Asymptotic approximations between the Hahn-type polynomials and Hermite, Laguerre and Charlier polynomials

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ABSTRACT

It has been shown in [1], [5], [7] that the three lower levels of the Askey table of hypergeometric orthogonal polynomials are connected by means of asymptotic expansions. In this paper we continue with that investigation and establish asymptotic connections between the fourth level and the two lower levels: we derive twelve asymptotic expansions of the Hahn, dual Hahn, continuous Hahn and continuous dual Hahn polynomials in terms of Hermite, Charlier and Laguerre polynomials. From these expansions, several limits between polynomials are derived. Some numerical experiments give an idea about the accuracy of the approximations and, in particular, about the accuracy in the approximation of the zeros of the Hahn, dual Hahn, continuous Hahn and continuous dual Hahn polynomials in terms of the zeros of the Hermite, Charlier and Laguerre polynomials.

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1. Introduction

It is well known that there exist several asymptotic relations between polynomials of the Askey scheme of hypergeometric orthogonal polynomials [1], [5], [7]. For example, the Meixner polynomials can be expressed in terms of the Hermite polynomials as follows [1]:

$$M_n(x;\beta,c)=\frac{n!B^n}{(\beta)_n}\sum_{k=0}^{n}\frac{B^k}{z^k}\frac{H_{n-k}(X)}{(n-k)!}, \tag{1}$$

where B and X are detailed in [1]. This expansion has asymptotic character for large β. From the first term of this expansion we can get the limit

$$\lim_{\beta\to\infty} (\beta)_n \left(\frac{2c}{\beta}\right)^{n/2} M_n\left(\frac{c\beta - x\sqrt{2c\beta}}{1-c}; \beta, c\right) = H_n(x),$$

which shows that, when the variable of the Meixner polynomials is properly scaled, the Meixner polynomials become the Hermite polynomials for large values of β. Moreover, this limit gives insight in the location of the zeros of the Meixner polynomials for large values of β in terms of the zeros of $H_n(x)$.

The asymptotic method from which expansions like (1) are obtained was introduced and developed in [1], [5], [6] and [7]. More precisely, the method to approximate orthogonal polynomials in terms of Hermite polynomials is described in [5], whereas [7] introduces the approximation in terms of Laguerre polynomials and [1] in terms of Charlier polynomials. In these references, asymptotic expansions of Laguerre and Charlier polynomials in terms of Hermite polynomials and asymptotic expansions of Meixner-Pollaczek, Jacobi, Meixner and Krawtchouk polynomials in terms of Laguerre, Charlier and Hermite polynomials are given. That is, [1], [5] and [7] contain the 14 possible asymptotic relations between the three lower levels of the Askey table.

Those asymptotic methods are based on the availability of a generating function for the polynomials and is different from the techniques described in [2], [3]. The techniques used in [2] and [3] are based on a connection problem and gives deeper information on the limit relations between classical discrete and classical continuous orthogonal polynomials. On the other hand, our method gives asymptotic expansions of polynomials situated at any level of the table in terms of polynomials located at lower levels. Our method is also different from the sophisticated uniform methods considered for example in [8] or [9], where asymptotic expansions of the Meixner $M_n(nx, b, c)$ or Charlier $C_n^a(nx)$ polynomials respectively are given for large values of n and fixed a, b, c, x. In our method we keep the degree n fixed and let some parameter(s) of the polynomial go to infinity. The purpose of this paper is the continuation of the asymptotic program started in [1], [5], [7] and derive asymptotic expansions (and limits when it is possible) between the fourth level and the two lower levels of the Askey tableau.

In the following section we summarize the asymptotic expansions and the limit relations obtained in this paper. In Section 3 we briefly summarize the principles of the Hermite-type, Laguerre-type and Charlier-type asymptotic approximations introduced in [1], [5] and [7]. In Section 4 we prove the formulas of Section 2. Some numerical experiments illustrating the accuracy of the approximations are given in Section 5.

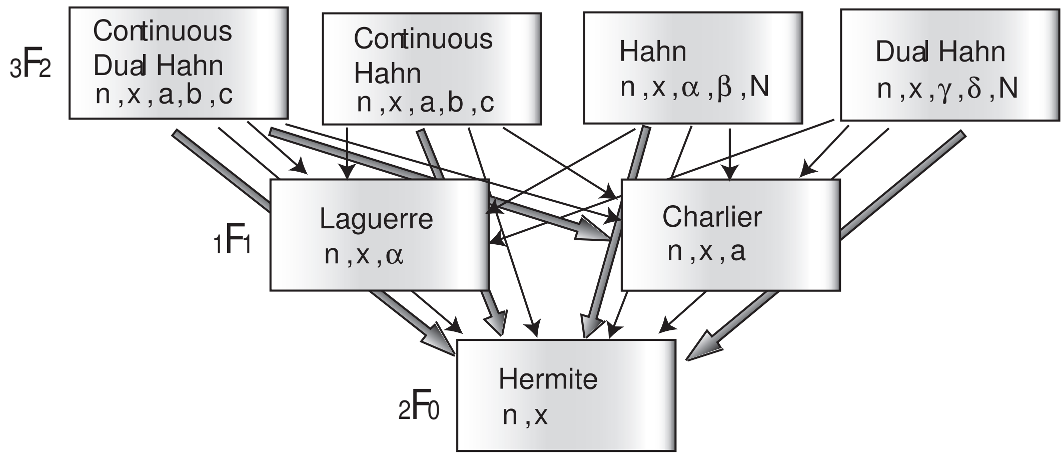

Figure 1. *Thin arrows indicate known limits, whereas thick arrows indicate new (as far as we know) limits derived in this paper.*

2. Descending asymptotic expansions and limits

The orthogonality property of the polynomials of the Askey table only holds when the variable x and other parameters which define the polynomials are restricted to certain real intervals [4]. The expansions that we resume below are valid for larger domains of the variable and the parameters and for any $n \in \mathbb{N}$. Nevertheless, for the shake of simplicity, we restrict ourselves to the orthogonality intervals. All of the square roots that appear in what follows assume real positive values for real positive argument. The coefficients c_k that appear below are the coefficients of the Taylor expansion at $w = 0$ of the given functions $f(w)$. The parameters α and β appearing in some formulas related to the continuous Hahn polynomials read $\alpha = a + ic$ and $\beta = b + id$ with a, b, c, $d \in \mathbb{R}$.

2.1. Continuous Dual Hahn to Hermite

2.1.1. Asymptotic expansion for large c:

$$\frac{S_n^{a,b,c}(x^2)}{(a+b)_n n!} = \sum_{k=0}^{n} \frac{c_k B^{n-k}}{(n-k)!} H_{n-k}(X), \tag{2}$$

$$B = \sqrt{\frac{x^4 + [a(a+1) + b(b+1) + (a+b)^2]x^2 - ab[a(1+a) + b(1+b) + ab]}{2(a+b)^2(1+a+b)} - \frac{c}{2}},$$

$$X = \frac{c(a+b) + ab - x^2}{2B(a+b)}, \qquad f(w) = e^{-2BXw + B^2w^2}(1-w)^{-c+ix}{}_2F_1\left(\left.\begin{matrix} a+ix, b+ix \\ a+b \end{matrix}\right| w\right).$$

2.1.2. Asymptotic property:

$$\frac{c_k B^{n-k}}{(n-k)!} H_{n-k}(X) = \mathcal{O}\left(c^{n+\left[\frac{k}{3}\right]-k}\right) \quad \text{when } c \to \infty \text{ uniformly in } a, b \text{ with } \frac{a}{c} \text{ and } \frac{b}{c} \text{ bounded.} \tag{3}$$

2.1.3. Limit:

$$\lim_{a\to\infty} \frac{S_n^{a,b,ac}\left(\sqrt{ca^2 - xa\sqrt{2a(1+c)c}}\right)}{\left[a\left(x+\sqrt{a/2}\right)\sqrt{c(1+c)}\right]^n} = H_n(x). \tag{4}$$

2.2. Dual Hahn to Hermite

2.2.1. Asymptotic expansion for large N:

$$\frac{(-N)_n R_n^{a,b,N}(\lambda(x))}{n!} = \sum_{k=0}^{n} \frac{c_k B^{n-k}}{(n-k)!} H_{n-k}(X), \quad \lambda(x) \equiv x(x+a+b+1), \tag{5}$$

$$B = \sqrt{\frac{N-x}{2} + \frac{x(x+b)[(2x+b-1)(1+a)+x(x+b)]}{2(a+1)^2(a+2)}}, \qquad X = \frac{\lambda(x) - N(a+1)}{2B(a+1)},$$

$$f(w) = e^{-2BXw+B^2w^2}(1-w)^{N-x}\,{}_2F_1\left(\left.\begin{matrix} -x, -b-x \\ a+1 \end{matrix}\,\right|\, w\right).$$

2.2.2. Asymptotic property:

$$\frac{c_k B^{n-k}}{(n-k)!} H_{n-k}(X) = \mathcal{O}\left(N^{n+\left[\frac{k}{3}\right]-k}\right) \text{ when } N\to\infty \text{ uniformly in } a, b \text{ with } \frac{a}{N} \text{ and } \frac{b}{N} \text{ bounded.} \tag{6}$$

2.2.3. Limit:

$$\lim_{a\to\infty}\left[\left(-\sqrt{2a}\right)^n R_n^{a,b,aN}\left(\frac{a}{2}\left[\sqrt{1+4N+4Nx\sqrt{2/a}}-1\right]\right)\right] = H_n(x). \tag{7}$$

2.3. Continuous Hahn to Hermite

2.3.1. Asymptotic expansion for large a and b:

$$\frac{(2a+2b-1)_n P_n^{\alpha,\beta,\overline{\alpha},\overline{\beta}}(x)}{(2a)_n(a+b+i(c-d))_n i^n} = \sum_{k=0}^{n} \frac{c_k B^{n-k}}{(n-k)!} H_{n-k}(X), \tag{8}$$

$$B = \sqrt{\frac{1}{2}p_1^2(x) - p_2(x)}, \qquad X = \frac{p_1(x)}{2B}, \tag{9}$$

where

$$\begin{aligned} p_1(x) =& i\,(1-2a-2b)\,\frac{bc+ad+x(a+b)}{a[a+b+i(c-d)]}, \\ p_2(x) =& (2a+2b-1)(a+b)\left\{1 - \frac{(2a+2b+1)[a+i(c+x)]}{a[a+b+i(c-d)]}\right. \\ &\left. + \frac{(2a+2b+1)(2a+2b+2)[a+i(c+x)][1+a+i(c+x)]}{2a(2a+1)[a+b+i(c-d))][a+b+1+i(c-d)]}\right\}, \end{aligned} \tag{10}$$

$$f(w) = e^{-2BXw+B^2w^2}(1-w)^{1-2(a+b)}{}_3F_2\left(\left.\begin{array}{c} a+b-\frac{1}{2}, a+b, a+i(c+x) \\ 2a, a+c+i(b-d) \end{array}\right| -\frac{4w}{(1-w)^2}\right).$$

2.3.2. Asymptotic property:

$$\frac{c_k B^{n-k}}{(n-k)!}H_{n-k}(X) = \mathcal{O}\left(a^{\frac{n-k}{2}+\left[\frac{k}{3}\right]}\right) \qquad \text{when } a, b \to \infty \text{ with } a \sim b. \tag{11}$$

2.3.3. Limit:

$$\lim_{a\to\infty}\left[\left(-\frac{2}{i}\sqrt{\frac{a(1+b)}{b}}\right)^n P_n^{\alpha,\beta_1,\overline{\alpha},\overline{\beta_1}}\left(x\sqrt{\frac{ab}{1+b}}\right)\right] = H_n(x), \quad \text{with} \quad \beta_1 = ab+id. \tag{12}$$

2.4. Hahn to Hermite

2.4.1. Asymptotic expansion for large a and N:

$$\frac{(-N)_n Q_n^{a,b,N}(x)}{(b+1)_n n!} = \sum_{k=0}^{n} \frac{c_k B^{n-k}}{(n-k)!}H_{n-k}(X), \tag{13}$$

with B and X given in (9) and

$$\begin{aligned} p_2(x) &= (a+b+1)(a+b+2)\left[\frac{1}{2} - \frac{x(a+b+3)}{N(a+1)} + \frac{x(x-1)(a+b+3)(a+b+4)}{2N(N-1)(a+1)(a+2)}\right], \\ p_1(x) &= (a+b+1)\left[1 - \frac{x(a+b+2)}{N(a+1)}\right], \end{aligned} \tag{14}$$

$$f(w) = e^{-2BXw+B^2w^2}(1-w)^{-1-a-b}{}_3F_2\left(\left.\begin{array}{c} (a+b+1)/2, (a+b+2)/2, -x \\ a+1, -N \end{array}\right| -\frac{4w}{(1-w)^2}\right).$$

2.4.2. Asymptotic property:

$$\frac{c_k B^{n-k}}{(n-k)!}H_{n-k}(X) = \mathcal{O}\left(N^{n+\left[\frac{k}{3}\right]-k}\right) \tag{15}$$

when a, $N \to \infty$ with $a \sim N$ uniformly in b with b/N bounded.

2.4.3. Limit:

$$\begin{aligned} \lim_{a\to\infty}&\left[\left(\frac{2a}{2x+\sqrt{2a}}\right)^n \left(\frac{N(1+b)}{b(1+b+N)}\right)^{\frac{n}{2}} \times \right. \\ &\left. Q_n^{a,ab,aN}\left(\frac{aN(1+b) - x\sqrt{2abN(1+b)(1+b+N)}}{(1+b)^2}\right)\right] = H_n(x). \end{aligned} \tag{16}$$

2.5. Continuous Dual Hahn to Laguerre

2.5.1. Asymptotic expansion for large a and b:

$$\frac{S_n^{a,b,c}(x^2)}{(a+b)_n n!} = \sum_{k=0}^{n} c_k L_{n-k}^{X}(A), \tag{17}$$

$$A = p_1(x) + p_1^2(x) - 2p_2(x), \qquad X = A + p_1(x) - 1, \tag{18}$$

where

$$\begin{aligned} p_1(x) &= c + \frac{ab - x^2}{a+b}, \\ p_2(x) &= \frac{c(c+1)}{2} + \frac{abc}{a+b} + \frac{ab(1+a)(1+b) - [c + 2ab + (1+c)(1+2(a+b))]x^2 + x^4}{2(a+b)(1+a+b)}, \end{aligned} \tag{19}$$

$$f(w) = e^{Aw/(1-w)}(1-w)^{X+1}(1-w)^{-c+ix}{}_2F_1\left(\left.\begin{matrix} a+ix, b+ix \\ a+b \end{matrix}\right| w\right).$$

2.5.2. Asymptotic property:

$$c_k L_{n-k}^{X}(A) = \mathcal{O}\left(a^{n+[\frac{k}{3}]-k}\right) \quad \text{when} \quad a,\, b \to \infty \quad \text{with} \quad a \sim b. \tag{20}$$

2.6. Dual Hahn to Laguerre

2.6.1. Asymptotic expansion for large a and N:

$$\frac{(-N)_n R_n^{a,b,N}(\lambda(x))}{n!} = \sum_{k=0}^{n} c_k L_{n-k}^{X}(A), \quad \lambda(x) \equiv x(x+a+b+1), \tag{21}$$

with A and X given in (18) and

$$\begin{aligned} p_1(x) &= \frac{\lambda(x)}{a+1} - N, \\ p_2(x) &= \frac{(N-x)(N-x-1)}{2} + \frac{x(b+x)\,[2(2+a)(x-N) + (b+x-1)(x-1)]}{2(a+1)(2+a)}, \end{aligned} \tag{22}$$

$$f(w) = e^{Aw/(1-w)}(1-w)^{X+1}(1-w)^{N-x}{}_2F_1\left(\left.\begin{matrix} -x, -b-x \\ a+1 \end{matrix}\right| w\right).$$

2.6.2. Asymptotic property:

$$c_k L_{n-k}^{X}(A) = \mathcal{O}\left(N^{n-k-1}\right) \quad \text{when} \quad N,\, a \to \infty \quad \text{with} \quad N \sim a. \tag{23}$$

2.7. Continuous Hahn to Laguerre

2.7.1. Asymptotic expansion for large a and b:

$$\frac{(2a+2b-1)_n P_n^{\alpha,\beta,\overline{\alpha},\overline{\beta}}(x)}{(2a)_n(a+b+i(c-d))_n i^n} = \sum_{k=0}^{n} c_k L_{n-k}^{X}(A), \tag{24}$$

with A and X given in (18), $p_1(x)$ and $p_2(x)$ given in (10) and

$$f(w) = e^{Aw/(1-w)}(1-w)^{X+2(1-a-b)}{}_3F_2\left(\left.\begin{matrix} a+b-\frac{1}{2}, a+b, a+i(c+x) \\ 2a, a+c+i(b-d) \end{matrix}\right| -\frac{4w}{(1-w)^2}\right).$$

2.7.2. Asymptotic property:

$$c_k L_{n-k}^{X}(A) = \mathcal{O}\left(b^{[\frac{n-k}{2}]+[\frac{k}{3}]}\right) \quad \text{when} \quad a,\, b \to \infty \quad \text{with} \quad a \sim b. \tag{25}$$

2.8. Hahn to Laguerre

2.8.1. Asymptotic expansion for large a, b, N:

$$\frac{(a+b+1)_n Q_n^{a,b,N}(x)}{n!} = \sum_{k=0}^{n} c_k L_{n-k}^{X}(A), \tag{26}$$

with A and X given in (18), $p_1(x)$ and $p_2(x)$ given in (14) and

$$f(w) = e^{Aw/(1-w)}(1-w)^{X-a-b}{}_3F_2\left(\left.\begin{matrix} (a+b+1)/2, (a+b+2)/2, -x \\ a+1, -N \end{matrix}\right| -\frac{4w}{(1-w)^2}\right).$$

2.8.2. Asymptotic property:

$$c_k L_{n-k}^{X}(A) = \mathcal{O}\left(N^{n-k}\right) \quad \text{when} \quad a, b, N \to \infty \quad \text{with} \quad a \sim b \sim N. \tag{27}$$

2.9. Continuous Dual Hahn to Charlier

2.9.1. Asymptotic expansion for large a and b:

$$\frac{S_n^{a,b,c}(x^2)}{(a+b)_n n!} = \sum_{k=0}^{n} \frac{c_k A^{n-k}}{(n-k)!} C_{n-k}(X,A), \tag{28}$$

$$A = p_1(x) + p_1^2(x) - 2p_2(x), \qquad X = p_1^2(x) - 2p_2(x), \tag{29}$$

with $p_1(x)$ and $p_2(x)$ given in (19) and

$$f(w) = e^{-Aw}(1-w)^{-X-c+ix}{}_2F_1\left(\left.\begin{matrix} a+ix, b+ix \\ a+b \end{matrix}\right| w\right).$$

2.9.2. Asymptotic property:

$$\frac{c_k A^{n-k}}{(n-k)!} C_{n-k}(X,A) = \mathcal{O}\left(a^{n+[\frac{k}{3}]-k}\right) \tag{30}$$

when $a, b \to \infty$ uniformly in c with $a \sim b$ and c/a bounded.

2.9.3. Limit:

$$\lim_{a\to\infty} \frac{S_n^{a,ab,\tilde{c}}(\tilde{x})}{c^n a^n (1+b)^n} = C_n(x,c), \tag{31}$$

where

$$\begin{aligned} \tilde{x} &= \frac{1}{2}\left[-a^2(1+b^2) + a(1+b)\sqrt{a^2(1-b)^2 + 4ac(1+b)}\right], \\ \tilde{c} &= \frac{1}{2}\left[2c - 2x - a(1-b) + \sqrt{a^2(1-b)^2 + 4ac(1+b)}\right]. \end{aligned} \tag{32}$$

2.10. Dual Hahn to Charlier

2.10.1 Asymptotic expansion for large a and N:

$$\frac{(-N)_n R_n^{a,b,N}(\lambda(x))}{n!} = \sum_{k=0}^{n} \frac{c_k A^{n-k}}{(n-k)!} C_{n-k}(X,A), \qquad \lambda(x) \equiv x(x+a+b+1), \tag{33}$$

with A and X given in (29), $p_1(x)$ and $p_2(x)$ given in (22) and

$$f(w) = e^{-Aw}(1-w)^{-X+N-x}\,{}_2F_1\left(\left.\begin{matrix} -x, -b-x \\ a+1 \end{matrix}\right| w\right).$$

2.10.2. Asymptotic property:

$$\frac{c_k A^{n-k}}{(n-k)!} C_{n-k}(X,A) = \mathcal{O}\left(a^{n-k-2}\right) \quad \text{when} \quad a, N \to \infty \quad \text{with} \quad a \sim N. \tag{34}$$

2.11. Continuous Hahn to Charlier

2.11.1. Asymptotic expansions for large a and b:

$$\frac{(2a+2b-1)_n P_n^{\alpha,\beta,\overline{\alpha},\overline{\beta}}(x)}{(2a)_n (a+b+i(c-d))_n i^n} = \sum_{k=0}^{n} \frac{c_k A^{n-k}}{(n-k)!} C_{n-k}(X,A), \tag{35}$$

with A and X given in (29), $p_1(x)$ and $p_2(x)$ given in (10) and

$$f(w) = e^{-Aw}(1-w)^{-X+1-2(a+b)}\,{}_3F_2\left(\left.\begin{matrix} a+b-\frac{1}{2}, a+b, a+i(c+x) \\ 2a, a+c+i(b-d) \end{matrix}\right| -\frac{4w}{(1-w)^2}\right).$$

2.11.2. Asymptotic property:

$$\frac{c_k A^{n-k}}{(n-k)!} C_{n-k}(X,A) = \mathcal{O}\left(b^{\left[\frac{n-k+1}{2}\right]+\left[\frac{k}{3}\right]}\right) \quad \text{when} \quad a,\, b \to \infty \quad \text{with} \quad a \sim b. \tag{36}$$

2.12. Hahn to Charlier

2.12.1. Asymptotic expansions for large a, b and N:

$$\frac{(a+b+1)_n Q_n^{a,b,N}(x)}{n!} = \sum_{k=0}^{n} \frac{c_k A^{n-k}}{(n-k)!} C_{n-k}(X,A), \tag{37}$$

with A and X given in (29), $p_1(x)$ and $p_2(x)$ given in (14) and

$$f(w) = e^{-Aw}(1-w)^{-X-a-b-1}\,{}_3F_2\left(\left.\begin{array}{c}(a+b+1)/2, (a+b+2)/2, -x\\ a+1, -N\end{array}\right| -\frac{4w}{(1-w)^2}\right).$$

2.12.2. Asymptotic property:

$$\frac{c_k A^{n-k}}{(n-k)!} C_{n-k}(X,A) = \mathcal{O}\left(a^{n-k}\right) \quad \text{when} \quad a, b, N \to \infty \quad \text{with} \quad a \sim b \sim N. \tag{38}$$

3. Principles of the asymptotic approximations

3.1. Expansions in terms of Hermite polynomials

To prove the results of Sections 2.1, 2.2, 2.3 and 2.4 we need the following formulas derived in [5]. If $F(w)$ is the generating function of the polynomials $p_n(x)$, then:

$$p_n(x) = B^n \sum_{k=0}^{n} \frac{c_k}{B^k} \frac{H_{n-k}(X)}{(n-k)!}, \tag{39}$$

where the coefficients c_k follow from

$$f(w) = \sum_{k=0}^{\infty} c_k w^k, \qquad\qquad f(w) = e^{B^2w^2-2BXw} F(w).$$

The choice of X and B is based on our requirement that $c_1 = c_2 = 0$. This happens if we take

$$B = \sqrt{\tfrac{1}{2}p_1^2(x) - p_2(x)}, \qquad\qquad X = \frac{p_1(x)}{2B}$$

and we assume that $F(0) = p_0(x) = 1$ (which implies $c_0 = 1$).

The quantities X and B may depend on x, and if B happens to be zero for a special $x-$value x_0, say, we write $p_n(x_0) = \sum_{k=0}^{n} \frac{c_k}{(n-k)!}\, p_1^{n-k}(x_0)$.

3.2. Expansions in terms of Laguerre polynomials

To prove the results of Sections 2.5, 2.6, 2.7 and 2.8 we need the following formulas derived in [7]. If $F(w)$ is the generating function of the polynomials $p_n(x)$, then

$$p_n(x) = \sum_{k=0}^{n} c_k L_{n-k}^{X}(A), \tag{40}$$

where the coefficients c_k follow from

$$f(w) = \sum_{k=0}^{\infty} c_k w^k, \qquad f(w) = (1-w)^{X+1} e^{Aw/(1-w)} F(w).$$

The choice of A and X is based on our requirement that $c_1 = c_2 = 0$. This happens if we take

$$A = p_1(x) + p_1^2(x) - 2p_2(x), \qquad X = A + p_1(x) - 1 \tag{41}$$

and we assume that $F(0) = p_0(x) = 1$ (which implies $c_0 = 1$).

3.3. Expansions in terms of Charlier polynomials

To prove the results of Sections 2.9, 2.10, 2.11 and 2.12 we need the following formulas derived in [7]. If $F(w)$ is the generating function of the polynomials $p_n(x)$, then

$$p_n(x) = A^n \sum_{k=0}^{n} \frac{c_k}{A^k} \frac{C_{n-k}(X, A)}{(n-k)!}, \tag{42}$$

where the coefficients c_k follow from

$$f(w) = \sum_{k=0}^{\infty} c_k w^k, \qquad f(w) = (1-w)^{-X} e^{-Aw} F(w).$$

The choice of A and X is based on our requirement that $c_1 = c_2 = 0$. This happens if we take

$$A = p_1(x) + p_1^2(x) - 2p_2(x), \qquad X = p_1^2(x) - 2p_2(x) \tag{43}$$

and we assume that $F(0) = p_0(x) = 1$ (which implies $c_0 = 1$). If A happens to be zero for a special $x-$value x_0, then we write $p_n(x_0) = \sum_{k=0}^{n} c_k (-1)^{n-k} \binom{X}{n-k}$.

3.4. Asymptotic properties of the coefficients c_k

The asymptotic nature of the expansions (39), (40) and (42) for large values of some of the parameters of the polynomial $p_n(x)$ depends on the asymptotic behaviour of the coefficients c_k. The following lemma is proved in [7]:

Lemma 1. *Let $\phi(w)$ be an analytic function at $w = 0$, with Maclaurin expansion of the form*

$$\phi(w) = \mu^s \omega^m (a_0 + a_1 w + a_2 w^2 + a_3 w^3 + \ldots),$$

where m is a positive integer, s is an integer number, and a_k are complex numbers that satisfy $a_k = \mathcal{O}(1)$ when $\mu \to \infty$, $a_0 \neq 0$. Let c_k denote the coefficients of the power series of $f(w) = e^{\phi(w)}$, that is,

$$f(w) = e^{\phi(w)} = \sum_{k=0}^{\infty} c_k w^k.$$

Then $c_0 = 1$, $c_k = 0$, $k = 1, 2, \ldots, m-1$, $c_k = \mathcal{O}\left(\mu^{[sk/m]}\right)$ if $s > 0$ and $c_k = \mathcal{O}\left(\mu^s\right)$ if $s \leq 0$ when $\mu \to \infty$.

4. Proofs of the formulae of Section 2

4.1. Proofs of the formulae of Section 2.1

Substitute:

$$F(x,w) = (1-w)^{-c+ix} {}_2F_1\left(\left.\begin{array}{c} a+ix, b+ix \\ a+b \end{array}\right| w\right) \quad \text{and} \quad p_n(x) = \frac{S_n^{a,b,c}(x^2)}{(a+b)_n n!}$$

in the formulae of Section 3.1 to obtain (2).

The function $\phi(w) = \log f(w)$ verifies Lemma 1 with $\mu = c$, $s = 1$ and $m = 3$. Therefore, we have $c_k = \mathcal{O}(c^{[k/3]})$. On the other hand, we trivially have $B = \mathcal{O}(\sqrt{c})$, $X = \mathcal{O}(\sqrt{c})$ and $H_{n-k}(X) = \mathcal{O}(c^{(n-k)/2})$ and we obtain the asymptotic behaviour 2.1.2. The limit (4) follows from the first term of the expansion (2) after obtaining $x(X)$.

4.2. Proofs of the formulae of Section 2.2

Substitute:

$$F(x,w) = (1-w)^{N-x} {}_2F_1\left(\left.\begin{array}{c} -x, -b-x \\ a+1 \end{array}\right| w\right) \quad \text{and} \quad p_n(x) = \frac{(-N)_n R_n^{a,b,N}(\lambda(x))}{n!}$$

in the formulae of Section 3.1 to obtain (5).

The function $\phi(w) = \log f(w)$ verifies Lemma 1 with $\mu = N$, $s = 1$ and $m = 3$. Therefore we have $c_k = \mathcal{O}(N^{[k/3]})$. On the other hand, we trivially have $B = \mathcal{O}(\sqrt{N})$, $X = \mathcal{O}(\sqrt{N})$ and $H_{n-k}(X) = \mathcal{O}(N^{(n-k)/2})$ and we obtain the asymptotic behaviour 2.2.2. The limit (7) follows from the first term of (5) after obtaining $x(X)$.

4.3. Proofs of the formulae of Section 2.3

Substitute:

$$F(x,w) = (1-w)^{1-2(a+b)}{}_3F_2\left(\begin{matrix} a+b-\frac{1}{2}, a+b, a+i(b+x) \\ 2a, a+c+i(b+d) \end{matrix}\,\middle|\, -\frac{4w}{(1-w)^2}\right)$$

and $$p_n(x) = \frac{(2a+2b-1)_n}{(2a)_n(a+b+i(c-b))_n} P_n^{\alpha,\beta,\overline{\alpha},\overline{\beta}}(x)$$

in the formulae of Section 3.1 to obtain (8).

In Section 4.7 we will show that the Taylor coefficients at $w=0$ of the logarithm of the above ${}_3F_2$ function are of the order $\mathcal{O}(a)$. Therefore, the function $\phi(w) = \log f(w)$ verifies Lemma 1 with $\mu = a$, $s=1$ and $m=3$ and we have $c_k = \mathcal{O}(a^{[k/3]})$. On the other hand, we have $B = \mathcal{O}(\sqrt{a})$, $X = \mathcal{O}(1/\sqrt{a})$ and $H_{n-k}(X) = \mathcal{O}(a^0)$ and we obtain the asymptotic behaviour 2.3.2. The limit (12) follows from the first term of (8) after obtaining $x(X)$.

4.4. Proofs of the formulae of Section 2.4

Substitute:

$$F(x,w) = (1-w)^{-a-b-1}{}_3F_2\left(\begin{matrix} (a+b+1)/2, (a+b+2)/2, -x \\ a+1, -N \end{matrix}\,\middle|\, -\frac{4w}{(1-w)^2}\right)$$

and $$p_n(x) = \frac{(-N)_n Q_n^{a,b,N}(x)}{(b+1)_n n!}$$

in the formulae of Section 3.1 to obtain (13).

The function $\phi(w) = \log f(w)$ verifies Lemma 1 with $\mu = N$, $s=1$ and $m=3$. Therefore we have $c_k = \mathcal{O}(N^{[k/3]})$. On the other hand, we trivially have $B = \mathcal{O}(\sqrt{N})$, $X = \mathcal{O}(\sqrt{N})$ and $H_{n-k}(X) = \mathcal{O}(N^{(n-k)/2})$ and we obtain the asymptotic behaviour 2.4.2. The limit (16) follows from the first term of (13) after obtaining $x(X)$.

4.5. Proofs of the formulae of Section 2.5

Substitute:

$$F(x,w) = (1-w)^{-c+ix}{}_2F_1\left(\begin{matrix} a+ix, b+ix \\ a+b \end{matrix}\,\middle|\, w\right) \quad \text{and} \quad p_n(x) = \frac{S_n^{a,b,c}(x^2)}{(a+b)_n n!}$$

in the formulae of Section 3.2 to obtain (17).

The function $y(w) = \log {}_2F_1\left(\begin{matrix} a+ix, b+ix \\ a+b \end{matrix}\,\middle|\, w\right)$ satisfies the following differential equation in the variable w:

$$w(1-w)(y+{y'}^2) + (a+b-(a+b+2ix+1)w)\,y' - (a+ix)(b+ix) = 0.$$

Substituting the Maclaurin series of $y(w) = \sum_1^\infty b_k w^k$ into this differential equation, we obtain

$$b_1 = \frac{(a+ix)(b+ix)}{a+b}, \quad b_2 = (a+ix)(b+ix)\frac{(a+b)(a+b+2ix+1)-(a+ix)(b+ix)}{2(a+b)^2(a+b+1)}$$

and

$$b_{k+1} = \frac{1}{(k+1)(k+a+b)}\Bigg\{k(k+a+b+2ix)b_k - kb_kb_1 + \sum_{j=0}^{k-2}(j+1)b_{j+1}\left[(k-j-1)b_{k-j-1}-(k-j)b_{k-j}\right]\Bigg\}.$$

Then, $b_1 = \mathcal{O}(a)$, $b_2 = \mathcal{O}(a)$ and using the above recurrence we can show by induction over k that $b_k = \mathcal{O}(a)$ for $k > 2$. Therefore, the function $\phi(w) = \log f(w)$ verifies Lemma 1 with $\mu = a$, $s = 1$ and $m = 3$. Therefore we have $c_k = \mathcal{O}(a^{[k/3]})$. On the other hand we trivially have $A = \mathcal{O}(a)$, $X = \mathcal{O}(a)$ and, taking into account that $\lim_{a\to\infty} A/X \neq 1$, we have $L_{n-k}^X(A) = \mathcal{O}(a^{n-k})$ and we obtain the asymptotic behaviour 2.5.2.

4.6. Proofs of the formulae of Section 2.6

Substitute:

$$F(x,w) = (1-w)^{N-x}{}_2F_1\left(\begin{matrix}-x,-b-x\\ a+1\end{matrix}\,\middle|\,w\right) \quad \text{and} \quad p_n(x) = \frac{(-N)_n R_n^{a,b,N}(\lambda(x))}{n!}$$

in the formulae of Section 3.2 to obtain (21).

The function $\phi(w) = \log f(w)$ verifies Lemma 1 with $\mu = N$, $s = -1$ and $m = 3$. Therefore we have $c_k = \mathcal{O}(N^{-1})$. On the other hand, we trivially have $A = \mathcal{O}(N^{-1})$, $X = \mathcal{O}(N)$ and $L_{n-k}^X(A) = \mathcal{O}(N^{n-k})$ and we obtain the asymptotic behaviour 2.6.2.

4.7. Proofs of the formulae of Section 2.7

Substitute:

$$F(x,w) = (1-w)^{1-2(a+b)}{}_3F_2\left(\begin{matrix}a+b-\frac{1}{2}, a+b, a+i(b+x)\\ 2a, a+c+i(b+d)\end{matrix}\,\middle|\,-\frac{4w}{(1-w)^2}\right)$$

$$\text{and} \qquad p_n(x) = \frac{(2a+2b-1)_n}{(2a)_n(a+b+i(c-b))_n}P_n^{\alpha,\beta,\overline{\alpha},\overline{\beta}}(x)$$

in the formulae of Section 3.2 to obtain (24).

Substituting the Maclaurin series of the logarithm of the above ${}_3F_2$ function into its differential equation [10], and following a similar argument as in subsection 4.5 we find that the Taylor coefficients at $w = 0$ of the logarithm of the above ${}_3F_2$ function are of the order $\mathcal{O}(b)$. Therefore, the function $\phi(w) = \log f(w)$ verifies Lemma 1 with $\mu = b$, $s = 1$ and $m = 3$. Then we have $c_k = \mathcal{O}(b^{[k/3]})$. On the other hand, $A = \mathcal{O}(b)$, $X = \mathcal{O}(b)$ and $L_0^X(A) = \mathcal{O}(b^0)$, $L_1^X(A) = X + 1 - A = p_1(x) = \mathcal{O}(b^0)$, and using the recurrence relation [4]

$$(n+1)L_{n+1}^X(A) - (2n+X+1-A)L_n^X(A) + (n+X)L_{n-1}^X(A) = 0,$$

we can show by induction over n that $L_{n-k}^X(A) = \mathcal{O}(b^{[(n-k)/2]})$ and we obtain the asymptotic behaviour 2.7.2.

4.8. Proofs of the formulae of Section 2.8

Substitute:

$$F(x,w) = (1-w)^{-a-b-1}{}_3F_2\left(\begin{matrix}(a+b+1)/2, (a+b+2)/2, -x\\ a+1, -N\end{matrix}\,\middle|\,-\frac{4w}{(1-w)^2}\right)$$

$$\text{and} \qquad p_n(x) = \frac{(-N)_n Q_n^{a,b,N}(x)}{(b+1)_n n!}$$

in the formulae of Section 3.2 to obtain (26).

The function $\phi(w) = \log f(w)$ verifies Lemma 1 with $\mu = N$, $s = 0$ and $m = 3$. Then we have $c_k = \mathcal{O}(N^0)$. On the other hand, we trivially have $A = \mathcal{O}(N^0)$, $X = \mathcal{O}(N)$ and $L_{n-k}^X(A) = \mathcal{O}(N^{n-k})$ and we obtain the asymptotic behaviour 2.8.2.

4.9. Proofs of the formulae of Section 2.9

Substitute:

$$F(x,w)=(1-w)^{-c+ix}{}_2F_1\left(\left.\begin{matrix}a+ix,b+ix\\a+b\end{matrix}\right|w\right)\quad\text{and}\quad p_n(x)=\frac{S_n^{a,b,c}(x^2)}{(a+b)_n n!}$$

in the formulae of Section 3.3 to obtain (28).

From Section 4.5 we have that the coefficients of the Taylor expansion at $w=0$ of the logarithm of the above ${}_2F_1$ function are of the order $\mathcal{O}(a)$. Then, the function $\phi(w)=\log f(w)$ verifies Lemma 1 with $\mu=a$, $s=1$ and $m=3$. Then we have $c_k=\mathcal{O}(a^{[k/3]})$. On the other hand, $A=\mathcal{O}(a)$, $X=\mathcal{O}(a)$ and, taking into account that $\lim_{a\to\infty}A/X\neq 1$, we have $C_{n-k}(X,A)=\mathcal{O}(a^0)$ and we obtain the asymptotic behaviour 2.9.2. The limit (31) follows from the first term of (28) after obtaining $x^2(X,A)$ and $c(X,A)$.

4.10. Proofs of the formulae of Section 2.10

Substitute:

$$F(x,w)=(1-w)^{N-x}{}_2F_1\left(\left.\begin{matrix}-x,-b-x\\a+1\end{matrix}\right|w\right)\quad\text{and}\quad p_n(x)=\frac{(-N)_nR_n^{a,b,N}(\lambda(x))}{n!}$$

in the formulae of Section 3.3 to obtain (33).

The function $\phi(w)=\log f(w)$ verifies Lemma 1 with $\mu=a$, $s=-2$ and $m=3$. Then we have $c_k=\mathcal{O}(a^{-2})$. On the other hand, we trivially have $A=\mathcal{O}(a^{-1})$, $X=\mathcal{O}(a)$ and $C_{n-k}(X,A)=\mathcal{O}(a^{2n-2k})$ and we obtain the asymptotic behaviour 2.10.2.

4.11. Proofs of the formulae of Section 2.11

Substitute:

$$F(x,w)=(1-w)^{1-2(a+b)}{}_3F_2\left(\left.\begin{matrix}a+b-\frac{1}{2},a+b,a+i(b+x)\\2a,a+c+i(b+d)\end{matrix}\right|-\frac{4w}{(1-w)^2}\right)$$

$$\text{and}\qquad p_n(x)=\frac{(2a+2b-1)_n}{(2a)_n(a+b+i(c-b))_n}P_n^{\alpha,\beta,\overline{\alpha},\overline{\beta}}(x)$$

in the formulae of Section 3.3 to obtain (35).

From Section 4.7, the coefficients of the Taylor expansion at $w=0$ of the logarithm of the above ${}_3F_2$ function are of the order $\mathcal{O}(b)$. Then, the function $\phi(w)=\log f(w)$ verifies Lemma 1 with $\mu=b$, $s=1$ and $m=3$. Therefore we have $c_k=\mathcal{O}(b^{[k/3]})$. On the other hand, $A=\mathcal{O}(b)$, $X=\mathcal{O}(b)$ and $C_0(X,A)=1=\mathcal{O}(b^0)$, $C_1(X,A)=(A-X)/A=p_1(x)/A=\mathcal{O}(b^{-1})$, and using the recurrence relation [4]

$$AC_{n+1}(X,A)+(X-A-n)C_n(X,A)+nC_{n-1}(X,A)=0,$$

we can show by induction over n that $C_{n-k}(X,A) = \mathcal{O}(b^{-[(n+k)/2]})$ and we obtain the asymptotic behaviour 2.11.2.

4.12. Proofs of the formulae of Section 2.12

Substitute:

$$F(x,w) = (1-w)^{-a-b-1}{}_3F_2\left(\left. \begin{array}{c} (a+b+1)/2, (a+b+2)/2, -x \\ a+1, -N \end{array} \right| -\frac{4w}{(1-w)^2} \right)$$

$$\text{and} \qquad p_n(x) = \frac{(-N)_n Q_n^{a,b,N}(x)}{(b+1)_n n!}$$

in the formulae of Section 3.3 to obtain (37).

The function $\phi(w) = \log f(w)$ verifies Lemma 1 with $\mu = a$, $s = 0$ and $m = 3$. Then we have $c_k = \mathcal{O}(a^0)$. On the other hand, we trivially have $A = \mathcal{O}(a^0)$, $X = \mathcal{O}(a)$ and $C_{n-k}(X,A) = \mathcal{O}(a^{n-k})$ and we obtain the asymptotic behaviour 2.12.2.

5. Numerical experiments

The following graphics illustrate the approximation supplied by the expansions given in Section 2. It is worthwhile to note the accuracy obtained in the approximation of the zeros of the polynomials. In all of the graphics, the degree of the polynomials is $n = 6$, dashed lines represent the exact polynomial and continuous lines represent the first order approximation given by the corresponding expansion.

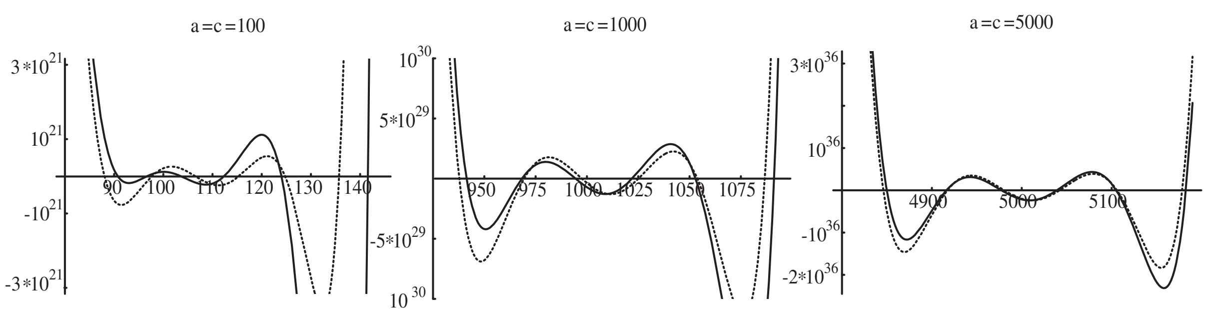

Figure 2. *Expansion (2) for* $b = 8$. $S_6^{a,b,c}(x^2)/(a+b)_6$ *versus* $B^6 H_6(X)$.

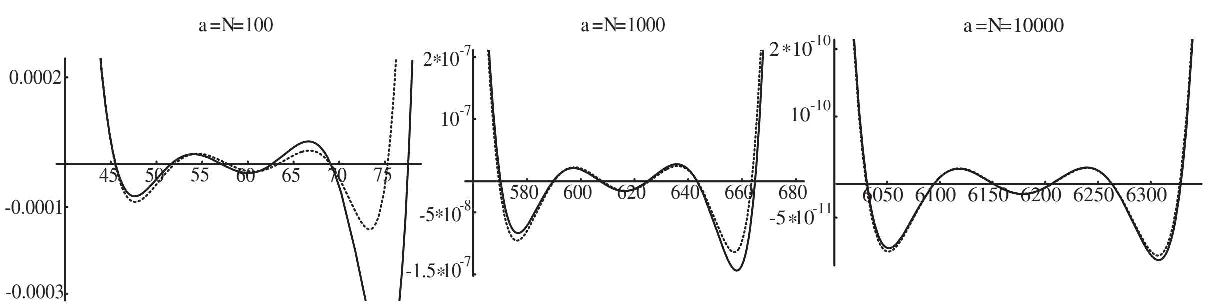

Figure 3. *Expansion (5) for* $b = 8$. $-(N)_6 R_6^{a,b,N}(\lambda(x))$ *versus* $B^6 H_6(X)$.

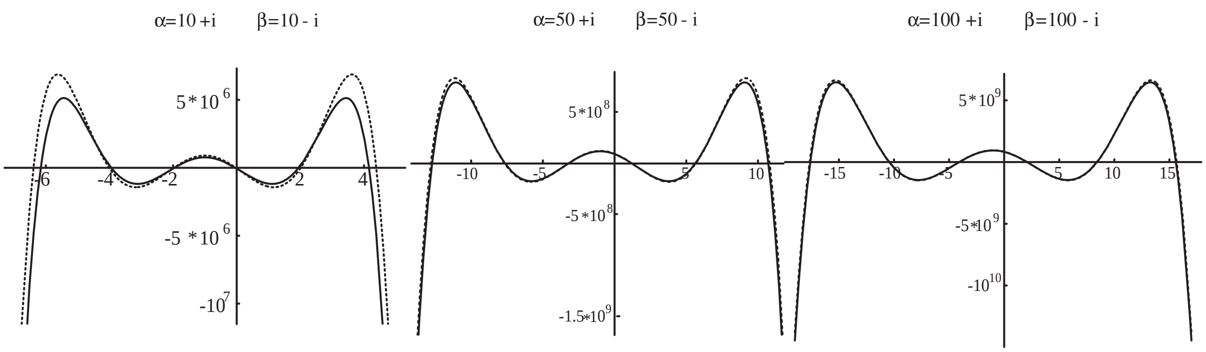

Figure 4. *Expansion (5).* $-(2a+2b-1)_6 P_6^{\alpha,\beta,\bar{\alpha},\bar{\beta}}(x)/[(2a)_6(a+b+i(c-d))_6]$ *versus* $B^6 H_6(X)/6!$.

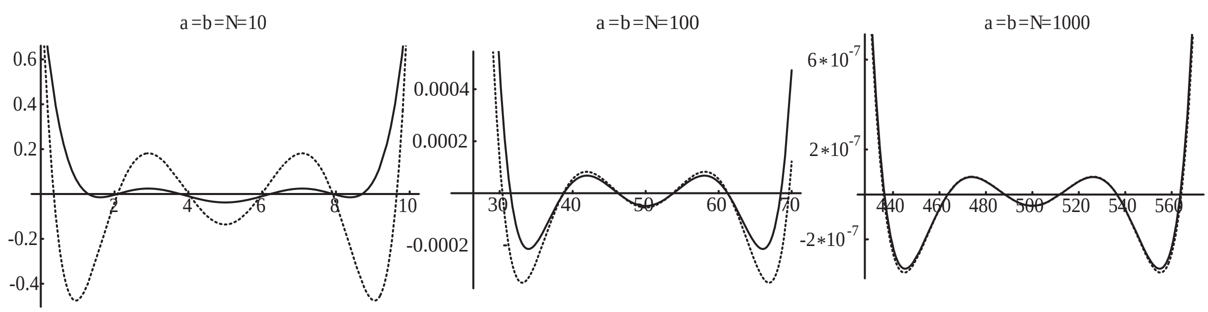

Figure 5. *Approximation (13).* $(-N)_6 Q_6^{a,b,c}(x)/(b+1)_6$ *versus* $B^6 H_6(X)$.

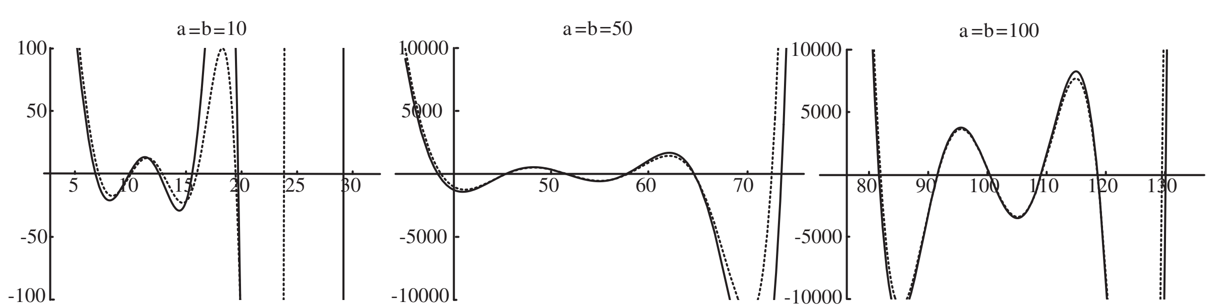

Figure 6. *Expansion (17) for* $c = 3/2$. $S_6^{a,b,c}(x^2)/[6!(a+b)_6]$ *versus* $L_6^X(A)$.

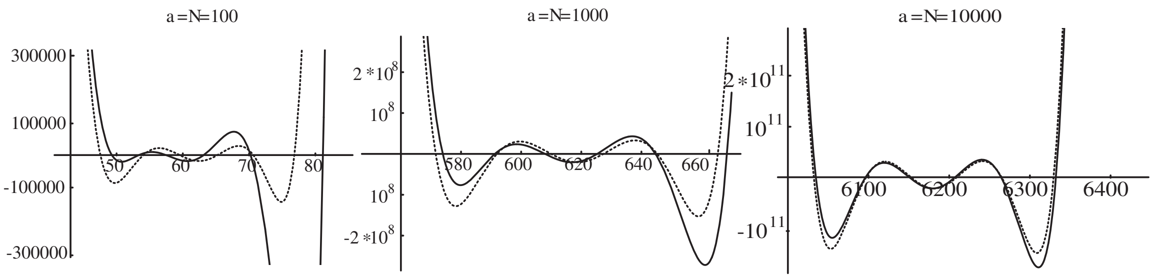

Figure 7. *Expansion (21) for* $b = 1$. $-(N)_6 R_6^{a,b,N}(\lambda(x))/6!$ *versus* $L_6^X(A)$.

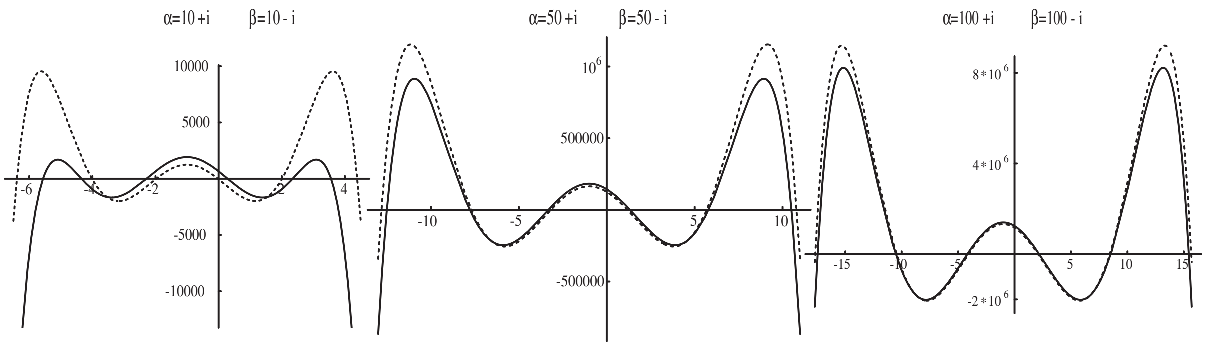

Figure 8. *Expansion (24)*. $-(2a + 2b - 1)_6 P_6^{\alpha,\beta,\bar{\alpha},\bar{\beta}}(x)/[(2a)_6(a + b + i(c - d))_6]$ *versus* $L_6^X(A)$.

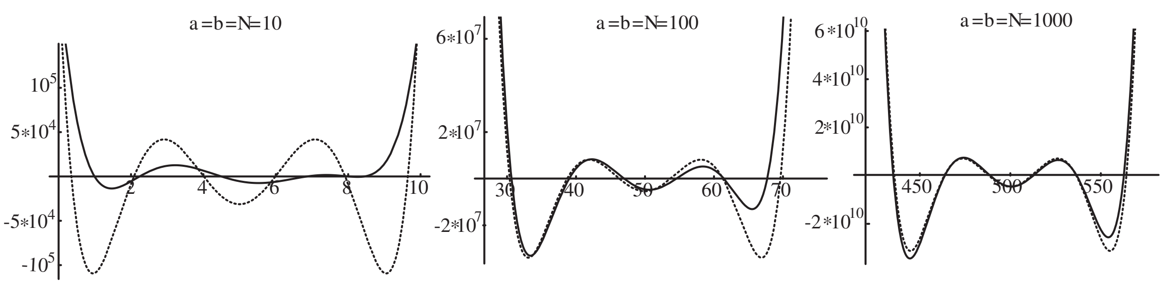

Figure 9. *Expansion (26)*. $(a + b + 1)_6 Q_6^{a,b,N}(x)/6!$ *versus* $L_6^X(A)$.

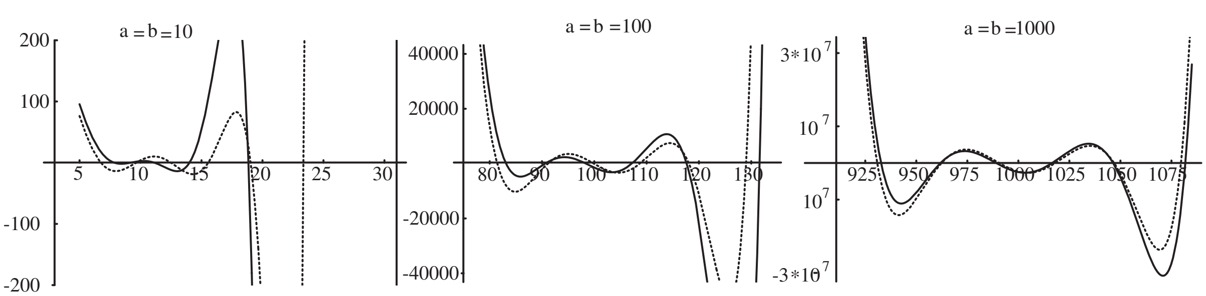

Figure 10. *Expansion (28) for* $c = 1$. $S_6^{a,b,c}(x^2)/(a + b)_6$ *versus* $A^6 C_6(X, A)$.

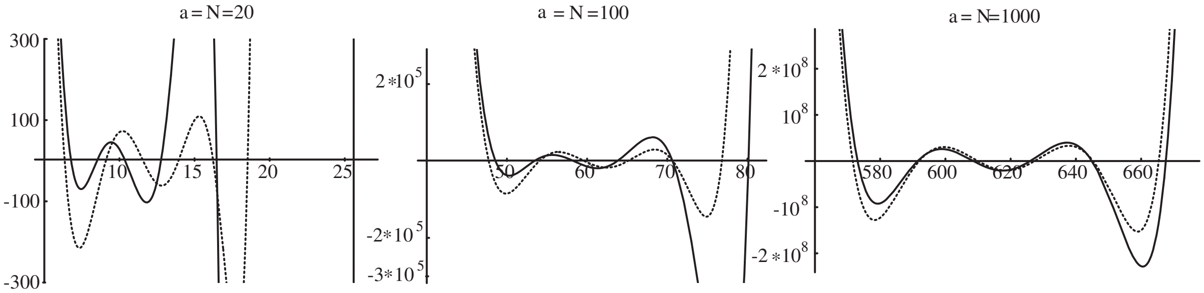

Figure 11. *Expansion (33) for* $b = 1$. $-(N)_6 R_6^{a,b,N}(\lambda(x))$ *versus* $A^6 C_6(X, A)$.

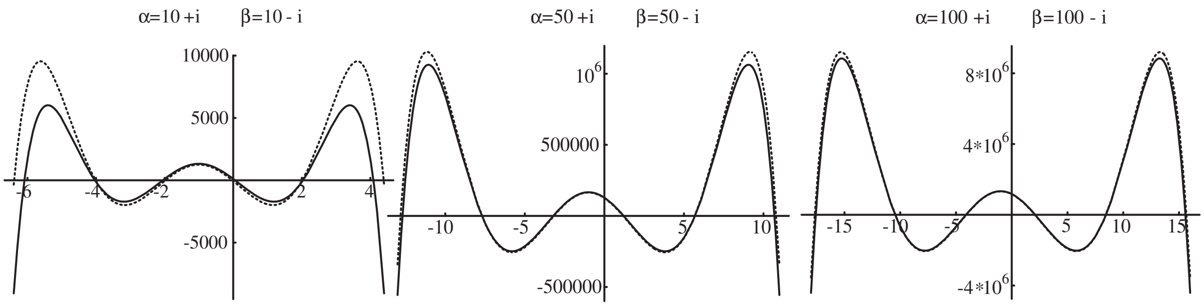

Figure 12. *Expansion (35)*. $-(2a + 2b - 1)_6 P_6^{\alpha,\beta,\bar{\alpha},\bar{\beta}}(x)/[(2a)_6(a + b + i(c - d))_6]$ *versus* $A^6 C_6(X, A)$.

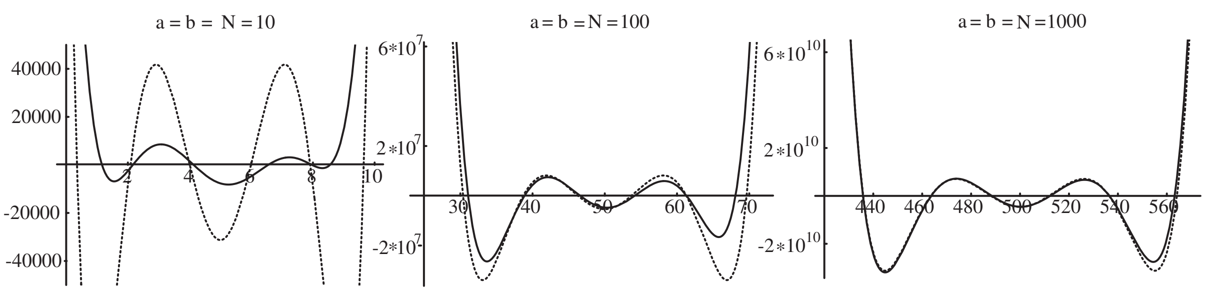

Figure 13. *Expansion (37)*. $(a + b + 1)_6 Q_6^{a,b,N}(x)$ *versus* $A^6 C_6(X, A)$.

6. Acknowledgments

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