Asymptotic relations in the Askey scheme for hypergeometric orthogonal polynomials

ABSTRACT

It has been recently pointed out that several orthogonal polynomials of the Askey table admit asymptotic expansions in terms of Hermite and Laguerre polynomials [4], [6]. From those expansions, several known and new limits between polynomials of the Askey table were obtained in [4], [6]. In this paper, we make an exhaustive analysis of the three lower levels of the Askey scheme which completes the asymptotic analysis performed in [4] and [6]: (i) We obtain asymptotic expansions of Charlier, Meixner-Pollaczek, Jacobi, Meixner and Krawtchouk polynomials in terms of Hermite polynomials. (ii) We obtain asymptotic expansions of Meixner-Pollaczek, Jacobi, Meixner and Krawtchouk polynomials in terms of Charlier polynomials. (iii) We give new proofs for the known limits between polynomials of these three levels and derive new limits.

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1. Introduction

It is well known that there exist several limit relations between polynomials of the Askey scheme of hypergeometric orthogonal polynomials [3]. For example, the limit

$$\lim_{\alpha\to\infty} \alpha^{-n/2} L_n^{\alpha}(x\sqrt{\alpha}+\alpha) = \frac{(-1)^n\, 2^{-n/2}}{n!}\, H_n(x/\sqrt{2}\,), \tag{1}$$

shows that, when the variable is properly scaled, the Laguerre polynomials become the Hermite polynomials for large values of the order parameter. Moreover, this limit gives insight in the location of the zeros of the Laguerre polynomials for large values of the order parameter.

It has been recently pointed out that this limit may be obtained from an asymptotic expansion of the Laguerre polynomials in terms of the Hermite polynomials for large α [4]:

$$L_n^{\alpha}(x) = (-1)^n\, B^n \sum_{k=0}^{n} \frac{c_k}{B^k}\, \frac{H_{n-k}(X)}{(n-k)!}, \tag{2}$$

where the value of X, B and some other details are given in Section 2.1. This expansion has an asymptotic character for large values of $|\alpha|$ (see Section 2.1). The limit (1) is obtained from the first order approximation of this expansion.

The asymptotic method from which expansions like (2) are obtained was introduced in [4], [5] and [6]. More precisely, the method to approximate orthogonal polynomials in terms of Hermite polynomials is described in [4], whereas the method to approximate orthogonal polynomials in terms of Laguerre polynomials is described in [6]. Based on those methods, asymptotic expansions of Laguerre and Jacobi polynomials in terms of Hermite polynomials are given in [4]. Asymptotic expansions of Meixner-Pollaczek, Jacobi, Meixner and Krawtchouk polynomials in terms of Laguerre polynomials are given in [6]. Moreover, asymptotic expansions in terms of Hermite polynomials of other polynomials not included in the Askey scheme, such as Tricomi-Carlitz and generalized Bernoulli, Euler, Bessel and Buchholz polynomials are given in [4] and [5].

Those asymptotic methods are based on the availability of a generating function for the polynomials and is different from the techniques described in [1], [2]. The method introduced in [1] is based on a connection problem and gives deeper information on the limit relations between classical discrete and classical continuous orthogonal polynomials in the Askey scheme. On the other hand, our method gives asymptotic expansions of polynomials situated at any level of the table in terms of polynomials located at lower levels. Our method is also different from the sophisticated uniform methods considered for example in [8] or [7], where asymptotic expansions of the Meixner $M_n(nx, b, c)$ or Charlier $C_n^a(nx)$ polynomials respectively are given for large values of n and fixed a, b, c, x. In our method we keep the degree n fixed and let some parameter(s) of the polynomial go to infinity.

In the following section we resume the descending asymptotic expansions and limit relations between all the polynomials situated at the three first levels of the Askey table.

For completeness, we resume in the next section not only the new expansions obtained in this paper, but also the known ones derived in [4] and [6]. In Section 3 we briefly resume the principles of the Hermite-type asymptotic approximations introduced in [4] and introduce the principles of the Charlier-type asymptotic approximations. In Section 4 we prove the new asymptotic expansions and the new limits presented in Section 2. Some numerical experiments illustrating the accuracy of the approximations are given in Section 5.

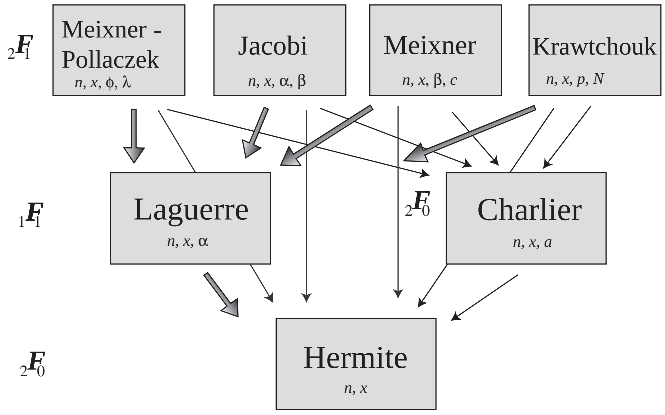

Figure 1. *Three lower levels of the Askey table of hypergeometric orthogonal polynomials. Thick arrows indicate known asymptotic expansions between polynomials, whereas thin arrows indicate new asymptotic expansions derived in this paper.*

2. Descending asymptotic expansions and limits

The orthogonality property of the polynomials of the Askey table only holds when the variable x and other parameters which appear in the polynomials are restricted to certain real intervals [3]. Nevertheless, the expansions that we resume below are valid for complex values of the variable and the parameters and for any $n \in \mathbb{N}$. All the square roots that appear in what follows assume real positive values for real positive argument.

2.1. Laguerre to Hermite

2.1.1. Asymptotic expansion for large α:

$$L_n^\alpha(x) = (-1)^n B^n \sum_{k=0}^{n} \frac{c_k}{B^k} \frac{H_{n-k}(X)}{(n-k)!}, \qquad B = \sqrt{x - \frac{\alpha+1}{2}}, \quad X = \frac{x-\alpha-1}{2B}. \tag{3}$$

2.1.2. Coefficients: $c_0 = 1, \quad c_1 = c_2 = 0, \quad c_3 = \frac{1}{3}(3x - \alpha - 1)$ and, for $k = 4, 5, 6, \ldots$,

$$kc_k = -2(k-1)c_{k-1} - (k-2)c_{k-2} + (3x - \alpha - 1)c_{k-3} + (2x - \alpha - 1)c_{k-4}. \tag{4}$$

2.1.3. Asymptotic property: $\frac{c_k}{B^k} H_{n-k}(X) = \mathcal{O}(\alpha^{n+\lfloor k/3 \rfloor - 3k/2}), \qquad \alpha \to \infty.$

2.1.4. Limit (known): $\lim_{\alpha\to\infty} \alpha^{-n/2} L_n^{\alpha}(x\sqrt{\alpha}+\alpha) = \frac{(-1)^n 2^{-n/2}}{n!} H_n\left(\frac{x}{\sqrt{2}}\right)$.

2.2. Charlier to Hermite

2.2.1. Asymptotic expansion for large a:

$$C_n(x,a) = B^n \sum_{k=0}^{n} \frac{c_k}{B^k} \frac{H_{n-k}(X)}{(n-k)!}, \qquad B = \sqrt{\frac{x}{2a^2}}, \quad X = \frac{a-x}{\sqrt{2x}}. \tag{5}$$

2.2.2. Coefficients: $c_0 = 1, \quad c_1 = c_2 = 0$ and, for $k = 3, 4, 5, \ldots,$

$$a^3 k c_k = a^2(k-1)c_{k-1} - x c_{k-3}. \tag{6}$$

2.2.3. Asymptotic property: $\frac{c_k}{B^k} H_{n-k}(X) = \mathcal{O}(a^{n-k}), \qquad a \to \infty.$

2.2.4. Limit (known): $\lim_{a\to\infty} (2a)^{n/2} C_n(\sqrt{2a}x + a, a) = (-1)^n H_n(x)$.

2.3. Krawtchouk to Hermite

2.3.1. Asymptotic expansion for large N:

$$\binom{N}{n} K_n(x;p,N) = B^n \sum_{k=0}^{n} \frac{c_k}{B^k} \frac{H_{n-k}(X)}{(n-k)!}, \qquad B = \sqrt{\frac{p^2N + x - 2xp}{2p^2}}, \ \ X = \frac{Np-x}{2pB}. \tag{7}$$

2.3.2. Coefficients: $c_0 = 1, \quad c_1 = c_2 = 0, \quad c_3 = \frac{3xp - 3xp^2 + p^3N - x}{3p^3}$ and, for $k = 4, 5, 6, \ldots,$

$$\begin{aligned} p^3 k c_k = & - p^2(2p-1)(k-1)c_{k-1} - p^2(p-1)(k-2)c_{k-2} \\ & + (-x + 3xp - 3xp^2 + p^3N)c_{k-3} \\ & + (-2xp^2 + 3xp + p^3N - p^2N - x)c_{k-4}. \end{aligned} \tag{8}$$

2.3.3. Asymptotic property: $\frac{c_k}{B^k} H_{n-k}(X) = \mathcal{O}(N^{n/2+[k/3]-k}), \qquad N \to \infty.$

2.3.4. Limit (known):

$$\lim_{N\to\infty} \binom{N}{n} \left(\sqrt{\frac{2p}{N(1-p)}}\right)^n K_n\left(pN + x\sqrt{2p(1-p)N}; p, N\right) = \frac{(-1)^n}{n!} H_n(x).$$

2.4. Meixner to Hermite

2.4.1. Asymptotic expansion for large β:

$$M_n(x;\beta,c) = \frac{n! B^n}{(\beta)_n} \sum_{k=0}^{n} \frac{B_k}{z^k} \frac{H_{n-k}(X)}{(n-k)!}, \qquad B = \sqrt{\frac{1-c^2}{2c^2}x - \frac{\beta}{2}}, \quad X = \frac{(c-1)x + \beta c}{2cB}. \tag{9}$$

2.4.2. Coefficients: $c_0 = 1, \quad c_1 = c_2 = 0, \quad c_3 = \frac{(c^3-1)x+c^3\beta}{3c^3}$ and, for $k = 4, 5, 6, ...,$

$$\begin{aligned} c^3 k c_k =& c^2(c+1)(k-1)c_{k-1} - c^2(k-2)c_{k-2} + (\beta c^3 + c^3 x - x)c_{k-3} \\ &+ (\beta c^2 + x - c^2 x)c_{k-4}. \end{aligned} \tag{10}$$

2.4.3. Asymptotic property: $\frac{c_k}{B^k}\, H_{n-k}(X) = \mathcal{O}(\beta^{n/2+[k/3]-k}), \qquad \beta \to \infty.$

2.4.4. Limit (new): $\lim_{\beta\to\infty} (\beta)_n \left(\frac{2c}{\beta}\right)^{n/2} M_n\left(\frac{c\beta - x\sqrt{2c\beta}}{1-c}; \beta, c\right) = H_n(x).$

2.5. Meixner-Pollaczek to Hermite

2.5.1. Asymptotic expansion for large λ:

$$P_n^{(\lambda)}(x;\phi) = B^n \sum_{k=0}^{n} \frac{c_k}{B^k} \frac{H_{n-k}(X)}{(n-k)!}, \tag{11}$$

$$B = \sqrt{-\lambda\cos(2\phi) - x\sin(2\phi)}, \quad X = \frac{\lambda\cos\phi + x\sin\phi}{B}. \tag{12}$$

2.5.2. Coefficients: $c_0 = 1, \quad c_1 = c_2 = 0, \quad c_3 = \frac{2}{3}[\lambda\cos(3\phi) + x\sin(3\phi)]$ and, for $k = 4, 5, 6, ...,$

$$\begin{aligned} k c_k =& 2\cos\phi(k-1)c_{k-1} - (k-2)c_{k-2} + 2[\lambda\cos(3\phi) + x\sin(3\phi)]c_{k-3} \\ &- 2[\lambda\cos(2\phi) + x\sin(2\phi)]c_{k-4}. \end{aligned} \tag{13}$$

2.5.3. Asymptotic property: $\frac{c_k}{B^k}\, H_{n-k}(X) = \mathcal{O}(\lambda^{n/2+[k/3]-k}), \qquad \lambda \to \infty.$

2.5.4. Limit (known): $\lim_{\lambda\to\infty} \lambda^{-n/2} P_n^{(\lambda)}\left(\frac{x\sqrt{\lambda} - \lambda\cos\phi}{\sin\phi}; \phi\right) = \frac{1}{n!} H_n(x).$

2.6. Jacobi to Hermite

2.6.1. Asymptotic expansion for large $\alpha + \beta$:

$$P_n^{(\alpha,\beta)}(x) = B^n \sum_{k=0}^{n} \frac{c_k}{B^k} \frac{H_{n-k}(X)}{(n-k)!}, \qquad X = \frac{2x + (\alpha+\beta)x + (\alpha-\beta)}{4B}, \tag{14}$$

$$B = \sqrt{\frac{1}{2} - x^2 + \frac{1}{8}(\alpha+\beta) - \frac{1}{4}(\alpha-\beta)x - \frac{3}{8}x^2(\alpha+\beta)}. \tag{15}$$

2.6.2 Coefficients:

$$c_0 = 1, \quad c_1 = c_2 = 0, \quad c_3 = \frac{4}{3x^3} - x - \frac{1}{12}(\alpha-\beta) - \frac{1}{4}x(\alpha+\beta) + \frac{5}{12}x^3(\alpha+\beta) + \frac{1}{4}x^2(\alpha-\beta).$$

2.6.3. Asymptotic property:

$$\frac{c_k}{B^k}\, H_{n-k}(X) = \mathcal{O}((\alpha+\beta)^{n/2+[k/3]-k}), \qquad \alpha+\beta \to \infty, \quad \frac{\alpha-\beta}{\alpha+\beta} \to 0.$$

2.6.4. Limit (new): $\lim_{\alpha+\beta\to\infty} \frac{P_n^{(\alpha,\beta)}\left(\frac{\sqrt{2\alpha+2\beta}x+\beta-\alpha}{\alpha+\beta+2}\right)}{(\alpha+\beta)^{n/2}} = \frac{H_n(x)}{2^{3n/2}n!}$, with $\frac{\alpha-\beta}{\alpha+\beta} \to 0$.

2.7. Krawtchouk to Laguerre

2.7.1. Asymptotic expansion for large N:

$$\binom{N}{n} K_n(x;p,N) = \sum_{k=0}^{n} c_k L_{n-k}^{(\alpha)}(X), \qquad X = \alpha + 1 - N + \frac{x}{p}. \tag{16}$$

2.7.2. Coefficients: $c_0 = 1, \quad c_1 = 0, \quad c_2 = \frac{1}{2}\left[1+\alpha-3N+\frac{1}{p}\left(4-\frac{1}{p}\right)x\right].$

2.7.3. Asymptotic property: $c_k\, L_{n-k}^{(\alpha)}(X) = \mathcal{O}(N^{n+[k/2]-k}), \qquad N \to \infty.$

2.8. Meixner to Laguerre

2.8.1. Asymptotic expansion for large β:

$$M_n(x;\beta,c) = \sum_{k=0}^{n} c_k L_{n-k}^{(\alpha)}(X), \qquad X = \alpha - \beta + 1 + \frac{1-c}{c}x. \tag{17}$$

2.8.2. Coefficients: $c_0 = 1, \quad c_1 = 0, \quad c_2 = \frac{1+\alpha-\beta}{2} + \frac{2c-c^2-1}{2c^2}x.$

2.8.3. Asymptotic property: $c_k L_{n-k}^{(\alpha)}(X) = \mathcal{O}(\beta^{n+[k/2]-k}), \qquad \beta \to \infty.$

2.8.4. Limit (known): $\lim_{c\to 1} M_n\left(\frac{cx}{1-c};\alpha+1,c\right) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}.$

2.9. Meixner-Pollaczek to Laguerre

2.9.1. Asymptotic expansion for large λ:

$$P_n^{(\lambda)}(x;\phi) = \sum_{k=0}^{n} c_k L_{n-k}^{(\alpha)}(X), \qquad X = \alpha + 1 - 2\lambda\cos\phi - 2x\sin\phi. \tag{18}$$

2.9.2. Coefficients:

$$c_0 = 1, \quad c_1 = 0, \quad c_2 = x\sin(2\phi) + \lambda\cos(2\phi) - 2(x\sin\phi + \lambda\cos\phi) + \frac{\alpha}{2}$$

and, for $k = 4, 5, 6, \ldots,$

$$\begin{aligned} kc_k =& 2(1+\cos\phi)(k-1)c_{k-1} + [\alpha + 1 - 2\lambda + 4(\cos\phi - 1)(\lambda\cos\phi + x\sin\phi) + \\ & 2(2-k)(1+2\cos\phi)]c_{k-2} + [4\lambda + 2(k-3)(1+\cos\phi) - \\ & 2(\alpha+1)\cos\phi]c_{k-3} + (\alpha + 5 - k - 2\lambda)c_{k-4}. \end{aligned} \tag{19}$$

2.9.3. Asymptotic property: $c_k L_{n-k}^{(\alpha)}(X) = \mathcal{O}(\lambda^{n+[k/2]-k}), \qquad \lambda \to \infty.$

2.9.4. Limit (known): $\lim_{\phi\to 0} P_n^{(\alpha+1)/2}\left(-\frac{x}{2\phi};\phi\right) = L_n^{(\alpha)}(x)$.

2.10. Jacobi to Laguerre

2.10.1 Asymptotic expansion for large $\alpha+\beta$:

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} c_k L_{n-k}^{(\alpha)}(X), \qquad X = \frac{1}{2}(\alpha+\beta+2)(1-x). \tag{20}$$

2.10.2. Coefficients:

$$c_0 = 1, \quad c_1 = 0, \quad c_2 = \frac{1}{8}[3\beta-\alpha-2(\alpha+3\beta+4)x+(3\alpha+3\beta+8)x^2].$$

2.10.3. Asymptotic property:

$$c_k\, L_{n-k}^{(\alpha)}(X) = \mathcal{O}((\alpha+\beta)^{n+[k/2]-k}), \qquad \alpha+\beta\to\infty, \qquad \frac{\alpha-\beta}{\alpha+\beta}\to 0. \tag{21}$$

2.10.4. Limit (known): $\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right) = L_n^{(\alpha)}(x)$.

2.11. Krawtchouk to Charlier

2.11.1. Asymptotic expansions for large N:

$$\binom{N}{n} K_n(x,p,N) = B^n \sum_{k=0}^{n} \frac{c_k}{B^k}\frac{C_{n-k}(X,A)}{(n-k)!}, \tag{22}$$

$$\begin{aligned} X &= \frac{\left(Np^2+x-2px\right)^3}{\left(Np^3+(-3p^2+3p-1)x\right)^2}, \qquad B = \frac{(p-1)(N-x)x}{Np^3+(3p-3p^2-1)x}, \\ A &= -\frac{(p-1)p(N-x)x\left(Np^2+x-2px\right)}{\left(Np^3+(-3p^2+3p-1)x\right)^2}. \end{aligned} \tag{23}$$

2.11.2. Coefficients: $c_0 = 1, \quad c_1 = c_2 = c_3 = 0, \quad c_4 = \frac{(p-1)^2x(x-N)}{4p^2(Np^2+x-2px)}$
and, for $k = 5, 6, 7, \ldots$,

$$kc_k = h_1(k-1)c_{k-1} + h_2(k-2)c_{k-2} + h_3(k-3)c_{k-3} + h_4c_{k-4}, \tag{24}$$

with

$$\begin{aligned} h_1 &= \frac{Np^2(1-3p)+(7p^2-7p+2)x}{p(Np^2+x-2px)}, \qquad h_2 = \frac{(2p-1)^3x-Np^3(3p-2)}{p^2(Np^2+x-2px)}, \\ h_3 &= \frac{(p-1)[(3p^2-3p+1)x-Np^3]}{p^2(Np^2+x-2px)}, \qquad h_4 = \frac{(p-1)^2x(x-N)}{p^2(Np^2+x-2px)}. \end{aligned}$$

2.11.3. Asymptotic property: $\frac{c_k}{B^k} C_{n-k}(X,A) = \mathcal{O}(N^{n-k}), \qquad N \to \infty.$

2.11.4. Limit (known): $\lim_{N\to\infty} K_n\left(x, \frac{a}{N}, N\right) = C_n(x,a).$

2.12. Meixner to Charlier

2.12.1. Asymptotic expansions for large β:

$$M_n(x,\beta,c) = \frac{B^n n!}{(\beta)_n} \sum_{k=0}^{n} \frac{c_k}{B^k} \frac{C_{n-k}(X,A)}{(n-k)!}, \tag{25}$$

$$\begin{aligned} X &= -\frac{\left(\beta c^2 + (c^2-1)x\right)^3}{\left(\beta c^3 + (c^3-1)x\right)^2}, \qquad B = -\frac{(c-1)^2 x(\beta+x)}{\beta c^3 + (c^3-1)x}, \\ A &= -\frac{(c-1)^2 c x(\beta+x)\left(\beta c^2 + (c^2-1)x\right)}{\left(\beta c^3 + (c^3-1)x\right)^2}. \end{aligned} \tag{26}$$

2.12.2. Coefficients: $c_0 = 1, \quad c_1 = c_2 = c_3 = 0, \quad c_4 = -\frac{(c-1)^2 x(\beta+x)}{4c^2(\beta c^2 + (c^2-1)x)}$
and, for $k = 5, 6, 7, \ldots$,

$$k c_k = h_1 (k-1) c_{k-1} + h_2 (k-2) c_{k-2} + h_3 (k-3) c_{k-3} + h_4 c_{k-4}, \tag{27}$$

with

$$\begin{aligned} h_1 &= \frac{\beta c^2(1+2c) - (2+c-c^2-2c^3)x}{\beta c^3 + c(c^2-1)x}, \qquad h_3 = \frac{\beta c^3 - x + c^3 x}{\beta c^4 + c^2(c^2-1)x}, \\ h_2 &= \frac{(1-c)(1+c)^3 x - \beta c^3(2+c)}{\beta c^4 + c^2(c^2-1)x}, \qquad h_4 = -\frac{(c-1)^2 x(\beta+x)}{\beta c^4 + c^2(c^2-1)x}. \end{aligned}$$

2.12.3. Asymptotic property: $\frac{c_k}{B^k} C_{n-k}(X,A) = \mathcal{O}(\beta^{n-k}), \qquad \beta \to \infty.$

2.12.4. Limit (known): $\lim_{\beta\to\infty} M_n\left(x, \beta, \frac{a}{a+\beta}\right) = C_n(x,a).$

2.13. Meixner-Pollaczeck to Charlier

2.13.1. Asymptotic expansions for large λ:

$$P_n^{(\lambda)}(x,\phi) = B^n \sum_{k=0}^{n} \frac{c_k}{B^k} \frac{C_{n-k}(X,A)}{(n-k)!}, \tag{28}$$

$$\begin{aligned} X &= -\frac{2\left(\lambda\cos(2\phi) + x\sin(2\phi)\right)^3}{\left(\lambda\cos(3\phi) + x\sin(3\phi)\right)^2}, \qquad B = -\frac{2\left(\lambda^2 + x^2\right)\sin^2\phi}{\lambda\cos(3\phi) + x\sin(3\phi)}, \\ A &= -\frac{2\left(\lambda^2 + x^2\right)\sin^2\phi\left(\lambda\cos(2\phi) + x\sin(2\phi)\right)}{\left(\lambda\cos(3\phi) + x\sin(3\phi)\right)^2}. \end{aligned} \tag{29}$$

2.13.2. Coefficients: $c_0 = 1, \quad c_1 = c_2 = c_3 = 0, \quad c_4 = -\frac{(\lambda^2+x^2)\sin^2\phi}{2(\lambda\cos(2\phi)+x\sin(2\phi))}$ and, for $k = 5, 6, 7, \ldots,$

$$kc_k = h_1(k-1)c_{k-1} - h_2(k-2)c_{k-2} + h_3(k-3)c_{k-3} + h_4 c_{k-4}, \tag{30}$$

with

$$\begin{aligned} h_1 &= \frac{\lambda\cos\phi + 2\lambda\cos(3\phi) + x\left[\sin\phi + 2\sin(3\phi)\right]}{\lambda\cos(2\phi) + x\sin(2\phi)}, \qquad & h_3 &= \frac{\lambda\cos(3\phi) + x\sin(3\phi)}{\lambda\cos(2\phi) + x\sin(2\phi)}, \\ h_2 &= \frac{2\lambda\cos(2\phi) + \lambda\cos(4\phi) + 8x\cos^3\phi\sin\phi}{\lambda\cos(2\phi) + x\sin(2\phi)}, & h_4 &= -\frac{2\left(\lambda^2 + x^2\right)\sin^2\phi}{\lambda\cos(2\phi) + x\sin(2\phi)}. \end{aligned}$$

2.13.3. Asymptotic property: $\frac{c_k}{B^k} C_{n-k}(X, A) = \mathcal{O}(\lambda^{\lfloor k/4\rfloor - k}), \qquad \lambda \to \infty.$

2.13.4. Limit (new): $\lim_{\lambda\to\infty} \lambda^{-n} P_n^{(\lambda)}(x, \phi) = \frac{2^n}{n!}\left(2\cos(2\phi) - 1\right)^n \cos^{2n}\phi\cos^{-n}(3\phi).$

2.14. Jacobi to Charlier

2.14.1. Asymptotic expansions for large β:

$$P_n^{(\alpha,\beta)}(x) = B^n \sum_{k=0}^{n} \frac{c_k}{B^k}\frac{C_{n-k}(X, A)}{(n-k)!}, \tag{31}$$

$$\begin{aligned} X &= \frac{1}{4}\left\{4 + \alpha + \beta - 2x(\alpha - \beta) - x^2[8 + 3(\alpha + \beta)]\right\}, \\ A = B &= \frac{1}{4}(1 - x)\left\{4 + \alpha + \beta + 2(\alpha - \beta) + x\left[8 + 3(\alpha + \beta)\right]\right\}. \end{aligned} \tag{32}$$

2.14.2. Coefficients:

$$\begin{aligned} c_0 &= 1, \quad c_1 = c_2 = 0, \\ c_3 &= \frac{1}{3} + \frac{\beta}{6} - \left(1 + \frac{5\alpha}{12} + \frac{\beta}{12}\right)x - \left(\frac{2}{3} + \frac{\beta}{2}\right)x^2 \\ &\quad + \left(\frac{4}{3} + \frac{5}{12}(\alpha + \beta)\right)x^3, \\ c_4 &= \frac{1}{2} + \frac{7}{64}\left(-\alpha + \beta\right) + \frac{5}{16}\left(-\alpha + \beta\right)x - \left(\frac{5}{2} + \frac{21}{32}(\alpha + \beta)\right)x^2 \\ &\quad + \frac{5}{16}\left(\alpha - \beta\right)x^3 + \left(2 + \frac{35}{64}(\alpha + \beta)\right)x^4. \end{aligned}$$

2.14.3. Asymptotic property: $\frac{c_k}{B^k} C_{n-k}(X, A) = \mathcal{O}(\beta^{[k/3] - k}), \qquad \beta \to \infty.$

2.14.4 Limit (new): $\lim_{\alpha+\beta\to\infty}(\alpha+\beta)^{-n} P_n^{(\alpha,\beta)}(x) = \frac{1}{n!}\left(\frac{x}{2}\right)^n$, with $(\alpha - \beta)/(\alpha + \beta) \to 0$.

3. Principles of the asymptotic approximations

The asymptotic expansions of polynomials in terms of polynomials listed above follow from an asymptotic principle based on the "matching" of their generating functions. That "matching" depends on the generating function of the polynomials used as approximant. Below we give details for the case when the basic approximant are the Charlier polynomials. The details for the case when the Hermite polynomials are chosen as the basic approximant are taken from [4].

3.1. Expansions in terms of Charlier polynomials

The Charlier polynomials are defined, for $n \in \mathbb{N}$, $a \neq 0$, by

$$C_n(x,a) = {}_2F_0\left(\left.\begin{matrix} -n, \ -x \\ - \end{matrix}\, \right| -\frac{1}{a}\right). \tag{33}$$

For $a > 0$ and $x \in \mathbb{N}$, they are orthogonal with respect to the discrete measure $a^x/x!$, $x = 0, 1, 2, \ldots$. They also follow from the generating function

$$e^w \left(1 - \frac{w}{a}\right)^x = \sum_{n=0}^{\infty} \frac{C_n(x,a)}{n!} w^n. \tag{34}$$

This formula gives the following Cauchy-type integral for the Charlier polynomials:

$$C_n(x,a) = \frac{n!}{2\pi i} \int_{\mathcal{C}} e^z \left(1 - \frac{z}{a}\right)^x \frac{dz}{z^{n+1}}, \tag{35}$$

where $\mathcal{C}$ is a circle around the origin of radius$< |a|$ and the integration is in positive direction.

The polynomials $p_n(x)$ of the Askey table (and many special functions) satisfy a relation in the form of a generating series, which usually has the form

$$F(x,w) = \sum_{n=0}^{\infty} p_n(x)\, w^n, \tag{36}$$

where $F(x,w)$ is a given function, which is analytic with respect to w in a domain that contains the origin, and $p_n(x)$ is independent of w.

The relation (36) gives for $p_n(x)$ the Cauchy-type integral

$$p_n(x) = \frac{1}{2\pi i} \int_{\mathcal{C}} F(x,w) \frac{dw}{w^{n+1}},$$

where $\mathcal{C}$ is a circle around the origin inside the domain where $F(x,w)$ is analytic (as a function of w). We write

$$F(x,w) = e^{Bw} \left(1 - \frac{Bw}{A}\right)^X f(x,w), \tag{37}$$

where A, B and X do not depend on w and can be chosen arbitrarily. This gives

$$p_n(x) = \frac{1}{2\pi i}\int_{\mathcal{C}} e^{Bw}\left(1-\frac{Bw}{A}\right)^X \frac{f(x,w)}{w^{n+1}}dw. \tag{38}$$

Since $f(x,w)$ is also analytic (as a function of w), we can expand

$$f(x,w) = \sum_{k=0}^{\infty} c_k w^k$$

and substitute this expansion in (38). By (35), the result is the finite expansion

$$p_n(x) = B^n \sum_{k=0}^{n} \frac{c_k}{B^k}\frac{C_{n-k}(X,A)}{(n-k)!}, \tag{39}$$

because terms with $k > n$ do not contribute in the integral in (38). The quantities A, B and X may depend on x and on other parameters which define the polynomials $p_n(x)$. If B happens to be zero for a special $x-$value x_0, say, and $A \neq 0$, then we write $p_n(x_0) = c_n$. If $A = B = 0$ for a given $x = x_0$, with $\lim_{x\to x_0}(B/A) = \alpha$, α finite, then we write $p_n(x_0) = \sum_{k=0}^{n} c_k(-\alpha)^{n-k}\begin{pmatrix} X \\ n-k \end{pmatrix}$.

In the examples considered in this paper, the choice of A, B and X is based on our requirement that $c_1 = c_2 = c_3 = 0$. This happens if we take

$$\begin{aligned}
X =& \frac{4\left(p_1(x)^2 - p_2(x)\right)^3}{\left(2p_1(x)^3 - 3p_1(x)p_2(x) + p_3(x)\right)^2}, \\
B =& \frac{p_1(x)^2 p_2(x) - 2p_2(x)^2 + p_1(x)p_3(x)}{2p_1(x)^3 - 3p_1(x)p_2(x) + p_3(x)}, \\
A =& \frac{2\left(p_2(x) - p_1(x)^2\right)\left(p_1(x)^2 p_2(x) - 2p_2(x)^2 + p_1(x)p_3(x)\right)}{\left(2p_1(x)^3 - 3p_1(x)p_2(x) + p_3(x)\right)^2}
\end{aligned} \tag{40}$$

and we assume that $F(x,0) = p_0(x) = 1$ (which implies $c_0 = 1$). This is easily verified from (36) and (37): from (36) we have

$$\begin{aligned}
\log[F(x,w)] =& p_1(x)w + \left[p_2(x) - \frac{1}{2}p_1^2(x)\right]w^2 \\
&+ \left[p_3(x) - p_1(x)p_2(x) + \frac{1}{3}p_1^3(x)\right]w^3 + \mathcal{O}(w^4), \quad w \to 0
\end{aligned}$$

and, on the other hand,

$$\log\left[e^{Bw}\left(1-\frac{Bw}{A}\right)^X\right] = B\left(1-\frac{X}{A}\right)w + \frac{XB^2}{2A^2}w^2 - \frac{XB^3}{3A^3}w^3 + \mathcal{O}(w^4), \quad w \to 0.$$

Equaling the coefficients of w, w^2 and w^3 in these two formulas we obtain (40). This choice of A, B and X makes the matching at the origin of the function $e^{Bw}\,(1-Bw/A)^X$ in (37) with $F(x,w)$ as best as possible.

We will show in Sections 4.6-4.9 that the finite sum in (39) gives the desired asymptotic representations from which the limits given in Sections 2.11-2.14 can be derived. The special choice of A, B and X is crucial for obtaining asymptotic properties. In some cases, we just require that the coefficients $c_1 = c_2 = 0$. In these cases we have to fix arbitrarily the value of one of the parameters A, B or X.

3.2. Expansions in terms of Hermite polynomials

To prove the results of Sections 2.2 to 2.6 we need the following formulas derived in [4] in a similar way to those of the preceding Section. If $F(x,w)$ satisfies (36), then:

$$p_n(x) = B^n \sum_{k=0}^{n} \frac{c_k}{B^k} \frac{H_{n-k}(X)}{(n-k)!}, \tag{41}$$

where the coefficients c_k follow from

$$f(x,w) = \sum_{k=0}^{\infty} c_k w^k, \qquad F(x,w) = e^{2BXw - B^2w^2}\, f(x,w).$$

The quantities X and B may depend on x, and if B happens to be zero for a special $x-$value x_0, say, we write $p_n(x_0) = \sum_{k=0}^{n} \frac{c_k}{(n-k)!}\, p_1^{n-k}(x_0)$.

In the examples considered in Sections 2.2 to 2.6, the choice of X and B is based on our requirement that $c_1 = c_2 = 0$. This happens if we take

$$B = \sqrt{\tfrac{1}{2}p_1^2(x) - p_2(x)}, \quad X = \frac{p_1(x)}{2B}$$

and we assume that $F(x,0) = p_0(x) = 1$ (which implies $c_0 = 1$).

We will show in Sections 4.1-4.5 that the finite sum in (41) gives the desired asymptotic representations from which the limits given in Sections 2.2-2.6 can be derived. The special choice of X and B is crucial for obtaining asymptotic properties.

The reader is referred to [6] for details about similar expansions in terms of Laguerre polynomials.

3.3. Asymptotic properties of the coefficients c_k

The asymptotic nature of the expansions (39) and (41) for large values of some of the parameters of the polynomial $p_n(x)$ depends on the asymptotic behaviour of the coefficients c_k. To prove the asymptotic character of the expansions given in Section 2 we will need the following lemma proved in [6]:

Lemma 1. *Let $\phi(w)$ be an analytic function at $w = 0$, with Maclaurin expansion of the form*

$$\phi(w) = \mu\omega^n(a_0 + a_1 w + a_2 w^2 + \ldots),$$

where n is a positive integer and a_k are complex numbers that satisfy $a_k = \mathcal{O}(1)$ when $\mu \to \infty$, $a_0 \neq 0$. Let c_k denote the coefficients of the power series of $f(w) = e^{\phi(w)}$, that is,

$$f(w) = e^{\phi(w)} = \sum_{k=0}^{\infty} c_k w^k.$$

Then $c_0 = 1$, $c_k = 0$, $k = 1, 2, \ldots, n-1$ and

$$c_k = \mathcal{O}\left(|\mu|^{[k/n]}\right), \quad \mu \to \infty.$$

4. Proofs of formulae given in Section 2

The proofs of the formulae of Sections 2.1, 2.7, 2.8, 2.9 and 2.10 are given in [4] and [6].

4.1. Proofs of formulae in Section 2.2

We substitute:

$$F(x, w) = e^w \left(1 - \frac{w}{a}\right)^x, \qquad p_n(x) = \frac{C_n(x, a)}{n!},$$

$$f(x, w) = e^{w - 2BXw + B^2 w^2} \left(1 - \frac{w}{a}\right)^x$$

in the formulae of Section 3.2. Then we obtain (5). The recursion (6) for c_k follows from substituting the Maclaurin series of f into the differential equation

$$xw^2 f = (w - a)a^2 \frac{df}{dw}.$$

From (6) we have $c_k = \mathcal{O}\left(a^{-k}\right)$ when $a \to \infty$. The asymptotic behaviour 2.2.3 follows from this and the asymptotic behaviour of X and B. Replacing x by $\sqrt{2a}x + a$ in the expansion given in (5) we have that $B = \mathcal{O}(a^{-1/2})$ and $X = -x + \mathcal{O}(a^{-1/2})$ when $a \to \infty$. From (6) we have that $c_k = \mathcal{O}(a^{-2(k+1)/3})$ when $a \to \infty$. Then, the asymptotic property 2.2.3 is replaced by $c_k B^{-k} H_{n-k}(X) = \mathcal{O}(a^{-(k+4)/6})$ when $a \to \infty$. Nevertheles, taking the limit $a \to \infty$ in (5), we obtain the limit 2.2.4.

4.2. Proofs of formulae in Section 2.3

We substitute:

$$F(x,w) = \left(1 - \frac{(1-p)}{p}\,w\right)^x (1+w)^{N-x}, \qquad\qquad p_n(x) = \binom{N}{n} K_n(x;p,N),$$

$$f(x,w) = \left(1 - \frac{(1-p)}{p}\,w\right)^x (1+w)^{N-x} e^{-2BXw+B^2w^2}$$

in the formulae of Section 3.2. Then we obtain (7). The recursion (8) for c_k follows from substituting the Maclaurin series of f into the differential equation

$$\begin{aligned} &w^2[(2xp^2 - 3xp - p^3N + p^2N + x)w + x - 3xp + 3xp^2 - p^3N]f + \\ &[(p-1)p^2w^2 + (2p-1)p^2w + p^3]\frac{df}{dw} = 0. \end{aligned}$$

From (8) we have $c_k = \mathcal{O}\left(N^{[k/3]}\right)$ when $N \to \infty$. The asymptotic behaviour 2.3.3 follows from this and the asymptotic behaviour of X and B. Replacing x by $pN + x\sqrt{2p(1-p)N}$ in the expansion given in (7) we have that $B = \sqrt{(1-p)N/(2p)} + \mathcal{O}(N^{-1/2})$ and $X = -x + \mathcal{O}(N^{-1})$ when $N \to \infty$. From (8) we have that $c_k = \mathcal{O}(N^{\lfloor k/3 \rfloor})$ when $N \to \infty$. Then, the asymptotic property 2.3.3 is replaced by $c_k B^{-k} H_{n-k}(X) = \mathcal{O}(N^{\lfloor k/3 \rfloor - k/2})$ when $N \to \infty$. Nevertheless, taking the limit $N \to \infty$ in (7), we obtain the limit 2.3.4.

4.3. Proofs of formulae in Section 2.4

We substitute:

$$F(x,w) = \left(1 - \frac{w}{c}\right)^x (1-w)^{-x-\beta}, \qquad\qquad p_n(x) = \frac{(\beta)_n}{n!} M_n(x;\beta,c),$$

$$f(x,w) = \left(1 - \frac{w}{c}\right)^x (1-w)^{-x-\beta} e^{-2BXw+B^2w^2}$$

in the formulae of Section 3.2. Then we obtain (9). The recursion (10) for c_k follows from substituting the Maclaurin series of f into the differential equation

$$w^2[x - \beta c^3 - c^3x + w(\beta c^2 - x - c^2x)]f = c^2[-c + w(c+1) - w^2]\frac{df}{dw}.$$

From (10) we have $c_k = \mathcal{O}\left(\beta^{[k/3]}\right)$ when $\beta \to \infty$. The asymptotic behaviour 2.4.3 follows from this and the asymptotic behaviour of X and B. Replacing x by $c(\beta - x\sqrt{2\beta/c})/(1-c)$ in the expansion given in (9) we have that $B = \sqrt{\beta/(2c)} + \mathcal{O}(\beta^{-1/2})$ and $X = x + \mathcal{O}(\beta^{-1})$ when $\beta \to \infty$. From (10) we have that $c_k = \mathcal{O}(\beta^{\lfloor k/3 \rfloor})$ when $\beta \to \infty$. Then, the asymptotic property 2.4.3 is replaced by $c_k B^{-k} H_{n-k}(X) = \mathcal{O}(\beta^{\lfloor k/3 \rfloor - k/2})$ when $\beta \to \infty$. Nevertheless, taking the limit $\beta \to \infty$ in (9), we obtain the limit 2.4.4.

4.4. Proofs of formulae in Section 2.5

We substitute:

$$F(x,w) = \left(1-e^{i\phi}w\right)^{-\lambda+ix}\left(1-e^{-i\phi}w\right)^{-\lambda-ix}, \qquad p_n(x) = P_n^{(\lambda)}(x;\phi),$$

$$f(x,w) = \left(1-e^{i\phi}w\right)^{-\lambda+ix}\left(1-e^{-i\phi}w\right)^{-\lambda-ix} e^{-2BXw+B^2w^2}$$

in the formulae of Section 3.2. Then we obtain (11) and (12). The recursion (13) for c_k follows from substituting the Maclaurin series of f into the differential equation

$$2w^2\left\{\lambda\cos(3\phi)+x\sin(3\phi)-[\lambda\cos(2\phi)+x\sin(2\phi)]w\right\}f = (1-2w\cos\phi+w^2)\frac{df}{dw}.$$

From (13) we have $c_k = \mathcal{O}\left(\lambda^{[k/3]}\right)$ when $\lambda\to\infty$. The asymptotic behaviour 2.5.3 follows from this and the asymptotic behaviour of X and B. Replacing x by $(\sin\phi)^{-1}(x\sqrt{\lambda}-\lambda\cos\phi)$ in the expansion given in (11) we have that $B=\sqrt{\lambda}+\mathcal{O}(\lambda^{-1/2})$ and $X = x+\mathcal{O}(\lambda^{-1})$ when $\lambda\to\infty$. From (13) we have that $c_k=\mathcal{O}(\lambda^{\lfloor k/3\rfloor})$ when $\lambda\to\infty$. Then, the asymptotic property 2.5.3 is replaced by $c_kB^{-k}H_{n-k}(X)=\mathcal{O}(\lambda^{\lfloor k/3\rfloor-k/2})$ when $\lambda\to\infty$. Nevertheless, taking the limit $\lambda\to\infty$ in (11), we obtain the limit 2.5.4.

4.5. Proofs of formulae in Section 2.6

We substitute:

$$F(x,w) = \frac{2^{\alpha+\beta}}{R(w)(1+R(w)-w)^{\alpha}(1+R(w)+w)^{\beta}}, \qquad R(w)\equiv\sqrt{1-2xw+w^2},$$

$$p_n(x) = P_n^{(\alpha,\beta)}(x), \qquad f(x,w) = \frac{2^{\alpha+\beta}e^{-2BXw+B^2w^2}}{R(w)(1+R(w)-w)^{\alpha}(1+R(w)+w)^{\beta}}$$

in the formulae of Section 3.2. Then we obtain (14) and (15). Using these values for X and B we have that

$$\phi(w)\equiv\log(f(x,w)) = w^3\left[\phi_1(x,w)+(\alpha+\beta)\phi_2(x,w)+(\alpha-\beta)\phi_3(x,w)\right],$$

where

$$\begin{aligned}
\phi_1(x,w) =& -\log\left(R(w)\right)-xw-(x^2-1/2)w^2,\\
\phi_2(x,w) =& -\frac{1}{2}\left\{\log[(1+R(w))^2-w^2]-2\log(2)+xw+\frac{1}{4}(3x^2-1)w^2\right\},\\
\phi_3(x,w) =& \frac{1}{2}\left[\log\left(\frac{1+R(w)+w}{1+R(w)-w}\right)-w-x\frac{w^2}{2}\right].
\end{aligned}$$

Then, the asymptotic behaviour $c_k=\mathcal{O}\left((\alpha+\beta)^{[k/3]}\right)$ when $\alpha+\beta\to\infty$, $(\alpha-\beta)/(\alpha+\beta)\to 0$ follows from Lemma 1 (in the remaining of the paragraph we assume that

$(\alpha-\beta)/(\alpha+\beta)\to 0)$. The asymptotic behaviour 2.6.3 follows from this and the asymptotic behaviour of X and B. Replacing x by $(\sqrt{2\alpha+2\beta}x+\beta-\alpha)/(2+\alpha+\beta)$ in the expansion of (14) we have that $B=\sqrt{(\alpha+\beta)/8}+\mathcal{O}((\alpha+\beta)^{-1/2})$ and $X=x+\mathcal{O}((\alpha+\beta)^{-1/2})$ when $\alpha+\beta\to\infty$. From the facts $x=\mathcal{O}((\alpha+\beta)^{-1/2})$ and $R(w)=\mathcal{O}(1)$ when $\alpha+\beta\to\infty$, we see that the asymptotic behaviour $c_k=\mathcal{O}((\alpha+\beta)^{\lfloor k/3\rfloor})$ when $\alpha+\beta\to\infty$ remains. Then, the asymptotic property 2.6.3 is replaced by $c_kB^{-k}H_{n-k}(X)=\mathcal{O}((\alpha+\beta)^{\lfloor k/3\rfloor-k/2})$ when $\alpha+\beta\to\infty$. Nevertheless, taking the limit $\alpha+\beta\to\infty$ in (14), with $(\alpha-\beta)/(\alpha+\beta)\to 0$, we obtain the limit 2.6.4.

4.6. Proofs of formulae in Section 2.11

We substitute:

$$F(x,w)=\left(1-\frac{1-p}{p}w\right)^x(1+w)^{N-x},\qquad p_n(x)=\binom{N}{n}K_n(x,p,N),$$

$$f(x,w)=\left(1-\frac{1-p}{p}w\right)^x(1+w)^{N-x}e^{-Bw}\left(1-\frac{B}{A}w\right)^{-X}$$

in the formulae of Section 3.1. Then we obtain (22) and (23). The recursion (24) for c_k follows from substituting the Maclaurin series of f into the differential equation

$$(w+1)(p-w+pw)\left\{Np^3(1+w)+\left[p-w+3pw-p^2(2+3w)\right]x\right\}\frac{df}{dw}+w^3(p-1)^2x(N-x)f=0.$$

From (24) we have $c_k=\mathcal{O}(1)$ when $N\to\infty$. The asymptotic behaviour 2.11.3 follows from this and the asymptotic behaviour of A, B and X. Replacing p by a/N in the expansion given in (22) we have that $A=a+\mathcal{O}(N^{-1})$, $B=N+\mathcal{O}(1)$ and $X=x+\mathcal{O}(N^{-1})$ when $N\to\infty$. From (24) we have that $c_k=\mathcal{O}(N^{k-1})$ when $N\to\infty$ for $k\geq 4$. Then, the asymptotic property 2.11.3 is lost, but, from (22) we have that $\binom{N}{n}K_n(x,a/N,N)=\frac{B^n}{n!}[C_n(X,A)+\mathcal{O}(N^{-1})]$ when $N\to\infty$. Taking the limit $N\to\infty$ in this expression we obtain the limit 2.11.4.

4.7. Proofs of formulae in Section 2.12

We substitute:

$$F(x,w)=\left(1-\frac{w}{c}\right)^x(1-w)^{-x-\beta},\qquad p_n(x)=\frac{(\beta)_n}{n!}M_n(x,\beta,c),$$

$$f(x,w)=\left(1-\frac{w}{c}\right)^x(1-w)^{-x-\beta}e^{-Bw}\left(1-\frac{B}{A}w\right)^{-X}$$

in the formulae of Section 3.1. Then we obtain (25) and (26). The recursion (27) for c_k follows from substituting the Maclaurin series of f into the differential equation

$$-w^3(c-1)^2x(\beta+x)f = (c-w)(w-1)\left\{\beta c^3(w-1) + \left[c + c^3(w-1) - w\right]x\right\}\frac{df}{dw}.$$

From (27) we have $c_k = \mathcal{O}\left(1\right)$ when $\beta \to \infty$. The asymptotic behaviour 2.12.3 follows from this and the asymptotic behaviour of A, B and X. Replacing c by $a/(a+\beta)$ in the expansion given in (25) we have that $A = a + \mathcal{O}(\beta^{-1})$, $B = \beta + \mathcal{O}(1)$ and $X = x + \mathcal{O}(\beta^{-1})$ when $\beta \to \infty$. From (27) we have that $c_k = \mathcal{O}(\beta^{k-1})$ when $\beta \to \infty$ for $k \geq 4$. Then, the asymptotic property 2.12.3 is lost, but, from (25) we have that $M_n(x, \beta, a/(a+\beta)) = \frac{B^n}{(\beta)_n}[C_n(X, A) + \mathcal{O}(\beta^{-1})]$ when $\beta \to \infty$. Taking the limit $\beta \to \infty$ in this expression we obtain the limit 2.12.4.

4.8. Proofs of formulae in Section 2.13

We substitute:

$$F(x,w) = \left(1 - e^{i\phi}w\right)^{-\lambda+ix}\left(1 - e^{-i\phi}w\right)^{-\lambda-ix}, \qquad p_n(x) = P_n^{(\lambda)}(x,\phi),$$

$$f(x,w) = \left(1 - e^{i\phi}w\right)^{-\lambda+ix}\left(1 - e^{-i\phi}w\right)^{-\lambda-ix} e^{-Bw}\left(1 - \frac{B}{A}w\right)^{-X}$$

in the formulae of Section 3.1. Then we obtain (28) and (29). The recursion (30) for c_k follows from substituting the Maclaurin series of f into the differential equation

$$-w^3 2r\sin^2\phi f = \left(2w\cos\phi - 1 - w^2\right)\left[w\sin(3\phi+\theta) - \sin(2\phi+\theta)\right]\frac{df}{dw},$$

where $x = r\cos\theta$, $\lambda = r\sin\theta$. From (30) we have $c_k = \mathcal{O}\left(\lambda^{[k/4]}\right)$ when $\lambda \to \infty$. The asymptotic behaviour 2.13.3 follows from this and the asymptotic behaviour of A, B and X. Taking the limit $\lambda \to \infty$ in the expansion given in (28) we obtain the limit 2.13.4. This is not given in terms of Charlier polynomials because $X = -2\lambda\cos^3(2\phi)\cos^{-2}(3\phi) + \mathcal{O}(1)$ and $A = -2\lambda\sin^2\phi\cos(2\phi)\cos^{-2}(3\phi) + \mathcal{O}(1)$ when $\lambda \to \infty$. Then, the limit $\lim_{\lambda\to\infty} C_n(X,A)$ is a number independent of x.

4.9. Proofs of formulae in Section 2.14

We substitute:

$$F(x,w) = \frac{2^{\alpha+\beta}}{R(w)(1+R(w)-w)^\alpha(1+R(w)+w)^\beta}, \qquad p_n(x) = P_n^{(\alpha,\beta)}(x),$$

$$f(x,w) = \frac{2^{\alpha+\beta}}{R(w)(1+R(w)-w)^\alpha(1+R(w)+w)^\beta}e^{-Bw}\left(1 - \frac{B}{A}w\right)^{-X}$$

in the formulae of Section 3.1 (we have set $A = B$ and then there are two free parameters instead of three). Then we obtain (31) and (32). Using this values for X and B we have that

$$\phi(w) \equiv \log(f(x,w)) = \beta\phi_1(x,w) + \phi_2(x,w),$$

where

$$\begin{aligned}
\phi_1(x,w) =& \frac{1}{4}w(x-1)(6x+2) + 2\log 2 + \frac{1}{4}(6x^2-2)\log(1-w) - \log\left[(R(w)+1)^2 - w^2\right],\\
\phi_2(x,w) =& w\left(-1 - \frac{3(\alpha-\beta)}{4} - x + 2x^2 + \frac{3(\alpha-\beta)x^2}{4}\right) + (\alpha-\beta)\log 2\\
&+ \left(-1 + 2x^2 + \frac{(\alpha-\beta)}{4}(-1+2x+3x^2)\right)\log(1-w) -\\
&\log R(w) - (\alpha-\beta)\log\left(1 - w + R(w)\right).
\end{aligned}$$

The series expansions of these functions in powers of w have the form

$$\phi_1(x,w) = w^3\left(a_0 + a_1 w + a_2 w^2 + ...\right), \qquad \phi_2(x,w) = w^3\left(b_0 + b_1 w + b_2 w^2 + ...\right)$$

with $a_0 \neq 0$, $b_0 \neq 0$.

Then, the asymptotic behaviour $c_k = \mathcal{O}\left((\alpha+\beta)^{[k/3]}\right)$ when $\alpha+\beta \to \infty$, $(\alpha-\beta)/(\alpha+\beta) \to 0$ follows from lemma 1. The asymptotic behaviour 2.14.3 follows from this and the asymptotic behaviour of X and B. Taking the limit $\alpha+\beta\to\infty$, with $(\alpha-\beta)/(\alpha+\beta)\to 0$, in the expansion given in (31) we obtain the limit 2.14.4. This limit is not given in terms of Charlier polynomials because of a similar reason to the one explained for Meixner-Pollaczeck→Charlier case.

5. Numerical experiments

The following graphics show the accuracy of the first order approximation supplied by the expansions given in Section 2. It is worthwhile to note the accuracy obtained in the approximation of the zeros of the polynomials. All the graphics are cut for extreme values of the polynomials.

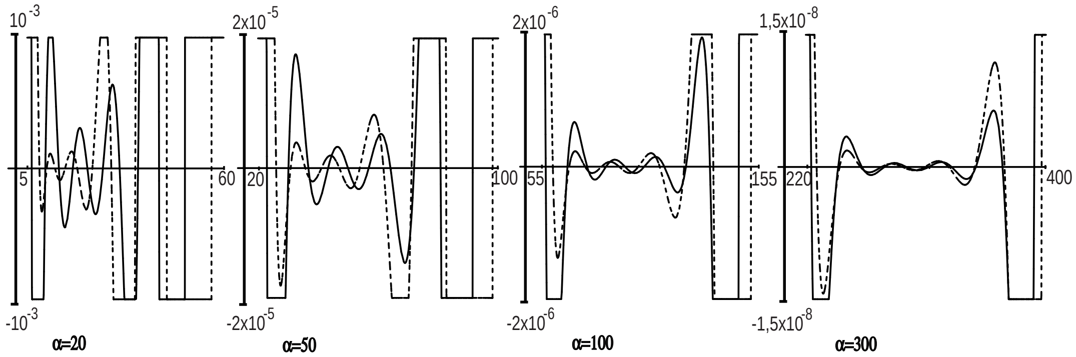

Figure 2. *Continuous lines represent the Charlier polynomial for* $n = 10$ *and several values of* a. *Dashed lines represent the first term in the right hand side of the expansion given in (5).*

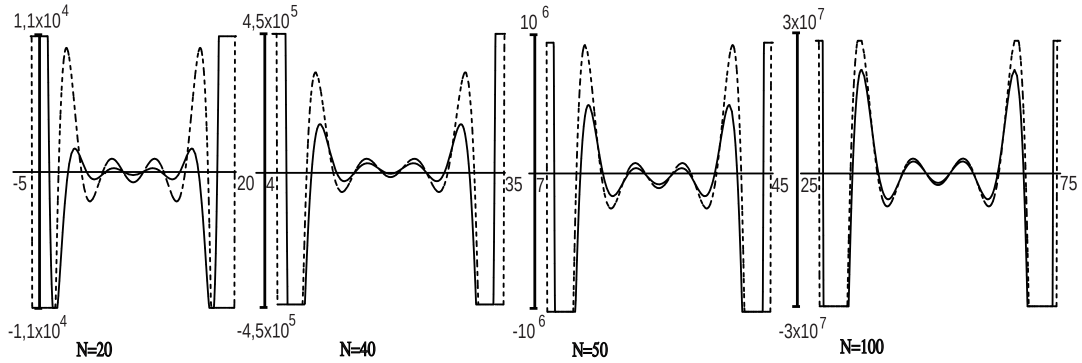

Figure 3. *Continuous lines represent the Krawtchouk polynomial for* $n = 10$, $p = 1/2$ *and several values of* N. *Dashed lines represent the first term in the right hand side of the expansion given in (7).*

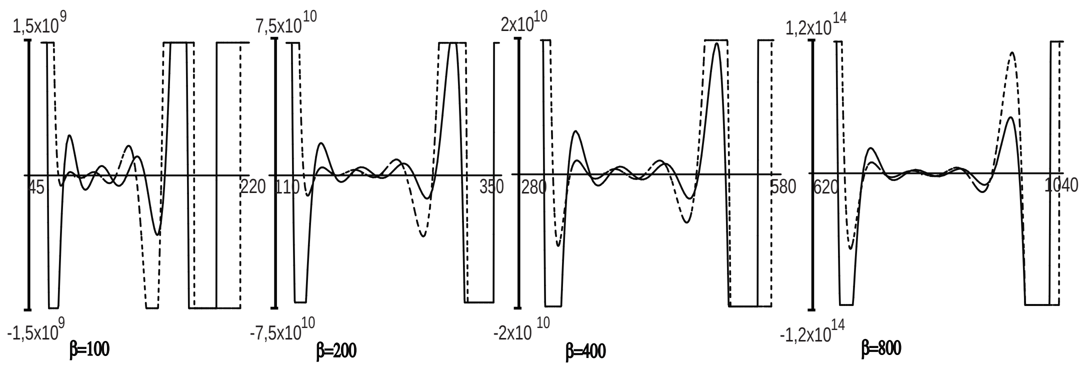

Figure 4. *Continuous lines represent the Meixner polynomial for* $n = 10$, $c = 1/2$ *and several values of* β. *Dashed lines represent the first term in the right hand side of the expansion given in (9).*

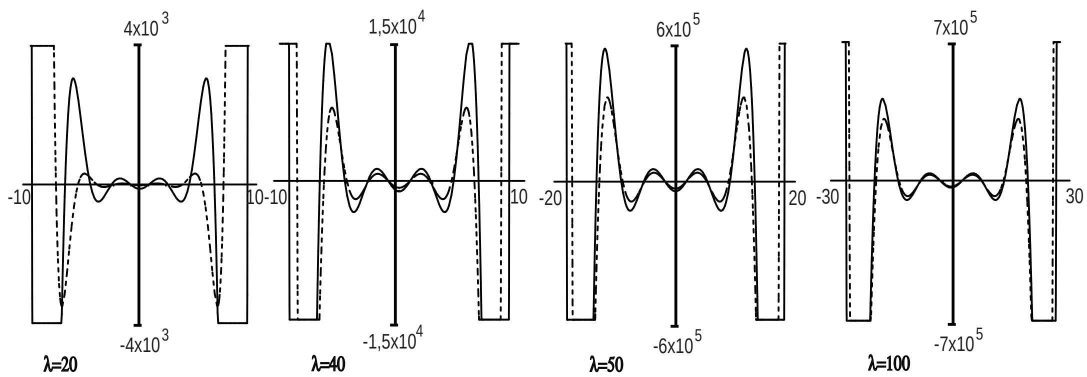

Figure 5. *Continuous lines represent the Meixner-Pollaczeck polynomial for $n = 10$, $\phi = \pi/2$ and several values of λ. Dashed lines represent the first term in the right hand side of the expansion given in (11)-(12).*

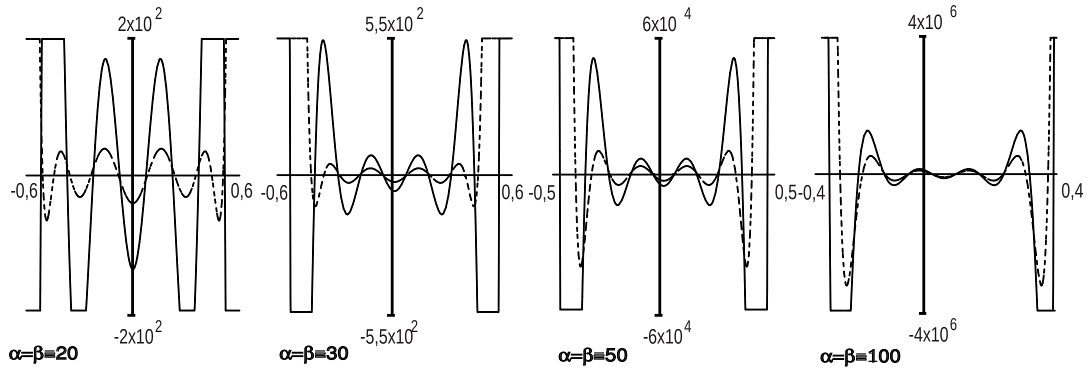

Figure 6. *Continuous lines represent the Jacobi polynomial for $n = 10$ and several values of $\alpha = \beta$. Dashed lines represent the first term in the right hand side of the expansion given in (14)-(15).*

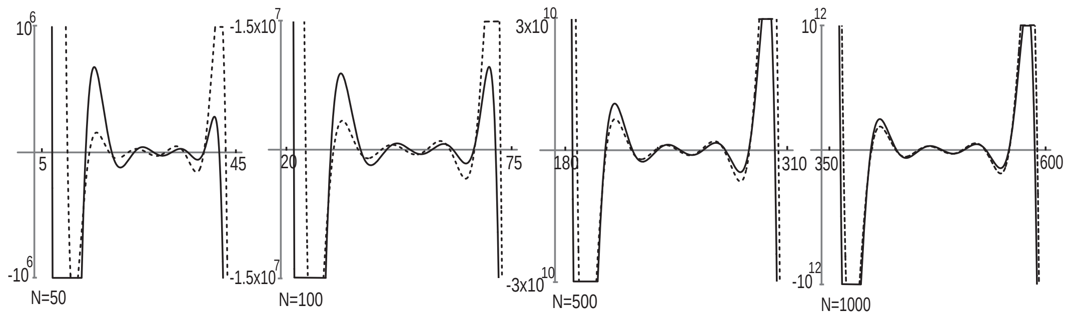

Figure 7. *Continuous lines represent the Krawtchouk polynomial for $n = 10$, $p = 1/2$ and several values of N. Dashed lines represent the first term in the right hand side of the expansion given in (22)-(23).*

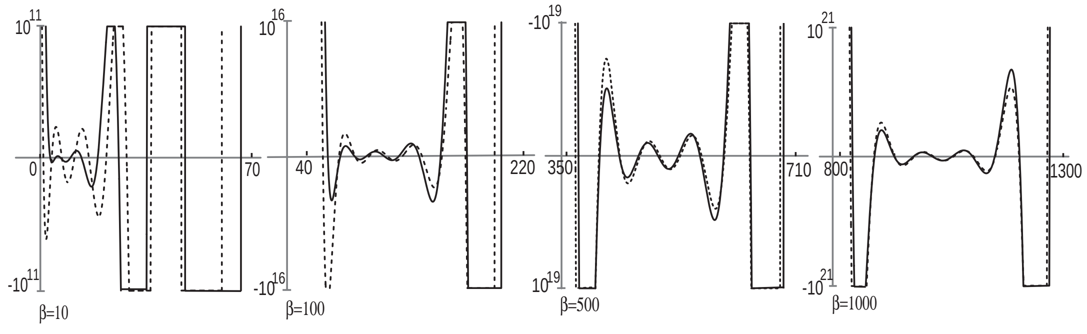

Figure 8. *Continuous lines represent the Meixner polynomial for $n = 10$, $c = 1/2$ and several values of β. Dashed lines represent the first term in the right hand side of the expansion given in (25)-(26).*

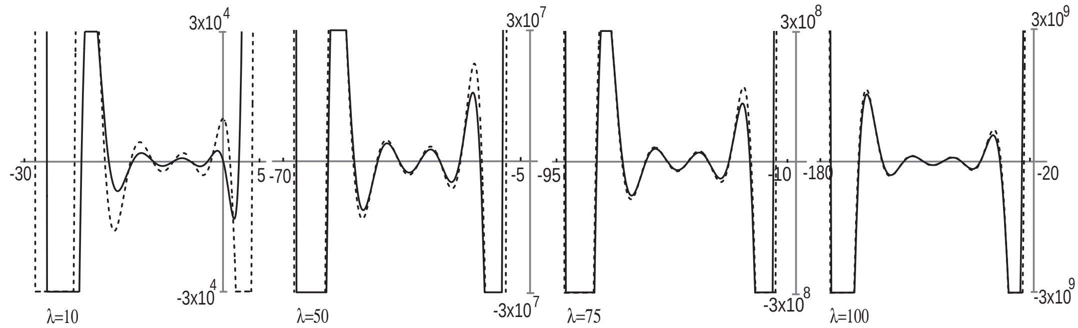

Figure 9. *Continuous lines represent the Meixner-Pollaczeck polynomial for $n = 10$, $\phi = 1$ and several values of λ. Dashed lines represent the first term in the right hand side of the expansion given in (28)-(29).*

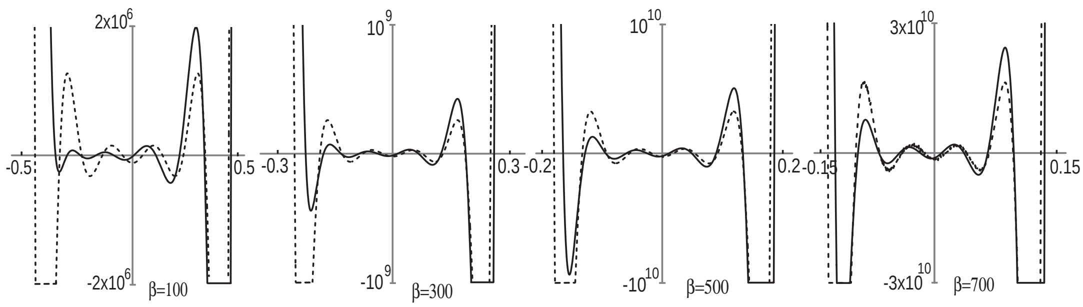

Figure 10. *Continuous lines represent the Jacobi polynomial for $n = 10$ and several values of $\alpha = \beta$. Dashed lines represent the first term in the right hand side of the expansion given in (31)-(32).*

6. Acknowledgments

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References

[1] E. Godoy, A. Ronveaux, A. Zarzo and I. Area, On the limit relations between classical continuous and discrete orthogonal polynomials, *J. Comp. Appl. Math.* **91** (1998) 97-105.

[2] E. Godoy, A. Ronveaux, A. Zarzo and I. Area, Transverse limits in the Askey tableau, *J. Comp. Appl. Math.* **99** (1998) 327-335.

[3] R. Koekoek and R. F. Swarttouw, Askey scheme or hypergeometric orthogonal polynomials, *http://aw.twi.tudelft.nl/koekoek/askey*,(1999)

[4] J.L. López and N. M. Temme, Approximations of orthogonal polynomials in terms of Hermite polynomials, *Meth. Appl. Anal.* **6** (1999) 131-146.

[5] J.L. López and N. M. Temme, Hermite polynomials in asymptotic representations of generalized Bernoulli, Euler, Bessel and Buchholz polynomials, *J. Math. Anal. Appl.* **239** (1999) 457-477.

[6] J.L. López and N. M. Temme, The Askey scheme for hypergeometric orthogonal polynomials viewed from asymptotic analysis, *J. Comp. Appl. Math.* **133** (2001) 623-633.

[7] B. Rui and R. Wong, Uniform asymptotic expansions of Charlier polynomials, *Meth. Appl. Anal.* **1** (1994) 294-313.

[8] Frenzen, C.L., and R. Wong, Uniform asymptotic expansions of Laguerre polynomials, *SIAM J. Math. Anal.* **19** (1988) 1232-1248.