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# ASYMPTOTIC APPROXIMATIONS OF INTEGRALS: AN INTRODUCTION, WITH RECENT DEVELOPMENTS AND APPLICATIONS TO ORTHOGONAL POLYNOMIALS*

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**Abstract.** In the first part we discuss the concept of asymptotic expansion and its importance in applications. We focus our attention on special functions defined through integrals and consider their approximation by means of asymptotic expansions. We explain the general ideas of the theory of asymptotic expansions of integrals and describe two classical methods (Watson's lemma and the saddle point method) and modern methods (distributional methods). In the second part we apply these ideas to approximate (in an asymptotic sense) polynomials of the Askey table in terms of simpler polynomials of the Askey table. We consider two different types of asymptotic expansions that have been recently developed: i) some parameter of the polynomial is large or ii) the degree (and perhaps the variable too) of the polynomial is large. For each situation we employ a different asymptotic method. In the first case we use the technique of "matching of the generating functions at the origin". In the second one we employ a modified version of the saddle point method together with the theory of two-point Taylor expansions. In the first case the asymptotic expansion results in a finite sum of polynomials. In the second one the asymptotic expansion is a convergent infinite series of polynomials. We conclude the paper with information on other recent developments in the research on asymptotic expansions of integrals.

**Key words.** Asymptotic expansions of integrals, asymptotics of orthogonal polynomials.

**AMS subject classifications.** 41A60, 33C65.

## 1. Introduction.

Asymptotic analysis is a useful mathematical tool which provides analytical insight and numerical information about the solutions of complicated problems in applied mathematics, engineering, physics and many other sciences, which require a mathematical framework for describing and modeling scientific problems.

Some examples of the power of this theory are the following:

- The Stirling formula for the factorial:

$$n! \sim n^n e^{-n}\sqrt{2\pi n}\left[1+\frac{1}{12n}+\frac{1}{288n^2}+...\right], \qquad n\to\infty.$$

- The approximation of the large harmonic numbers:

$$1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\sim \log n, \qquad n\to\infty.$$

- The approximation of solutions of boundary value problems. For example, the solution of the boundary problem:

$$\begin{cases} -\epsilon \triangle U + U_y = 0, \\ U(x,0)=0, U(0,y)=1, (x,y)\in(0,\infty)\times(0,\infty) \end{cases}$$

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may be approximated by

$$U \sim \operatorname{erfc}\sqrt{\frac{r-y}{2\epsilon}}, \qquad \epsilon \to 0, \qquad\qquad U \sim \frac{2\phi}{\pi}, \qquad r \to 0,$$

where erfc is the error function and $\phi$ is the polar angle.

- The approximate calculation of the electron charge: $e^- \sim 1.6021917 \cdot 10^{-19}$C.

In general, a practical definition of asymptotics may be *Asymptotics = Approximation of a complicated function $\mathcal{F}(z)$ by a more simple function $f(z)$, with improving approximation for large values of a parameter.* For example,

$$\frac{(\log(1+z) - \log z) z^z \sqrt{2\pi z}}{\Gamma(z+1) e^z \sinh(1/z)} \sim 1, \qquad z \to \infty.$$

A precise definition of asymptotics, and of the symbol "$\sim$", is given in the next section.

There are two main areas of investigation in asymptotics. The first one is concerned with solutions of differential equations, when $\mathcal{F}(z)$ is a solution of a $2^{nd}$ order linear O.D.E. [22]. For example, one solution of the equation $z^2 w'' + z w' + (z^2 - \nu^2) w = 0$ is the Bessel function $J_\nu(z)$. The asymptotic theory of $2^{nd}$ order linear O.D.E, together with information on initial values, tells us that

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}} \left(\frac{ez}{2\nu}\right)^\nu, \qquad \nu \to \infty.$$

The second area deals with functions $\mathcal{F}(z)$ that are expressible in the form of definite integrals or contour integrals [31]. For example, the integral

$$\frac{(z/2)^\mu}{\sqrt{\pi}\,\Gamma(\nu + 1/2)} \int_0^\pi \cos(z \cos t) \sin^{2\nu} t dt \sim \frac{1}{\sqrt{2\pi\nu}} \left(\frac{ez}{2\nu}\right)^\nu, \qquad \nu \to \infty.$$

The asymptotic behaviour of this integral for large $\nu$ agrees with the one of $J_\nu(z)$ above. Of course, this integral is nothing but an integral representation of $J_\nu(z)$ ([1], (9.1.20)).

In this paper we present an introductory overview of the asymptotic theory of integrals and apply this theory to obtain several asymptotic expansions of polynomials of the Askey table.

The paper is organised as follows. In Section 2 we give a brief introduction to asymptotics and, in particular, to the asymptotic theory of integrals. In Section 3 we give an overview of the classical methods for integrals based on Watson's lemma. In Sections 4 and 5 we explain the basic ideas of the summability and distributional methods, respectively. In Section 6 we present an asymptotic method to obtain asymptotic expansions of polynomials in the Askey table when one parameter is large. In Section 7 we derive asymptotic expansions of some polynomials of the Askey table when the degree (and perhaps the variable too) is large.

We conclude the paper with information on other recent developments in the research on asymptotic expansions of integrals.

**2. Asymptotic theory of integrals.** We motivate the precise definition of the symbol $\sim$ by means of an example. We define, for $x > 0$, the function $F(x) = x e^x E_1(x)$, where $E_1(x)$ denotes the exponential integral ([1], (5.1.1)). Then

$$F(x) \equiv x \int_0^\infty \frac{e^{-xt}}{1+t} dt = \int_0^\infty \frac{e^{-t}}{1+t/x} dt.$$

This integral is not expressible in terms of elementary functions and we wish to approximate it for large $x$. For large $x$ and fixed $t$ we can expand the factor $(1+t/x)^{-1}$ in powers of $t/x$. At this moment we do not pay attention to the fact that this expansion is not convergent for $t \geq x$ and proceed formally:

$$F(x) \sim \int_0^\infty e^{-t} \sum_{n=0}^\infty \left(-\frac{t}{x}\right)^n dt.$$

We proceed formally again and interchange sum and integral:

$$F(x) \sim \sum_{n=0}^\infty \frac{(-1)^n n!}{x^n} = 1 - \frac{1}{x} + \frac{2}{x^2} + ... \tag{2.1}$$

Define the sum of the first $N$ terms in the above expansion by

$$S_N(x) \equiv \sum_{n=0}^{N-1} \frac{(-1)^n n!}{x^n}.$$

The expansion (2.1) is not convergent ($\lim_{N\to\infty} S_N(x)$ does not exists). Then one is tempted to throw the expansion into the garbage and forget about those formal manipulations. Before doing that, let us give $S_N(x)$ a chance and plot the functions $S_4(x)$ and $f(x)$ together:

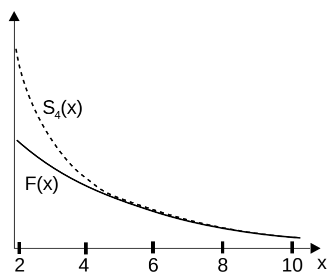


FIG. 2.1. *Graphics of the approximation to the integral $F(x)$ supplied by $S_4(x)$.*

Surprisingly, even though the series $\sum_{n=0}^\infty \frac{(-1)^n n!}{x^n}$ is not convergent, the sum of the first few terms of this series provides a good approximation of $F(x)$ for $x$ moderately large. An easy explanation of this phenomena may be obtained putting some rigour in the above formal manipulations. Instead of expanding $(1+t/x)^{-1}$ as a geometric series with an infinite number of terms, just expand up to $N$ terms:

$$\frac{1}{1+t/x} = \sum_{n=0}^{N-1} \left(-\frac{t}{x}\right)^n + \frac{(-t/x)^N}{1+t/x}.$$

Now to introduce this decomposition of $(1+t/x)^{-1}$ into the integral $F(x)$ and interchange summation and integration:

$$F(x) = \int_0^\infty e^{-t} \left[\sum_{n=0}^{N-1} \left(-\frac{t}{x}\right)^n + \frac{(-t/x)^N}{1+t/x}\right] dt =$$

$$1 - \frac{1}{x} + \frac{2}{x^2} - ... + \frac{(-1)^N N!}{x^N} + \frac{(-1)^N}{x^N} \int_0^\infty \frac{e^{-t} t^N}{x+t} dt = S_N(x) + R_N(x),$$

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where we have defined

$$R_N(x) \equiv \frac{(-1)^N}{x^N}\int_0^\infty \frac{e^{-t}t^N}{x+t}dt. \tag{2.2}$$

Using the bound $t + x \geq x$, we see immediately that

$$|R_N(x)| \leq \frac{N!}{x^N},$$

that is, $S_N(x)$ approximates very well the integral $F(x)$ for large $x$, although the approximation is not good for arbitrarily large $N$ because of $\lim_{N\to\infty} N!/x^N = \infty$. For fixed $x$, $\mathrm{Inf}\big(N!/x^N\big)$ is achieved for $\psi(N+1) = \log x$. For example, for $x = 5$ this equation has the approximate solution $N \simeq 4.5$ and $|R_4(5)/F(5)| \leq 0.045$. The expansion (2.1) is divergent and, for a given $x$, the best asymptotic approximation to $F(x)$ is obtained by truncating that series at a term $N$ such that $\psi(N+1) \sim \log x$. It usually happens that this optimal truncation occurs at the smallest term (in absolute value) of the series:

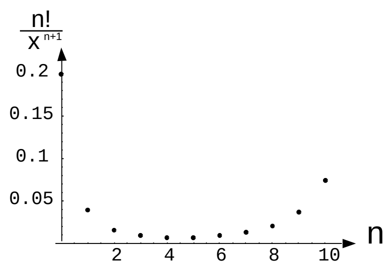


FIG. 2.2. *Plot of the absolute value $n!/x^n$ of the terms of the series (2.1) for $x = 5$. The smallest terms correspond to $n = 4, 5$. From $n = 6$ the terms grow indefinitely.*

Now we will give a meaning to the symbol $\sim$ appropriate to the above discussion [31]:

DEFINITION 2.1. *Let $F(z)$ and $\Phi_n(z)$, $n = 0, 1, 2, ...$ be functions defined in an unbounded set $\Omega$ of the complex plane. The formal series $\sum_{n=0}^{\infty}\Phi_n(z)$ is called an asymptotic expansion of $F(z)$ when $z \to \infty$ in $\Omega$, and we write*

$$F(z) \sim \sum_{n=0}^{\infty}\Phi_n(z), \qquad z \to \infty,$$

*if, for every $N > 0$,*

$$\mathcal{F}(z) = \sum_{n=0}^{N-1}\Phi_n(z) + R_N(z),$$

*where $\Phi_n(z)$ is an asymptotic sequence:*

$$\Phi_{n+1}(z) = o(\Phi_n(z)), \quad n = 0, 1, 2, .... \tag{2.3}$$

*and the remainder is of the order of the first neglected term:*

$$R_N(z) = \mathcal{O}(\Phi_N(z)), \qquad z \to \infty. \tag{2.4}$$

Nothing precludes the series $\sum_{n=0}^{\infty}\Phi_n(z)$ from being divergent. As a matter of fact, the asymptotic expansions of many special functions are divergent.

In the example of the exponential integral studied in this section, $\Phi_n(x)=(-1)^n n!/x^n$ and $R_N(x)$ is given in (2.2). This one is just an easy example from which the asymptotic analysis is quite clear. But in practice, to obtain an asymptotic expansion from an interesting integral is not always a straightforward task. For this reason, for a long time several asymptotic methods for integrals have been investigated. There is not a general method for integrals, and many methods are discussed in [31]. We classify them in several categories depending on the asymptotic principles from which they are obtained: classical methods based on Watson's lemma, summability methods, distributional methods, Mellin transform techniques, integration by parts, and so on. In the following sections we give a brief introduction to the first three methods.

**3. Classical methods for integrals based on Watson's lemma.** It is impossible to survey here all the methods based on Watson's lemma (Watson's Lemma, Laplace's method, the saddle point method, Perron's method, to mention the important ones). We just explain in this section the asymptotic ideas connected with these methods by means of two well-known examples: Watson's lemma and the saddle point method.

**3.1. Watson's Lemma [Watson, 1918].** Watson's lemma is used to obtain asymptotic expansions of Laplace transforms for large values of the parameter of the transformation. The simplest form is

$$F(z)\equiv\int_0^\infty e^{-zt}g(t)dt,\qquad z\to\infty,\qquad -\frac{1}{2}\pi<\text{phase}\,z<\frac{1}{2}\pi, \tag{3.1}$$

where we assume that the function $g(t)$ has an asymptotic expansion at $t=0$:

$$g(t)=\sum_{n=0}^{N-1}a_n t^{n+\alpha}+g_N(t),\qquad g_N(t)=\mathcal{O}(t^{N+\alpha}),\quad t\to 0^+,\quad \alpha>-1. \tag{3.2}$$

An intuitive argument tells us what to do to approximate this kind of integrals for large $z$. As $\Re(z)$ increases, the absolute value of the exponential, $e^{-t\Re(z)}$, concentrates more at $t=0$. Then, for large $z$, the main contribution of the integrand to the integral comes from $t$ near zero. Therefore, one hopes that only the value of $g(t)$ near $t=0$ is relevant. Thence, we approximate $g(t)$ at $t=0$ as in (3.2), substitute this approximation into the integral (3.1) and integrate term by term:

$$F(z)=\sum_{n=0}^{N-1}\frac{a_n n!}{z^{n+\alpha+1}}+R_N(z),\qquad R_N(z)\equiv\int_0^\infty e^{-zt}g_N(t)dt.$$

It is clear that $\Phi_n(z)\equiv z^{-n-\alpha-1}$ is an asymptotic sequence (2.3). Moreover, Watson proved [30] that, under suitable conditions on $g(t)$, the remainder satisfies (2.4): $R_N(z)=\mathcal{O}(z^{-N-\alpha-1})$ as $z\to\infty$. Hence we can write:

$$F(z)\sim\sum_{n=0}^{\infty}\frac{a_n n!}{z^{n+\alpha+1}}. \tag{3.3}$$

See [22], for the conditions on $g(t)$ (with on p. 71-72 details for real $z$ and on p. 106-109 for complex $z$).

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Many times in practice it happens that the remainder in the expansion of $g(t)$ may be bounded in the form $|g_N(t)| \le c_N t^{N+\alpha}$, where $c_N$ is a positive constant. Then,

$$|R_N(z)| \le \frac{c_N N!}{(\Re(z))^{N+\alpha+1}}.$$

The integral $F(x)$ studied in the preceding section is an example of application of Watson's lemma. For this integral $g(t) = (1+t)^{-1}$, $|g_N(t)| \le t^N$ and $|R_N(x)| \le \frac{N!}{x^{N+1}}$.

Another way to obtain the asymptotic expansion (3.3), more pragmatically than the intuitive argument described above, is the following. Just make the change of variable $t \to tz$ (suppose that $z$ is positive) in (3.1), substitute (3.2) and integrate term by term:

$$F(z) = \frac{1}{z}\int_0^\infty e^{-t} g\left(\frac{t}{z}\right) dt = \sum_{n=0}^{N-1} \frac{a_n n!}{z^{n+\alpha+1}} + R_N(z).$$

The remaining methods that we call *Classical methods based on Watson's lemma* consist of reducing the integral to another integral to which Watson's lemma may be applied. We show this procedure with the important example of the saddle point method.

**3.2. The saddle point method [Debye, 1909].** This method applies to contour integrals of the form:

$$F(z) \equiv \int_L e^{zh(w)} g(w) dw, \qquad z \to \infty, \tag{3.4}$$

where $L$ is a path in the complex plane and the functions $g$ and $h$ are analytic wherever it is needed to perform the manipulations described below. The saddle points of the integrand are the points where $h'(w)$ vanishes. These points may give the main contributions to the asymptotic behavior of the function $F(z)$ for large values of $z$. Other points that may give significant contributions are the endpoints of the path $L$, if these are finite. In fact one tries to determine if $\Re(zh(w))$ is maximal at one of the saddle points or at the endpoints of $L$. If a saddle point, say $w_0 \in L$, gives the main contribution, and $e^{\Re(zh(w))}$ has its peak value at $w_0$, the asymptotic expansion is obtained from local expansions of $h(w)$ and $g(w)$ at $w_0$. This method is also called Laplace's method. See [22] (pp. 121 - 127) and [31] (pp. 55 - 66).

If one of the saddle points is not located on $L$, and one has verified that this point will give the main contributions, one tries to deform the path $L$ such that the new path goes through that saddle point, say $w_0$.

The method is difficult to describe in all its generality, because several special situations may occur. We give a few examples to demonstrate the method.

An extension of the saddle point method is the method of steepest descent, in which the new path through $w_0$ is defined by $\Im(zh((w)) = \Im(zh(w_0))$, again, if this is possible, which depends on the original path $L$ and on the functions $h(w)$, $g(w)$, but also on the phase of the complex parameter $z$. Observe that, in fact, the path of steepest descent is a path where $e^{zh(w)}$ does not oscillate and concentrates at $w_0$, the concentration being greater for larger values of $|z|$. We explain in Figure 3.1 the situation that the new path $L'$, a steepest descent path, runs through $w_0$.

If the saddle point $w_0$ is simple ($h'(w_0) = 0$, $h''(w_0) \ne 0$), then there are just one steepest descent path and one steepest ascent path of the function $\Re(h(w))$ through $w_0$ (see Figure 3.1). Moreover, over both the steepest descent and ascent paths, we have $\Im(zh(w)) = \Im(zh(w_0))$, a constant (this may be taken as the definition of these paths). For a detailed discussion we refer to [22] (pp. 125 - 127) or [31] (pp. 84 - 103).

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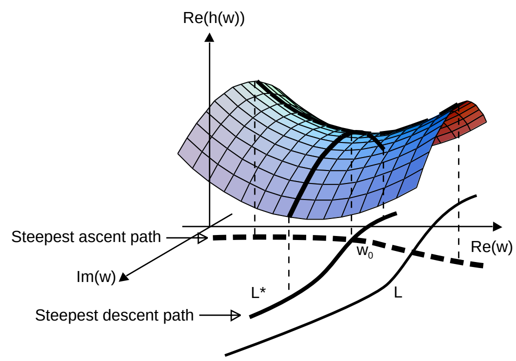


FIG. 3.1. *Typical plot of the real part of $h(w)$ over a simple saddle point $w_0$. Crossing this point we can find a steepest descent path (dashed lines) and a steepest ascent path ($L^*$).*

Difficulties may arise when the new path cannot be reached without disturbing the convergence of the integral, or without passing singular points of $h(w)$ or $g(w)$. Also, it is possible that $L$ can be deformed into the path of steepest descent only by introducing extra paths. For example,

$$F(z) = \int_{-1}^{1} e^{izw^2} g(w)\, dw \tag{3.5}$$

has a saddle point at the origin, and when $z > 0$ the steepest descent path is the diagonal $u = v$, where $w = u + iv$. (Integrals with purely imaginary phase functions are usually handled by the method of stationary phase.) The best strategy for using the steepest descent path seems to be as follows. Integrate from $w = -1$ along a path $L_{-1}$ defined by $u = -\sqrt{1+v^2}, v \le 0$, on which $\Im(iw^2)$ has the constant value 1 (as at $w = -1$), then integrate along the diagonal, and then along the path $L_1$ defined by $u = \sqrt{1+v^2}, v \ge 0$ to $w = 1$ (see Figure 3.2). We assume that $g(w)$ is analytic in the domain where the integration occurs, and that the function $g(w)$ is of limited growth at infinity. The integrals along $L_{\pm 1}$ can be handled by using Watson's lemma.

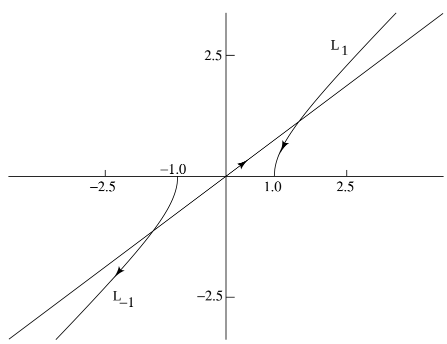


FIG. 3.2. *Steepest descent path contour for $F(z)$ of (3.5).*

For a further demonstration we consider the integral representation for the gamma function

$$\Gamma(z) = \int_0^\infty u^{z-1} e^{-u}\, du, \qquad \Re(z) > 0$$

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We are interested in approximating $\Gamma(z)$ for large $z$. First we consider positive $z$. After the change of variable $u = zt$, we have

$$\Gamma(z) = z^z \int_0^\infty e^{z(\log t - t)} \frac{1}{t} dt.$$

This integral is of the form (3.4) with $g(w) = w^{-1}$, $h(w) = \log w - w$ and $L = [0, \infty)$. The function $h(w)$ has a unique saddle point: $w_0 = 1$. At this point $h(w_0) = -1 \in \mathbb{R}$ and therefore, $\Im(h(w_0)) = 0$. The steepest descent/ascent paths through $w_0 = 1$ consist of the points $w$ in the complex plane that satisfy $\Im(h(w)) = \Im(h(w_0)) = 0$. In polar coordinates $w = re^{i\theta}$ we have $\Im(h(w)) = \theta - r \sin\theta = 0$. This equation has two solutions:

$$r(\theta) = \frac{\theta}{\sin\theta}, \qquad \theta \in (-\pi, \pi), \qquad (L')$$

and

$$\theta = 0, \qquad r \in (0, \infty). \qquad (L^*)$$

The first solution is the steepest ascent path and the second one the steepest descent path (see Figures 3.3 and 3.4).

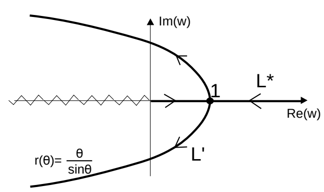


FIG. 3.3. *The steepest ascent path $L'$ is defined by the curve $r(\theta) = \theta/\sin\theta$ $\theta \in (-\pi, \pi)$ and the steepest descent path $L^*$ is the positive real axis $\theta = 0$, $r \in (0, \infty)$.*

The saddle point method consists of the steps (A), (B) and (C) described below.

(A) Deform $L \to L^*$.
In this example this step is not necessary because $L = L^*$.

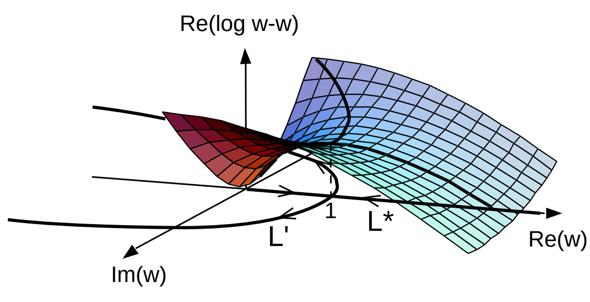


FIG. 3.4. *Plot of $\Re(\log w - w) = \log r - r\cos\theta$. This function has a simple saddle point at $w_0 = 1$, a steepest ascent path $L'$ and a steepest descent path $L^*$.*

Observe that over the steepest descent path $L^*$, the real part of $h(w)$ is (see Figure 3.5):

$$\Re(h(w)) = h(t) = \log t - t, \qquad t \in (0, \infty).$$

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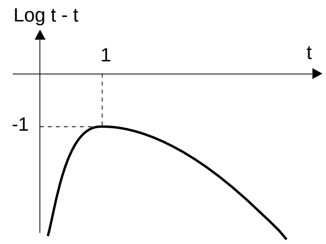


FIG. 3.5. *Plot of* $\Re(h(w)) = h(t) = \log t - t$, $t \in (0, \infty)$.

The function $h(t)$ has a maximum at $t = 1$, $(h(1) = -1)$ and then it decreases to $-\infty$ when $t \to 0^+$ or $t \to \infty$.

(B) We carry out the change of variable to obtain a standard form

$$t - \log t - 1 = \frac{1}{2}x^2, \quad \operatorname{sign}(t-1) = \operatorname{sign} x. \tag{3.6}$$

In fact this transformation can be written as

$$x = (t-1)\sqrt{\frac{2(t - \log t - 1)}{(t-1)^2}},$$

where the square root is positive for all $t > 0$.
Then, the function $\Gamma(z)$ reads:

$$\Gamma(z) = z^z e^{-z} \int_{-\infty}^{\infty} e^{-\frac{1}{2}zx^2} f(x)\,dx, \quad f(x) \equiv \frac{1}{t}\frac{dt}{dx} = \frac{x}{t-1}. \tag{3.7}$$

This representation is also valid for $\Re(z) > 0$. For large $z$ (with $\Re(z) > 0$), the main contribution of the integrand to the integral comes from $x$ near zero. Therefore, only the value of $f(x)$ near $x = 0$ is relevant. As in Watson's lemma, we approximate $f(x)$ at $x = 0$ (the function $t(x)$ is defined implicitly in (3.6)):

$$f(x) = 1 + a_1 x + a_2 x^2 + \dots .$$

(C) Substitute this approximation in (3.7) and integrate term by term to obtain the expansion:

$$\Gamma(z) \sim z^z e^{-z} \sum_{k=0}^{\infty} a_{2k} \int_{-\infty}^{\infty} e^{-\frac{1}{2}zx^2} x^{2k}\,dx,$$

and by evaluating the integrals we obtain Stirling's expansion

$$\Gamma(z) \sim \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z} \sum_{k=0}^{\infty} \frac{\gamma_k}{z^k}, \quad \gamma_k = a_{2k} \frac{2^k \Gamma(k+1/2)}{\Gamma(1/2)}.$$

The first few coefficients are (cf. [1], p. 257)

$$\gamma_0 = 1,\ \gamma_1 = \frac{1}{12},\ \gamma_2 = \frac{1}{288},\ \gamma_3 = -\frac{139}{51840},\ \gamma_4 = -\frac{571}{2488320}.$$

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It is of interest to observe that the following integral for the reciprocal gamma function can be expanded with the same transformation. We consider

$$\frac{1}{\Gamma(z+1)} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} u^{-z-1} e^{u}\, du,$$

where (see [28], p. 48) the second integral starts at $-\infty$, with phase $u = -\pi$, encircles the origin in positive direction, and terminates at $-\infty$, with phase $u = \pi$. This integral is defined for all complex $z$. For $z > 0$ we have

$$\frac{1}{\Gamma(z+1)} = \frac{z^{-z}e^{z}}{2\pi i}\int_{-\infty}^{(0+)} e^{z(t-\log t-1)}\frac{dt}{t}.$$

For this integral the steepest descent path is the curve $L'$ defined by $r(\theta) = \theta/\sin\theta\ \theta \in (-\pi,\pi)$ (see Figure 3.3). We use the transformation (3.6); the path $L'$ is mapped onto the imaginary axis in the $z-$plane. The result is the expansion

$$\frac{1}{\Gamma(z)} \sim \frac{z^{-z+\frac{1}{2}}e^{z}}{\sqrt{2\pi}}\sum_{k=0}^{\infty}(-1)^k\frac{\gamma_k}{z^k},$$

with the same coefficients $\gamma_k$ as in Stirling's expansion.

**3.3. Interchanging summation and integration.** All the classical methods based on Watson's lemma and Laplace's method, at some point, deal with integrals of the form

$$\int_L h(w) g\left(\frac{w}{z}\right) dw,$$

where $g(t)$ has an asymptotic expansion in powers of $w$ at $w = 0$:

$$g(w) \sim \sum_{n=0}^{\infty} a_n w^n, \qquad w \to 0.$$

Then we substitute this expansion of $g(w)$ into the integral and interchange sum and integral obtaining an asymptotic expansion for large $z$:

$$\int_L h(w) g\left(\frac{w}{z}\right) dw \sim \sum_{n=0}^{\infty}\frac{a_n}{z^n}\int_L h(w) w^n\, dw, \qquad z \to \infty.$$

But something goes wrong when the integration contour $L$ goes to infinity and the kernel $h(w)$ converges slowly to 0 when $w \to \infty$. In this case, the coefficients of the expansion, $\int_L h(w)w^n dw$, may not be defined and the classical expansion makes no sense. Consider the following example, an integral representation of a combination of Bessel functions (the Hankel function):

$$\begin{aligned} H_\nu^{(1)}(z) &= J_\nu(x) + iY_\nu(x) \\ &= -\frac{2i(x/2)^{-\nu}e^{ix}}{\sqrt{\pi}\Gamma(1/2-\nu)}\int_0^\infty \frac{e^{ixt}}{[t(t+2)]^{\nu+1/2}}dt, \quad x > 0, \quad |\Re(\nu)| < \frac{1}{2}. \end{aligned}$$

We are interested in the asymptotic approximation of the integral

$$F(x) \equiv \frac{2^{\nu+1/2}}{x^{\nu-1/2}}\int_0^\infty \frac{e^{ixt}}{[t(2+t)]^{\nu+1/2}}dt$$

for large $x$. After the change of variable $t \to t/x$, we have

$$F(x) = \int_0^\infty \frac{e^{it}}{[t(1+t/(2x))]^{\nu+1/2}} dt.$$

If we substitute the expansion $[1+t/(2x)]^{-\nu-1/2} \sim \sum_{n=0}^\infty a_n t^n x^{-n}$ into the integral and interchange summation and integration, we obtain, formally:

$$F(x) \sim \sum_{n=0}^\infty \left[ \int_0^\infty \frac{e^{it}}{\sqrt{t}} t^{n-\nu} dt \right] \frac{a_n}{x^n}$$

This expansion does not make sense, not even formally, because the moments of $e^{it}$ are not defined. Classical methods based on Watson's lemma do not apply to this integral, although in this case, because the integrand is analytic, we can turn the path of integration to the positive imaginary axis to obtain a Laplace integral. With the following method, we can obtain asymptotic expansions of Fourier transforms.

**4. Summability methods [Olver, 1974].** This method applies to Fourier transforms for large values of the parameter. We consider (see [23])

$$F(x) \equiv \int_0^\infty e^{ixt} f(t) dt, \qquad x \in \mathbb{R}, \qquad x \to \infty.$$

We require for $f(t)$:

i) $f(t) = \sum_{n=0}^{N-1} a_n t^{n+\alpha} + f_N(t)$, where $f_N(t) = \mathcal{O}(t^{N+\alpha})$ when $t \to 0^+$, $\alpha > -1$ and this expansion may be differentiated term by term an infinite number of times.

ii) $f(t) \in \mathcal{C}^\infty(0,\infty)$ and $\int_1^\infty f^{(k)}(t) e^{ixt} dt$ converges uniformly at infinity for $x > x_0$.

Just substituting the expansion i) into the integral and interchanging summation and integration makes no sense:

$$\int_0^\infty e^{ixt} f(t) dt = \sum_{n=0}^{N-1} a_n \int_0^\infty t^{n+\alpha} e^{ixt} dt + \int_0^\infty f_N(t) e^{ixt} dt,$$

because the integrals $\int_0^\infty t^{n+\alpha} e^{ixt} dt$ are not defined. But we modify the kernel $e^{ixt}$ to make these integrals convergent: multiply it by a negative exponential $e^{-\epsilon t}$, $\epsilon > 0$. Then:

$$\int_0^\infty e^{ixt-\epsilon t} f(t) dt = \sum_{n=0}^{N-1} a_n \int_0^\infty t^{n+\alpha} e^{(ix-\epsilon)t} dt + \int_0^\infty f_N(t) e^{(ix-\epsilon)t} dt. \tag{4.1}$$

This is a trivial equality (which makes sense). But, is the right hand side an asymptotic expansion for large $x$? If it is, it is an asymptotic expansion of $\int_0^\infty e^{ixt-\epsilon t} f(t) dt$. Then, does it have something to do with an asymptotic expansion of $F(x)$? The answer to both questions is yes and this is explained below. Roughly speaking, the asymptotic expansion for $\int_0^\infty e^{ixt} f(t) dt$ is obtained from the above asymptotic expansion for $\int_0^\infty e^{ixt-\epsilon t} f(t) dt$ by taking the limit $\epsilon \to 0^+$.

Integrating by parts in the last term in the above equation, we obtain:

$$\int_0^\infty f_N(t) e^{(ix-\epsilon)t} dt = \frac{1}{(\epsilon - ix)^N} \int_0^\infty f_N^{(N)}(t) e^{(ix-\epsilon)t} dt.$$

This make sense thanks to hypothesis ii). At this point we need to apply the following lemma [29]:

LEMMA 4.1. *If the integral $\int_0^\infty f(t)dt$ exists as an improper Riemann integral, then*

$$\lim_{\epsilon\to 0^+}\int_0^\infty e^{-\epsilon t}f(t)dt=\int_0^\infty f(t)dt.$$

From ii) and this lemma we have:

$$\lim_{\epsilon\to 0^+}\int_0^\infty e^{ixt-\epsilon t}f(t)dt=\int_0^\infty e^{ixt}f(t)dt$$

and

$$\lim_{\epsilon\to 0^+}\int_0^\infty f_N^{(N)}(t)e^{(ix-\epsilon)t}dt=\int_0^\infty f_N^{(N)}(t)e^{ixt}dt.$$

On the other hand, straightforward computation shows that

$$\int_0^\infty t^{n+\alpha}e^{(ix-\epsilon)t}dt=\frac{\Gamma(n+\alpha+1)}{(\epsilon-ix)^{n+\alpha+1}}.$$

Then, taking the limit $\epsilon\to 0^+$ in (4.1) we obtain:

$$\int_0^\infty e^{ixt}f(t)dt=\sum_{n=0}^{N-1}a_n\frac{i(n+\alpha)!}{x^{n+1+\alpha}}e^{\frac{i\pi(n+\alpha)}{2}}+\frac{i^N}{x^N}\int_0^\infty f_N^{(N)}(t)e^{ixt}dt. \tag{4.2}$$

This is a formal asymptotic expansion for large $x$. It is a valid asymptotic expansion if we can show that $\int_0^\infty f_N^{(N)}(t)e^{ixt}dt=o(1)$ when $x\to\infty$. But this is just the Riemann-Lebesgue lemma for Riemann integrals [31]. Therefore:

$$\frac{i^N}{x^N}\int_0^\infty f_N^{(N)}(t)e^{ixt}dt=o(x^{-N}),\qquad x\to\infty$$

and (4.2) is a valid asymptotic expansion for large $x$.

Summability methods apply to other integral transforms with oscillatory kernels such as Hankel transforms [31].

But the situation may be worse when the integral does not have an oscillatory kernel, but an integrand with just an algebraic decay at infinity. Then neither the classical methods described above nor the summability methods apply. Consider the following example, the third symmetric elliptic integral:

$$R_J(x,y,w,z)\equiv\frac{3}{2}\int_0^\infty\frac{dt}{\sqrt{(t+x)(t+y)(t+w)}(t+z)},\qquad x,y,w,z>0.$$

We are interested in approximations of this function for large $z$. The intuitive argument used in the classical methods fails: if we expand

$$\frac{1}{\sqrt{(t+x)(t+y)(t+w)}}\sim\sum_{n=0}^\infty a_n t^n$$

and substitute this expansion into the integral we obtain, formally,

$$R_J(x,y,w,z)\sim\sum_{n=0}^\infty a_n\left[\int_0^\infty\frac{t^n dt}{t+z}\right].$$

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But this is not a formal asymptotic expansion for large $z$. Moreover, all the terms of the sum are infinite.

The pragmatic procedure produces a formal asymptotic expansion: if we substitute the expansion

$$\frac{1}{t+z} \sim \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-t)^n}{z^n}$$

into the integral, we obtain the expansion

$$R_J(x,y,w,z) \sim \sum_{n=0}^{\infty}\left[\int_0^{\infty}\frac{t^n dt}{\sqrt{(t+x)(t+y)(t+w)}}\right]\frac{(-1)^n}{z^{n+1}}.$$

This is a formal asymptotic expansion but, for $n \geq 1$ the coefficients are infinite. Summability methods do not apply because we do not have an oscillatory kernel.

McClure and Wong gave a solution [21], [31]. If an expansion at $t=0$ is not successful, try an expansion at $t=\infty$:

$$\frac{1}{\sqrt{(t+x)(t+y)(t+w)}} \sim \frac{1}{t^{3/2}}\sum_{n=0}^{\infty}\frac{a_n}{t^n}, \qquad t\to\infty.$$

Substituting this expansion into the integral and interchanging summation and integration, we obtain the formal asymptotic expansion:

$$R_J(x,y,w,z) \sim \sum_{n=0}^{\infty} a_n\int_0^{\infty}\frac{dt}{t^{n+3/2}(t+z)} = \sum_{n=0}^{\infty}\frac{a_n}{z^{n+3/2}}\int_0^{\infty}\frac{dt}{t^{n+3/2}(t+1)}.$$

Once more, the coefficients of the expansion are divergent. But now the origin of the divergence is different: it is at $t=0$, not at $t=\infty$. And this divergence may be repaired by using the theory of distributions.

**5. Distributional methods [McClure and Wong, 1978].** This technique applies to several kinds of integrals. We consider here the important example of the Stieltjes transforms [21]:

$$F(z) \equiv \int_0^{\infty}\frac{f(t)dt}{t+z}, \qquad |\text{phase}\, z| < \pi, \qquad z\to\infty. \tag{5.1}$$

We require for the function $f(t)$:
i) $f(t) \in L_{Loc}[0,\infty)$

ii) $f(t) = \sum_{n=0}^{N-1}\frac{a_n}{t^{n+\alpha}} + f_N(t)$, where $f_N(t) = \mathcal{O}(t^{-N-\alpha})$ when $t\to\infty$ and $0<\alpha<1$.

Substituting the expansion ii) into (5.1) and replacing summation and integration does not make sense, because in

$$\int_0^{\infty}\frac{f(t)dt}{t+z} = \sum_{n=0}^{N-1} a_n\int_0^{\infty}\frac{dt}{t^{n+\alpha}(t+z)} + \int_0^{\infty}\frac{f_N(t)dt}{t+z},$$

the integrals $\int_0^{\infty}\frac{dt}{t^{n+\alpha}(t+z)}$ are not defined. But observe that if we proceed formally, integrating these integrals by parts, and forgetting about the boundary terms, we obtain the finite integrals:

$$\frac{n!(-1)^n}{(\alpha)_n}\int_0^{\infty}\frac{dt}{t^{\alpha}(t+z)^{n+1}}.$$

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These formal manipulations may be made rigorous by means of the theory of distributions. The expansion ii) is just an identity between the functions $f(t)$, $t^{-n-\alpha}$ and $f_N(t)$. Consider the space of rapidly decreasing functions $\mathcal{S}$ and the set of tempered distributions acting on the functions $\varphi$ of this space. Each of the functions $f(t)$, $t^{-n-\alpha}$ and $f_N(t)$ has an associated tempered distribution that we denote by the same symbol, but in bold font:

$$< \mathbf{f}, \varphi > \equiv \int_0^\infty f(t)\varphi(t)dt, \qquad \varphi \in \mathcal{S},$$

$$< \mathbf{t}^{-\mathbf{n}-\alpha}, \varphi > \equiv \frac{1}{(\alpha)_n}\int_0^\infty t^{-\alpha}\varphi^{(n)}(t)dt, \qquad \varphi \in \mathcal{S},$$

$$< \mathbf{f_n}, \varphi > \equiv (-1)^n\int_0^\infty f_{n,n}(t)\varphi^{(n)}(t)dt, \qquad \varphi \in \mathcal{S},$$

where $f_{n,n}(t)$ is the $n-$th integral of $f_n(t)$: $f_0(t) = f(t)$ and, for $k = 0, 1, ..., n-1$:

$$f_{n,k+1}(t) = -\int_t^\infty f_{n,k}(u)du = \frac{(-1)^{k+1}}{k!}\int_t^\infty (u-t)^k f_n(u)du.$$

Now the key point is to find a relation (similar to ii)) between these distributions. This relation was obtained by McClure and Wong [21]:

$$\mathbf{f} = \sum_{n=0}^{N-1} a_n \mathbf{t}^{-\mathbf{n}-\alpha} + \sum_{n=0}^{N-1} M[f; n+1]\delta^{(\mathbf{n})} + \mathbf{f_N}, \tag{5.2}$$

where $M[f;z]$ is the Mellin transform of $f(t)$, $M[f;z] \equiv \int_0^\infty t^{z-1}f(t)dt$ or its analytic continuation.

We apply both sides of (5.2) to the function $\varphi(t) \equiv \frac{e^{-\epsilon t}}{t+z}$, $\epsilon > 0$, of $\mathcal{S}$, and we obtain:

$$\int_0^\infty \frac{e^{-\epsilon t}f(t)}{t+z}dt = \sum_{n=0}^{N-1}\frac{a_n}{(\alpha)_n}\int_0^\infty \frac{(-1)^n e^{-\epsilon t}}{t^\alpha (t+z)^{n+1}}dt$$
$$+ \sum_{n=0}^{N-1}\frac{M[f;n+1]}{n!} < \delta^{(n)}, \frac{e^{-\epsilon t}}{t+z} > + N!\int_0^\infty \frac{f_{N,N}(t)e^{-\epsilon t}}{(t+z)^{N+1}}dt + \mathcal{O}(\epsilon),\ \epsilon \to 0^+.$$

We take the limit $\epsilon \to 0^+$ and use the dominated convergence theorem to obtain:

$$\int_0^\infty \frac{f(t)}{t+z}dt = \frac{\pi}{\sin(\pi\alpha)}\sum_{n=0}^{N-1}\frac{a_n(-1)^n}{z^{n+\alpha}} + \sum_{n=0}^{N-1}\frac{M[f;n+1]}{z^{n+1}} + R_N(z),$$

where

$$R_N(z) \equiv N!\int_0^\infty \frac{f_{N,N}(t)}{(t+z)^{N+1}}dt.$$

We have obtained a formal asymptotic expansion containing two asymptotic sequences: $\{z^{-n-\alpha}\}$ and $\{z^{-n-1}\}$. It remains to be shown that it is a valid asymptotic expansion. In [8] it is shown that in fact $R_N(z) = \mathcal{O}(z^{-N-\alpha})$ when $z \to \infty$. Moreover, in [31] and in [8] it is shown that if the remainder in the expansion ii) of $f(t)$ satisfies the bound $|f_N(t)| \le c_N t^{-N-\alpha}$, where $c_N$ is a constant, then

$$|R_N(z)| \le \frac{\pi c_N}{\sin(\pi\alpha)}\frac{1}{|z|^{N+\alpha}}.$$

We have considered in hypothesis ii) only the case $0 < \alpha < 1$. The case $\alpha = 1$ is similar although a little bit more cumbersome. In this case the expansion contains logarithmic terms in $z$ [31].

The distributional approach applies to other integral transforms, such as the Laplace and Fourier transforms when the parameter of the transformation is small: $\int_0^\infty e^{-zt} f(t)dt$, $z \to 0$ or $\int_0^\infty e^{ixt} f(t)dt$, when $x \to 0$ [31].

For the example of the third symmetric elliptic integral, considered as a motivation for this section, we have (we only show the dominant term of the expansion) [18]:

$$R_J(x, y, w, z) \sim 3\frac{R_F(x, y, w)}{z}, \qquad z \to \infty,$$

where $R_F(x, y, w)$ is the first symmetric elliptic integral.

All the asymptotic methods described up to here are based on an expansion of the integrand at some appropriate point. There are some other methods, that are not based on such an expansion, but on a very different idea: Mellin transform techniques, integration by parts, and so on. See the references [22], [27], and [31].

**6. Asymptotic relations in the Askey scheme I.** Taking a look at the Askey table (see Figure 6.1) we see that there are several known limits between polynomials located at different levels in the table. The limit is always taken over a parameter contained in the polynomials, the variable $x$ and the degree $n$ remain fixed.

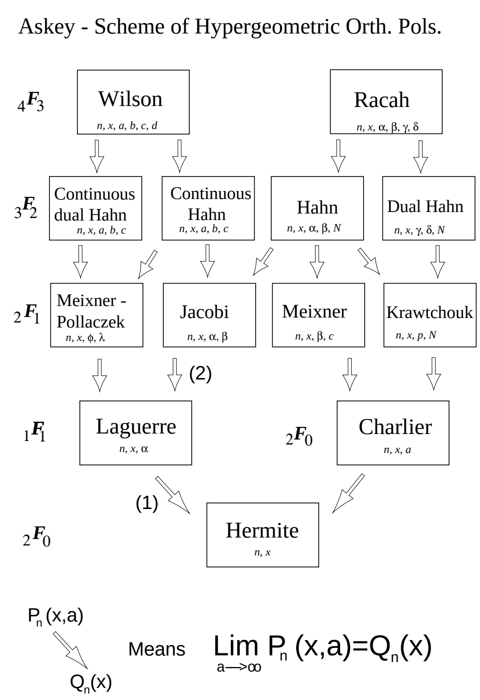


FIG. 6.1. *Askey table of hypergeometric polynomials. Every arrow between two polynomials indicates a known limit between those polynomials.*

For example, the arrow between the Laguerre and the Hermite polynomials means that a limit exists,

$$\lim_{\alpha\to\infty} \alpha^{-n/2} L_n^{(\alpha)}(x\sqrt{\alpha}+\alpha) = \frac{(-2)^n}{n!} H_n\left(\frac{x}{\sqrt{2}}\right),$$

The arrow between the Jacobi and the Laguerre polynomials can be illustrated by

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right) = L_n^{(\alpha)}(x).$$

We wonder if there is something else beyond these limits. That is, we wonder if there is a complete asymptotic expansion of the form:

$$\alpha^{-n/2} L_n(\alpha)(x\sqrt{\alpha}+\alpha) \sim \frac{(-2)^n}{n!} H_n\left(\frac{x}{\sqrt{2}}\right) + \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \ldots.$$

If this is the case, the limits are obtained just from the first term of this expansion. We restrict ourselves in this paper to the limits of the type: any polynomial $p_n(x) \to$ Hermite polynomial $H_n(x)$. The theory developed here may be extended to the case: any polynomial $\to$ Charlier polynomial [9]. In general, it may be extended to any case: polynomial A $\to$ polynomial B, where polynomial B is below polynomial A in the Askey table [10].

The key point to answer the question stated in the above paragraph is to consider the generating function $F(x,w)$ of a generic family of polynomials $p_n(x)$:

$$F(x,w) = \sum_{n=0}^{\infty} p_n(x) w^n \Rightarrow p_n(x) = \frac{1}{2\pi i}\int_\Gamma F(x,w)\frac{dw}{w^{n+1}},$$

and the generating function for the Hermite polynomials:

$$e^{2xw-w^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} w^n \Rightarrow H_n(x) = \frac{n!}{2\pi i}\int_\Gamma \frac{e^{2xw-w^2}}{w^{n+1}} dw.$$

In both integrals above the contour $\Gamma$ is a small circle around the origin $w=0$ ($\Gamma$ must be inside the analyticity region of the generating functions). For convenience we replace the variable $x$ in the Hermite polynomials by $X$ and introduce the new variable $B$, replacing $w \to Bw$ in the above expansion:

$$e^{2XBw-B^2w^2} = \sum_{n=0}^{\infty} \frac{H_n(X)}{n!} B^n w^n \Rightarrow H_n(X) = \frac{n!}{2\pi i B^n}\int_\Gamma e^{2XBw-B^2w^2}\frac{dw}{w^{n+1}}.$$

We define the function $f(x,w)$ as the ratio of the generating functions $F(x,w)$ and $e^{2XBw-B^2w^2}$:

$$F(x,w) = e^{2XBw-B^2w^2} f(x,w).$$

Because the contour $\Gamma$ can be taken close to the origin, the polynomials $p_n(x)$ and $H_n(x)$ are defined by the values of their respective generating functions around the origin. Then, "if both generating functions are similar around the origin, the polynomials will be similar, too". Therefore, the key point is to choose $X$ and $B$ (free parameters up to now) such that these generating functions are as similar as possible around $w=0$, that is, $f(x,w) \simeq 1$ (in an optimal way) near $w=0$. For this purpose we write:

$$F(x,w) = e^{\tilde{F}(x,w)},$$

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and expand $\tilde{F}(x,w)$ at $w=0$: $\tilde{F}(x,w)=\log F(x,w)=a_1w+a_2w^2+a_3w^3+...$

We assume now that $\tilde{F}(x,w)$ contains a large parameter $\alpha$ in such a way that $a_k=\mathcal{O}(\alpha)$ when $\alpha\to\infty$ for $k=1,2,3,....$ This is the case for all the polynomials in the Askey table. But instead of constructing a general and surely not quite clear theory, we continue the explanation of the method by means of one concrete example: the Laguerre polynomials. The method for the remaining polynomials of the Askey table is similar.

The generating function of the Laguerre polynomial is:

$$(1-w)^{-\alpha-1}e^{-wx/(1-w)}=\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)w^n. \tag{6.1}$$

Therefore,

$$\tilde{F}(x,w)=-\frac{wx}{1-w}-(\alpha+1)\log(1-w)$$

and it is clear in this example that $a_k=\mathcal{O}(\alpha)$ when $\alpha\to\infty$. The function $f(x,w)$ reads

$$f(x,w)=e^{\tilde{F}(x,w)-2XBw+B^2w^2}\equiv e^{\tilde{f}(x,w)},$$

and we expand

$$\tilde{f}(x,w)=\log f(x,w)=b_1w+b_2w^2+b_3w^3+\dots\ .$$

Obviously $b_1=a_1-2XB$, $b_2=a_2+B^2$ and $b_k=a_k$ for $k=3,4,5,\dots$ . Therefore, $b_k=\mathcal{O}(\alpha)$ when $\alpha\to\infty$ for $k=3,4,5,....$

The notion "$f(x,w)\simeq 1$ (in an optimal way) for $w\to 0$" means that we set as many coefficients $b_k$ as possible to 0. By choosing $X$ and $B$ to be the solution of the system

$$\begin{cases} a_1-2XB=0,\\ a_2+B^2=0,\end{cases}$$

we have $b_1=b_2=0$ and also $A=\mathcal{O}(\sqrt{\alpha})$ and $B=\mathcal{O}(\sqrt{\alpha})$ when $\alpha\to\infty$. Moreover, under these circumstances $f(x,w)=e^{b_3w^3+b_4w^4+\cdots}$ and then:

$$\begin{aligned} f(x,w)&=1+\left[b_3w^3+b_4w^4+...\right]+\frac{1}{2}\left[b_3w^3+b_4w^4+...\right]^2+...=\\ &1+c_3w^3+c_4w^4+c_5w^5+c_6w^6+...,\end{aligned}$$

where $c_3=b_3$, $c_4=b_4$, $c_5=b_5$, $c_6=b_6+b_3^2/2$,... But $b_k=\mathcal{O}(\alpha)$ when $\alpha\to\infty$ then means that

$$c_k=\mathcal{O}(\alpha^{\lfloor k/3\rfloor}),\qquad \alpha\to\infty.$$

Taking into account all these preliminaries, we can write the Laguerre polynomials $L_n^{(\alpha)}(x)$ in the form:

$$\begin{aligned} L_n^{(\alpha)}(x)&=\frac{1}{2\pi i}\int_\Gamma F(x,w)\frac{dw}{w^{n+1}}=\frac{1}{2\pi i}\int_\Gamma e^{2XBw-B^2w^2}f(x,w)\frac{dw}{w^{n+1}}\\ &=\frac{1}{2\pi i}\int_\Gamma e^{2XBw-B^2w^2}\sum_{k=0}^{n}c_kw^k\frac{dw}{w^{n+1}}\\ &=\sum_{k=0}^{n}c_k\frac{1}{2\pi i}\int_\Gamma e^{2XBw-B^2w^2}\frac{dw}{w^{n-k+1}}=B^n\sum_{k=0}^{n}\frac{c_k}{B^k}\frac{H_{n-k}(X)}{(n-k)!}.\end{aligned} \tag{6.2}$$

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And taking into account that $c_k = \mathcal{O}(\alpha^{\lfloor k/3 \rfloor})$, $B^k = \mathcal{O}(\alpha^{k/2})$ and $H_{n-k}(X) = \mathcal{O}(\alpha^{(n-k)/2})$ when $\alpha \to \infty$, we conclude that the terms of the expansion (6.2) are of the order

$$\frac{c_k}{B^k} H_{n-k}(X) = \mathcal{O}(\alpha^{n/2+\lfloor k/3 \rfloor - k}), \qquad \alpha \to \infty.$$

It follows that the expansion (6.2) of Laguerre polynomials in terms of Hermite polynomials is an asymptotic expansion for large $\alpha$.

The preceding discussion is not only valid for Laguerre polynomials; it may be generalized to other polynomials $p_n(x)$ of the Askey table [9], [16] . Hermite polynomials are not the only polynomials that may be taken as basic approximants. Laguerre or Charlier may be taken instead [9], [17]. From these expansions, known and new limits between polynomials may be obtained (see details in [9], [16], [17]).

**7. A simplified saddle point method.** In this section we introduce a new and simpler version of the saddle point method and apply it to three examples of orthogonal polynomials: Charlier, Laguerre and Jacobi polynomials to obtain asymptotic (and convergent) expansions of these polynomials for large $n$ and/or $x$.

From the generating function of the Laguerre polynomials (6.1), we have that

$$L_n^{(\alpha)}(x) = \frac{1}{2\pi i} \int_\Gamma e^{xw/(w-1)} (1-w)^{-\alpha-1} \frac{dw}{w^{n+1}},$$

where $\Gamma$ is a circle of radius $r < 1$ with center at $w = 0$. In general, we consider integrals of the form

$$F(n) \equiv \int_\Gamma f(w) e^{n g(w)} \frac{dw}{w^{n+1}} = \int_\Gamma f(w) e^{n\varphi(w)} \frac{dw}{w}, \tag{7.1}$$

where we have defined $\varphi(w) \equiv g(w) - \log w$ and

i) $\Gamma$ is a small circle with center at $w = 0$.

ii) $f(w)$ and $g(w)$ are analytic on and inside $\Gamma$.

For example, the above integral representation of the Laguerre polynomials fits into the form (7.1) if we write

$$L_n^{(\alpha)}(nx) = \frac{1}{2\pi i} \int_\Gamma e^{n[xw/(w-1) - \log w]} \frac{1}{(1-w)^{\alpha+1}} \frac{dw}{w}.$$

We have $\varphi(w) = xw/(w-1) - \log w$ and $f(w) = (1-w)^{-\alpha-1}$.

The standard saddle point method consists of:

(A) To deform $\Gamma$ to another contour $\Gamma^*$ that crosses the saddle points of $\varphi(w)$;

(B) a suitable change of the variable of integration;

(C) application of Laplace's method or Watson's lemma.

In particular when more parameters are involved, the transformation of the given integral into a standard form gives complicated integrands, and the analysis is hard work. Examples of these expansions are available for the Jacobi, Laguerre, Meixner, Charlier, Krawtchouk polynomials, and so on by R. Wong and co-workers [4], [11], [14]. For example, [14] gives the asymptotic expansion for the Krawtchouk polynomials:

$$K_n^{(\lambda n)}(\nu n; p, q) \sim \frac{p^n}{n!} \left(\frac{q}{p}\right)^{\nu n} n^{n/2} e^{\gamma n} \times \left( V_n(\beta\sqrt{n}) \sum_{k=0}^{N-1} \frac{a_k}{n^k} + \frac{1}{\sqrt{n}} V_n'(\beta\sqrt{n}) \sum_{k=0}^{N-1} \frac{b_k}{n^k} \right),$$

where $V_n(z)$ is a parabolic cylinder function, $\lambda > 0$ and $\nu^{-1} < 1$. The parameters $\beta$ and $\gamma$ are solutions of the system

$$\nu F(w_+,\lambda) = G(z_+,\beta)+\gamma,$$
$$\nu F(w_-,\lambda) = G(z_-,\beta)+\gamma,$$

where

$$F(w,\lambda) \equiv (1-\lambda)\log(1-w)+\lambda\log(p/q+w)-\nu^{-1}\log w,$$
$$G(z,\beta) \equiv -\log z+\beta z-\frac{z^2}{2}$$

and

$$w_\pm \equiv (1-\nu^{-1})^{-1}\left\{[\lambda(1+p/q)-p/q-\nu^{-1}+p/(q\nu)]\pm \sqrt{[\lambda(1+p/q)-p/q-\nu^{-1}+p/(q\nu)]^2-4p/(q\nu)(1-\nu^{-1})}\right\},$$

$$z_\pm \equiv \frac{\beta\pm\sqrt{\beta^2-4}}{2}.$$

The coefficients $a_k$, $b_k$ are complicated functions of the parameters $\beta$ and $\gamma$.

At this point we wonder if it is possible to have simpler expansions for large $n$ of the polynomials of the Askey table than the expansions obtained from the standard saddle point method. For this purpose we will formulate a simplified version of the saddle point method. We want to approximate the integral

$$F(n)=\int_\Gamma f(w)e^{n\varphi(w)}\frac{dw}{w}$$

for large $n$. We consider first the case of one relevant saddle point $w_0$, and we assume that the main contribution of the integrand to the integral comes from a region close to the saddle point $w_0$ of the exponent: $\varphi'(w_0)=0$. We expand $f(w)$ at $w_0$ by means of its Taylor series:

$$f(w)=\sum_{k=0}^{\infty}\frac{f^{(k)}(w_0)}{k!}(w-w_0)^k, \tag{7.2}$$

which is convergent in a disc $D_r(w_0)$ centered at $w_0$ and some radius $r$ (see Figure 7.1).

Suppose that $|w_0| < r$. Then $0 \in D_r(w_0)$ and we can shrink the circle $\Gamma$ as much as necessary to have $\Gamma \in D_r(w_0)$, that is, the integration variable $w$ in (7.1) is always in $D_r(w_0)$. But in that disc the series (7.2) converges uniformly. Therefore,

$$F(n)=\sum_{k=0}^{\infty}\frac{f^{(k)}(w_0)}{k!}\Phi_k(n), \qquad \Phi_k(n)\equiv\int_\Gamma (w-w_0)^k e^{n\varphi(w)}\frac{dw}{w}. \tag{7.3}$$

Now, two natural questions arise:

i) Are the integrals $\Phi_k(n)$ simpler than the integral $F(n)$ itself?

ii) Is the expansion (7.3) asymptotic when $n\to\infty$? That is, is $\Phi_{k+1}(n)=o(\Phi_k(n))$ for large $n$?

On the other hand, whether or not i) and ii) hold, an extra feature is that

iii) The expansion (7.3) is convergent.

We are going to see in the next subsections that for the examples $F(n) = C_n^a(nx)$, $L_n^{(\alpha)}(nx)$ or $P_n^{(\alpha,\beta)}(x)$, the answer to questions i) and ii) is yes.

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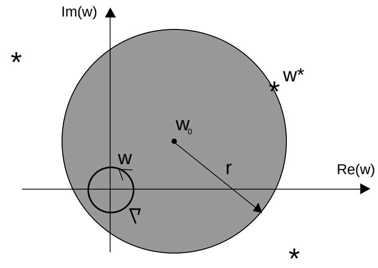


FIG. 7.1. *Disc of convergence $D_r(w_0)$ for the expansion (7.2). If $|w_0| < r$, then we can take a circle $\Gamma$ completely contained in $D_r(w_0)$.*

**7.1. Charlier polynomials.** From the generating function of the Charlier polynomials we have:

$$C_n^a(nx) = \frac{n!}{2\pi i}\int_\Gamma e^{-aw} e^{n\varphi(x,w)} \frac{dw}{w}, \tag{7.4}$$

where $\varphi(x,w) \equiv x\log(1+w) - \log w$ and $\Gamma$ is a small circle around $w = 0$. An asymptotic expansion of these polynomials in terms of $J-$Bessel functions has been obtained in [4]. The function $\varphi(w)$ has a unique saddle point $w_0 = (x-1)^{-1}$. We assume in this case then $x \neq 1$, and we expand

$$f(w) \equiv e^{-aw} = \sum_{k=0}^{\infty} \frac{(-a)^k e^{-aw_0}}{k!}(w-w_0)^k. \tag{7.5}$$

This series has an infinite radius of convergence, $r = \infty$. Then, for $a \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$C_n^a(x) = n!\, e^{a/(1-x)} \sum_{k=0}^{\infty} \frac{(-a)^k}{k!}\Phi_k(x,n), \tag{7.6}$$

where

$$\Phi_k(x,n) \equiv \frac{1}{2\pi i}\int_\Gamma (w-w_0)^k (1+w)^{xn} \frac{dw}{w^{n+1}}. \tag{7.7}$$

After straightforward calculations, we obtain

$$\Phi_0(x,n) = \frac{\Gamma(nx+1)}{n!\,\Gamma(nx+1-n)},$$

$$\Phi_1(x,n) = \frac{\Phi_0(x,n)}{(1-x)(n(x-1)+1)} = \mathcal{O}\left(\frac{\Phi_0(x,n)}{n}\right),$$

and, for $k = 2, 3, 4, \ldots$,

$$\Phi_k(x,n) = \frac{\Gamma(nx+1)}{\Gamma(nx-n+1)} \frac{{}_2F_1(-k,-n,nx-n+1;1-x)}{n!\,(1-x)^k}.$$

An application of the saddle point method to (7.7) shows that when $k$ is even

$$\Phi_k(x,n) = [x^x(x-1)^{1-x}]^n \mathcal{O}(n^{-(k+1)/2}), \quad n \to \infty.$$

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Hence, $\{\Phi_k(x,n)\}$ constitute an asymptotic sequence. On the other hand, integrating by parts in (7.7) we obtain the following recurrence

$$\Phi_k(x,n) = \frac{1-k}{n(x-1)+k}\left[\frac{x+2}{x-1}\Phi_{k-1}(x,n) + \frac{x\Phi_{k-2}(x,n)}{(x-1)^2}\right],$$

from which we see again that $\{\Phi_k(x,n)\}$ constitute an asymptotic sequence.

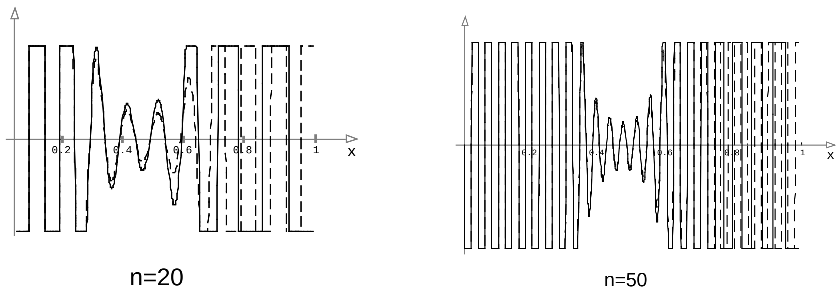


FIG. 7.2. *Numerical experiment with the approximation supplied by (7.6) for $a = 1$. Solid lines represent the Charlier polynomial, whereas dashed lines represent the first term in (7.6).*

**7.2. Laguerre polynomials.** From the generating function of the Laguerre polynomials we have:

$$L_n^{(\alpha)}(nx) = \frac{1}{2\pi i}\int_\Gamma f(w)\frac{e^{n\varphi(x,w)}}{(1-w)^{3/2}}\frac{dw}{w},$$

where

$$\varphi(x,w) \equiv \frac{xw}{w-1} - \log w, \quad f(w) \equiv (1-w)^{1/2-\alpha}.$$

An asymptotic expansion of the Laguerre polynomials for large $n$ in terms of $J-$Bessel functions near the origin and in terms of Airy functions near the largest zero is given in [11].

When we want to apply the modified saddle point method described in the beginning of this section, we have the extra feature that $\varphi(x,w)$ has two (conjugate) saddle points, not just one:

$$w^{\pm} = 1 - \frac{x}{2} \pm \frac{i}{2}\sqrt{x(4-x)}.$$

When $0 < x < 4$ both saddle points are of the same importance when approximating the integral. It is known that the zeros of $L_n^{(\alpha)}(nx)$ occur for these values of $x$. In this case we must expand $f(w)$ at both points simultaneously (a two-point Taylor expansion) [19]. In general, consider that we want to approximate the integral (7.1) for large $n$ and that $\varphi(w)$ has two saddle points $w_1$ and $w_2$. Then we expand [19]

$$f(w) = \sum_{n=0}^{\infty}\left[A_n + B_n w\right](w-w_1)^n(w-w_2)^n. \tag{7.8}$$

This expansion is convergent for $w$ inside the Cassini oval $O$ [19] (see Figure 7.3):

$$O \equiv \{w \in \Omega, \ \ |(w-w_1)(w-w_2)| < r\},$$

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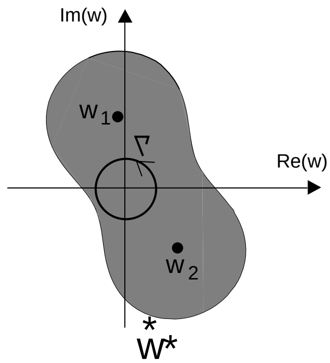


FIG. 7.3. *Oval $O$ of convergence for the expansion (7.8). The circle $\Gamma$ is completely contained in this oval if* $|w_1 w_2| < r$.

where $r \equiv \mathrm{Inf}_{w\in C\setminus\Omega}\,\{|(w-w_1)(w-w_2)|\}$.
If $|w_1 w_2| < r$, we can shrink the circle $\Gamma$ as much as necessary to have $\Gamma \in O$, that is, the integration variable $w$ in (7.1) is always inside $O$. But there, the series (7.8) converges uniformly. Therefore, substituting (7.8) in (7.1) and interchanging summation and integration we have

$$F(n) = \sum_{k=0}^{\infty}\left[A_k \Phi_k(x,n) + B_k \Psi_k(x,n)\right], \tag{7.9}$$

where

$$\Phi_k(x,n) \equiv \int_\Gamma (w-w_1)^k (w-w_2)^k e^{n\varphi(w)} \frac{dw}{w}$$

and

$$\Psi_k(x,n) \equiv \int_\Gamma (w-w_1)^k (w-w_2)^k e^{n\varphi(w)} dw.$$

We formulate again the same questions i) and ii) for this expansion. And again, iii) holds, the expansion (7.9) is convergent.

Applying the above theory to the Laguerre polynomials we have

$$L_n^{(\alpha)}(xn) = \sum_{k=0}^{\infty}\left[A_k \Phi_k(x,n) + B_k \Psi_k(x,n)\right], \tag{7.10}$$

where

$$\Phi_k(x,n) = \frac{1}{2\pi i}\int_\Gamma (w-w^+)^k (w-w^-)^k \frac{e^{n\varphi(x,w)}}{(1-w)^{3/2}} \frac{dw}{w},$$

$$\Psi_k(x,n) = \frac{1}{2\pi i}\int_\Gamma (w-w^+)^k (w-w^-)^k \frac{e^{n\varphi(x,w)}}{(1-w)^{3/2}} dw.$$

After straightforward computations we obtain that

$$\Phi_0(x,n) \equiv L_n^{(1/2)}(nx) = \frac{(-1)^n}{n!\,2^{2n+1}\sqrt{nx}} H_{2n+1}\left(\sqrt{nx}\right),$$
$$\Psi_0(x,n) \equiv L_{n-1}^{(1/2)}(nx) = \frac{(-1)^{n-1}}{(n-1)!\,2^{2n-1}} H_{2n-1}\left(\sqrt{nx}\right),$$

and, for $k = 1, 2, 3, ...,$

$$\Phi_k(x,n) \equiv \sum_{j=0}^{k} \binom{k}{j} x^{k-j} L_{n-k+j}^{(1/2-2j)}(nx),$$

$$\Psi_k(x,n) \equiv \sum_{j=0}^{k} \binom{k}{j} x^{k-j} L_{n-k+j-1}^{(1/2-2j)}(nx).$$

From the recurrence relation [13] for the $L_k^{(j)}(x)$, we immediately see that $L_{n-k+j-1}^{(1/2-2j)}(nx)$ are expressible in terms of the Hermite polynomials $H_m(\sqrt{nx})$ and their derivatives. Consequently, the terms in the expansion (7.10) are given in terms of Hermite polynomials. Moreover, these terms satisfy the recurrences

$$\Phi_k = \frac{a_1\Phi_{k-1} + a_2\Phi_{k-2} + b_1\Psi_{k-1} + b_2\Psi_{k-2}}{n - 2k + 3/2},$$

$$\Psi_k = \frac{c_0\Phi_k + c_1\Phi_{k-1} + c_2\Phi_{k-2} + d_1\Psi_{k-1} + d_2\Psi_{k-2},}{n - 2k},$$

where $a_1$, $a_2$,...,$d_2$ are certain coefficients independent of $n$ [20]. From these recurrences we obtain immediately:

$$\Phi_k(x,n) = \mathcal{O}\left(n^{-\lfloor (k+1)/2 \rfloor}\right) \left[|\Phi_0(x,n)| + |\Psi_0(x,n)|\right],$$
$$\Psi_k(x,n) = \mathcal{O}\left(n^{-\lfloor (k+1)/2 \rfloor}\right) \left[|\Phi_0(x,n)| + |\Psi_0(x,n)|\right],$$

The singularity of $f(w) = (1-w)^{1/2-\alpha}$ closest to the origin is $w = 1$. Therefore

$$r = |1 - w^+||1 - w^-| = x.$$

On the other hand $w^+w^- = 1$. Hence, the expansion (7.10) converges for $|w^+w^-| < r$, that is, for $x > 1$.

**7.3. Jacobi polynomials.** An asymptotic expansion for large $n$ in terms of $J-$Bessel functions for $x \to \pm 1$ and in terms of sine and cosine away from $\pm 1$ is given in [33].

From Rodrigues' formula for the Jacobi polynomials we have

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2\pi i}\frac{(-1)^n}{2^n}\int_\Gamma \frac{(1-w-x)^\alpha (1+w+x)^\beta}{(1-x)^\alpha (1+x)^\beta} e^{n\varphi(x,w)} \frac{dw}{w},$$

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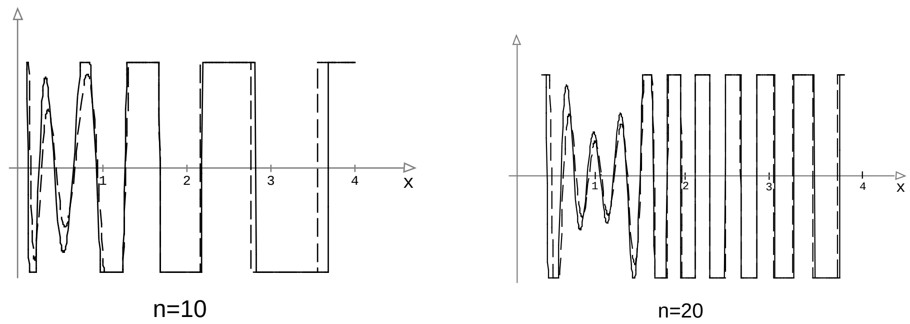


FIG. 7.4. *Numerical experiment with the approximation supplied by (7.10) for $\alpha = 4$. Solid lines represent the Laguerre polynomial, whereas dashed lines represent the first term in (7.10).*

where $x \in (-1,1)$, $w = -x \pm 1$ are outside $\Gamma$ and $\varphi(x,w) \equiv \log(1+w+x) + \log(1-w-x) - \log w$.

After steps similar to those explained for the Laguerre polynomials we obtain

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{\infty} \left[A_k \Phi_k(x,n) + B_k \Psi_k(x,n)\right], \tag{7.11}$$

where

$$\Phi_0(x,n) = \frac{2^{-2n}(2n)!}{(n!)^2} T_n(x), \quad \Psi_0(x,n) = -\frac{1}{2}(1-x^2)\,\frac{2^{2-2n}(2n-1)!}{(n-1)!\,n!} U_{n-1}(x)$$

are Chebyshev polynomials. Furthermore, for $k = 2, 3, 4, \ldots$,

$$\Phi_k(x,n) \equiv \sum_{j=0}^{k} \binom{k}{j} \frac{(1-x^2)^{k+j}}{4^j} P_{n-2j}^{(2j-1/2,2j-1/2)}(x),$$

$$\Psi_k(x,n) \equiv -\frac{1}{2}(1-x^2) \sum_{j=0}^{k} \binom{k}{j} \frac{(1-x^2)^{k+j}}{4^j} P_{n-1-2j}^{(2j+1/2,2j+1/2)}(x).$$

The polynomials $P_m^{(2j\pm1/2,2j\pm1/2)}(x)$ are expressible in terms of $T_m(x)$, $U_m(x)$ and their derivatives [13]. Hence, the terms of the expansion (7.11) can be expressed in terms of Chebyshev polynomials. Moreover, they satisfy the recurrences

$$\Phi_k = \frac{a_1\Phi_{k-1} + a_2\Phi_{k-2} + b_1\Psi_{k-1} + b_2\Psi_{k-2}}{n+2k-1},$$

$$\Psi_k = \frac{c_0\Phi_k + c_1\Phi_{k-1} + c_2\Phi_{k-2} + d_1\Psi_{k-1} + d_2\Psi_{k-2}}{n+2k},$$

where $a_1$, $a_2$,...,$d_2$ are independent of $n$. Also,

$$\Phi_k(x,n) = \mathcal{O}\left(n^{-\lfloor (k+1)/2 \rfloor}\right) \left[|\Phi_0(x,n)| + |\Psi_0(x,n)|\right],$$

$$\Psi_k(x,n) = \mathcal{O}\left(n^{-\lfloor (k+1)/2 \rfloor}\right) \left[|\Phi_0(x,n)| + |\Psi_0(x,n)|\right],$$

as $n \to \infty$ and $k = 0, 1, 2, \ldots$. The expansion (7.11) converges for $x \in (-1,1)$.

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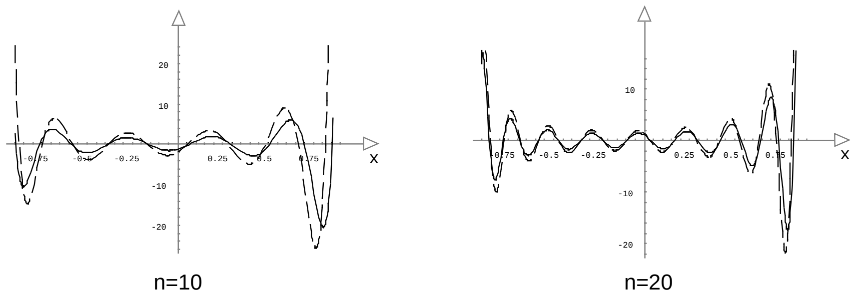


FIG. 7.5. *Numerical experiment on the approximation supplied by (7.11) for* $\alpha = 3$ *and* $\beta = 4$*. Solid lines represent the Jacobi polynomial, whereas dashed lines represent the first term in (7.11).*

**8. Further reading.** The new developments discussed in this paper are related to our own recent research on asymptotic expansions of integrals. In the last ten years significant other developments have occurred in the general theory of asymptotic expansions and in a great number of applications to special functions and orthogonal polynomials.

In the general theory of asymptotics, new insight in the Stokes phenomenon was given by Berry (see [2]), and many publications were devoted to this topic. Together with these activities more attention arose for the exponential small terms behind the dominant terms in asymptotic expansions. Exponentially improved expansions were given for the exponential integral (incomplete gamma function) by Olver for the Kummer function $U(a;c;z)$ ([24], [25]), and so-called hyperasymptotic expansions were studied [3]. For expositions on these new theories see the review paper [6], [26] and the recent excellent book by Paris and Kaminski [27]. This book is also of interest in connection with the asymptotics of Mellin-Barnes integrals, and related issues.

Papers on uniform asymptotic expansions for integrals (in particular for non-classical orthogonal polynomials) were written by Wong and co-workers; see [4], [5], [11], [12], [15], [14], [32], in which steepest descent methods are discussed in great detail for complicated contour integrals.

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REFERENCES

[1] M. Abramowitz and I.A. Stegun (1964): *Handbook of mathematical functions with formulas, graphs and mathematical tables*, Nat. Bur. Standards Appl. Series, **55**, U.S. Government Printing Office, Washington, D.C.

[2] M. V. Berry (1989): *Uniform asymptotic smoothing of Stokes' discontinuities*. Proc. R. Soc. Lond. A, **422** 7–21.

[3] M. V. Berry and C. J. Howls (1990): *Hyperasymptotics*. Proc. R. Soc. Lond., Ser. A, **430** 653–668.

[4] Rui Bo and R. Wong (1994): *Uniform asymptotic expansion of Charlier polynomials*. Methods Appl. Anal. **1** 294-313.

[5] Rui Bo and R. Wong (1996): *Asymptotic behavior of the Pollaczek polynomials and their zeros*. Stud. Appl. Math. **96** 307–338.

[6] J. P. Boyd (1999): *The devil's invention: asymptotic, superasymptotic and hyperasymptotic series*. Acta Appl. Math. **56** 1–98.

[7] P. Debye (1909): *Näherungsformeln für die Zylinderfunktionen für grosse Werte des Arguments und unbeschränkt veränderliche Werte des Index*. Math. Ann. **67** 535–558.

[8] C. Ferreira and J. L. López (2002): *Asymptotic expansions of generalized Stieltjes transforms of algebraically decaying functions*. Stud. Appl. Math. **108** 187-215.
[9] C. Ferreira, E. Mainar and J. L. López (2002): *Asymptotic relations in the Askey scheme of hypergeometric orthogonal polynomials*. To be published in Adv. Appl. Math.
[10] C. Ferreira and J. L. López, *Asymptotic relations in the Askey scheme of hypergeometric orthogonal polynomials II*. In preparation.
[11] C. L. Frenzen and R. Wong (1988): *Uniform asymptotic expansions of Laguerre polynomials*. SIAM J. Math. Anal. **19** 1232–1248.
[12] X. S. Jin and R. Wong (1998): *Uniform asymptotic expansions for Meixner polynomials*. Constr. Approx. **14** 113–150.
[13] R. Koekoek and R. F. Swarttouw (1999): *The Askey scheme of hypergeometric orthogonal polynomials and its $q-$analogue*. *Reports of the Faculty of Technical Mathematics and Informatics* **No. 98-17**. Delft University of Technology, Delft. Available at *http://aw.twi.tudelft.nl/koekoek/askey*
[14] X. C. Li and R. Wong (2000):*A uniform asymptotic expansions for Krawtchouk polynomials*. J. Approx. Theor. **106** 155–184.
[15] X. C. Li and R. Wong (2001): *On the asymptotics of the Meixner-Pollaczek polynomials and their zeros*. Constr. Approx. **17** (2001) 59–90.
[16] J. L. López and N. M. Temme (1999): *Approximations of orthogonal polynomials in terms of Hermite polynomials*. Meth. Appl. Anal. **6** 131–146.
[17] J. L. López and N. M. Temme (2001): *The Askey scheme for hypergeometric orthogonal polynomials viewed from asymptotic analysis*. J. Comp. Appl. Math. **133** (2001) 623–633.
[18] J. L. López (2000): *Asymptotic expansions of symmetric standard elliptic integrals*. SIAM J. Math. Anal. **31** n. 4 754–775.
[19] J. L. Lopez and N.M. Temme: *Two-point Taylor expansions of analytic functions*. To be published in Stud. Appl. Math.
[20] J. L. Lopez and N.M. Temme: *Convergent expansions of Charlier, Laguerre and Jacobi polynomials. Submitted*.
[21] J. P. McClure and R. Wong (1978): *Explicit error terms for asymptotic expansions of Stieltjes transforms*. J. Inst. Math. Appl. **22** 129–145.
[22] F. W. J. Olver: *Asymptotics and special functions*, *Academic Press, New York*, 1974. Reprinted by A.K. Peters, Wellesley, 1997.
[23] F. W. J. Olver (1974): *Error bounds for stationary phase approximation*. SIAM J. Math. Anal. **5** 19–29.
[24] F. W. J. Olver (1991): *Uniform, exponentially improved, asymptotic expansions for the generalized exponential integral*. SIAM J. Math. Anal. **22** 1460–1474.
[25] F. W. J. Olver(1991): *Uniform, exponentially improved, asymptotic expansions for the confluent hypergeometric function and other integral transforms*. SIAM J. Math. Anal. **22** 1475–1489.
[26] R. B. Paris and A. D. Wood (1995): *Stokes phenomenon demystified*. *IMA Bulletin*, **31** 21–28.
[27] R. B. Paris and D. Kaminski:, *Asymptotics and Mellin-Barnes integrals*, Cambridge Universty Press, Cambridge, 2001.
[28] N. M. Temme: *Special functions: An introduction to the classical functions of mathematical physics*, Wiley, New York, 1996.
[29] E. C. Titchmarsh: *The Theory of Functions, Second Edition*, Oxford University Press, London, 1939.
[30] G. N. Watson (1918): *Harmonic functions associated with the parabolic cylinder*. Proc. Lond. Math. Soc. (2) **17**, 116–148.
[31] R. Wong, Asymptotic Approximations of Integrals, *Academic Press, New York*, 1989. Reprinted by SIAM, Philadelphia, 2001.
[32] R. Wong and J-M. Zhang (1997): *Asymptotic expansions of the generalized Bessel polynomials*. J. Comput. Appl. Math. **85** 87–112.
[33] J-M. Zhang and R. Wong (1996): *A uniform asymptotic expansion for the Jacobi polynomials with explicit remainder*. Applicable Analysis **61** 17–29.