

A note on the error bound for the remainder of an asymptotic expansion of the Double Gamma function

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Using a recent result [2], we can improve the error bound for an asymptotic expansion for the double gamma function $G(z)$ given in [[1], Theorem 2] when $|\text{Arg}(z)| < \pi/2$. This new bound is simpler as well as more accurate.

Theorem 1 *For $|\text{Arg}(z)| < \pi/2$, an error bound for the remainder $R_N(z)$ in the expansion [[1], Theorem 1] of $\log G(z+1)$ is given by [1, (9)]*

$$|R_N(z)| \leq \frac{|B_{2N+2}|}{2N(2N+1)(2N+2)(\text{Re}(z))^{2N}}, \quad N = 1, 2, 3, \dots \quad (1)$$

Proof. For $|\text{Arg}(z)| < \pi/2$ we can take $\varphi = 0$ in formula [[1], (11)]. In [2] the author proves that the inequality $(-1)^N r_{2N+2}(x) \geq 0$, for $r_{2N+2}(x)$ given in [[1], (9)], holds when $x \geq 0$. Then, $r_{2N+2}(x)$ verifies the error test for all positive x and the bound given in [[1], (13)] is valid for $0 \leq x < \infty$. Using this bound in [[1], (11)] we obtain the result. ‡

In the following table, we compare the relative error bounds given in [[1], Theorem 2] (namely “old”) with (1) (namely “new”). This new bound is in particular more accurate for small values of z . The new bound (1) has analytical as well as computational advantages for lower values of z .

z	$I_2(z)$	Relat. err. (1st ord)	Old relat. er. bound	New relat. er. bound	Relat. err. (2nd ord)	Old relat. er. bound	New relat. er. bound
1	-0.0012455	0.115	2.24	0.16	0.0442	13.4	0.08
2	-0.0003361	0.0332	0.0425	0.037	0.00374	0.032	0.0046
5	-0.0000552	0.00564	0.00574	0.00574	0.000110	0.000115	0.0001149
10	-0.0000139	0.00142	0.00143	0.00143	7.009e-6	7.18e-6	7.18e-6
20	-3.471e-6	0.0003568	0.0003572	0.0003572	4.45e-7	4.47e-7	4.47e-7
50	-5.55552e-7	5.713e-5	5.715e-5	5.715e-5	1.142e-8	1.143e-8	1.143e-8
100	-1.38887e-7	1.42852e-5	1.42859e-5	1.42859e-5	7.142e-10	7.143e-10	7.143e-10

Table 1: Approximation supplied by [[1], Theorem 1] taking $z \in \mathbb{R}$ and error bounds given by [[1], Theorem 2] and (1).

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References

- [1] C. Ferreira and J. L. López, An asymptotic expansion of the double gamma function, *Journal of Approx. Theory*, **111** (2001) 54-63.
- [2] H. L. Pedersen, On the remainder in an asymptotic expansion of the double gamma function, *manuscript*.