	1.64492907	0.03654	0.1672	1.64492907 - 0.06010285i	0.0008225	0.01444
0.02	1.644712609 - 0.02403975i	1.64492907	0.014615	0.06687	1.64492907 - 0.02404113i	0.00013	0.0023
0.01	1.64487495 - 0.0120204i	1.64492907	0.007307	0.03344	1.64492907 - 0.01202057i	3.29e-5	5.8e-4
0.005	1.64491554- 0.00601026i	1.64492907	0.00365	0.0167	1.64492907 - 0.00601028i	8.23e-6	1.44e-4

5. Conclusions

We have obtained two integral representations, eqs. (2) and (3), for the double zeta function from the double series definition given in (1). In that definition, the double zeta function $\zeta_2(v,z,w)$ is defined for $\Re(z)$, $\Re(w)>0$ and $\Re(v)>2$. For $w>0$, these integrals define the analytical continuation of $\zeta_2(v,z,w)$ to the complex z and v planes with $|\text{Arg}(z)|<\pi$ and $v\neq 1$, 2.

Appropriate Taylor expansions of the integrand in either of these integrals let us to obtain asymptotic expansions of $\zeta_2(v,z,w)$ for large or small z or w in theorems 1-6. These Taylor expansions verify the error test in a large domain of v, z and w. Accurate error bounds for the asymptotic expansions have been obtained in those theorems using this property.

Complete asymptotic expansions with error bounds for the double gamma function $\Gamma_2(z,w)$ should follow from the asymptotic expansions given in theorems 1-6 for the double zeta function. This is the subject of further investigations.

6. Acknowledgments

The financial support of the *Dirección General de Ciencia y Tecnologia, BFM2000-0803*, of the *Government of Navarra, Res. 92/2002* and of the University of La Rioja is acknowledged. The useful comments and suggestions for improvement of Nico Temme are also acknowledged.

References

[1] M. Abramowitz and I.A. Stegun, Handbook of mathematical functions, *Dover, New York*, 1970.

[2] E. W. Barnes, The theory of the G-function, *Quart. J. Math.*, **31** (1899) 264-314.

[3] E. W. Barnes, Genesis of the double gamma function, *Proc. London Math. Soc.*, **31** (1900) 358-381.

[4] E. W. Barnes, The theory of the double gamma function, *Philos. Trans. Roy. Soc. London Ser A*, **196** (1901) 265-388.

[5] E. L. Basor and P. J. Forrester, Formulas for the evaluation of Toeplitz determinants with rational generating functions, *Math. Nachr.*, **170** (1994) 5-18.

[6] J. Billingham and A. C. King, Uniform asymptotic expansions for the Barnes double gamma function, *Proc. Roy. Soc. London Ser A*, **453** (1997) 1817-1829.

[7] J. Choi, Determinant of laplacians on S^3, *Math. Japan.*, **40** (1994) 155-166.

[8] J. Choi and H. M. Srivastava, Summs associated with the zeta function, *J. Math. Anal. Appl.*, **206** (1997) 103-120.

[9] J. Choi and H. M. Srivastava, Certain classes of series involving the zeta function, *J. Math. Anal. Appl.*, **231** (1999) 91-117.

[10] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenk and S. Zerbini, Zeta regularization techniques with applications, *World Scientific*, 1994.

[11] C. Ferreira and J. L. Lopez, An asymptotic expansion of the double gamma function, J. Approx. Theor. **111** (2001) 298-314.

[12] K. Matsumoto, Asymptotic series for double zeta and double gamma functions of Barnes, *RIMS Kokyuroku.*, **958** (1996) 162-165.

[13] K. Matsumoto, Asymptotic series for double zeta, double gamma, and Hecke L-functions, *Math. Proc. Camb. Philos. Soc.*, **123** N^o 3, (1998) 385-405.

[14] B. Osgood, R. Phillips and P. Sarnak, Extremal of determinants of Laplacians, *J. Funct. Anal.*, **80** (1988) 148-211.

[15] J. R. Quine and J. Choi, Zeta regularized products and functional determinants on spheres, *Rocky Mountain J. Math.*, **26** (1996) 719-729.

[16] P. Sarnak, Determinants of Laplacians, *Commun. Math. Phys.* , **110** (1987) 113-120.

[17] T. Shintani, A proof of the classical Kronecker limit formula, *Tokyo J. Math.*, **3** (1980) 191-199.

[18] H. M. Srivastava, A unified presentation of certain classes of series of the Riemann zeta function, *Riv. Mat. Univ. Parma (4)*, **14** (1988) 1-23.

[19] N.M. Temme, Special functions: An introduction to the classical functions of mathematical physics, *Wiley and Sons, New York*, 1996.

[20] I. Vardi, Determinants of Laplacians and multiple gamma functions, *SIAM J. Math. Anal.*, **19** (1988) 493-507.

[21] A. Voros, Spectral functions, special functions and the Selberg zeta function, *Commun. Math. Phys.*, **110** (1987) 439-465.

[22] R. Wong, Asymptotic approximations of integrals, *Academic Press, New York*, 1989.